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PHYSICS-BASED MULTI-TIME STEPPING ALGORITHM FOR THREE-DIMENSIONAL QUASI-STATIC ELECTROPOROELASTICITY EQUATIONS

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Abstract. Electroporoelasticity equations comprise of Maxwell's equations and Biot's equations, playing an important role in geophysical areas such as oil-gas exploration and earthquake early warning. The development of electromagnetic waves and elastic waves presents distinct time scales due to the multi-physics nature. In this paper, we propose a multi-time stepping numerical algorithm to approximate electroporoelasticity equations, in which we use a smaller time step to compute Maxwell's equations and a larger time step to calculate Biot's equations. We prove the stability of this algorithm and derive its error estimates. Numerical experiments are conducted to demonstrate the theoretical analysis.

Key words. Electroporoelasticity equations, multi-time stepping algorithm, finite element method, error estimates.

1. Introduction

Natural resource reservoirs such as water, oil and gas are predominantly poroelastic media, comprising of solid skeletons and pore fluids [7]. The seismoelectric coupling phenomenon emerges from the relative motion between the solid matrix and fluid in fluid-saturated porous media, induced by the presence of an electric double layer consisting of an adsorbed layer and a diffuse layer. This phenomenon includes both the seismoelectric effect and the electroseismic effect and leads to seismoelectric coupling waves [25]. In the absence of fluid flow, the porous medium is electrically neutral overall. However, when seismic waves propagate through the porous medium, fluid flow occurs within the pores, causing charges in the diffuse layer to move relatively to those in the adsorbed layer and thereby forming a streaming current, which is the seismoelectric effect. And vice versa, the electroseismic effect occurs when the electric field in the porous medium changes, charges in the diffuse layer move within the electric field to generate a conduction current, which simultaneously drags the fluid in the diffuse layer into motion.

Seismoelectric coupling is modeled by electroporoelasticity equations which consist of Maxwell's equations [2] and Biot's equations via an electrokinetic coupling coefficient. Seismoelectric coupling waves integrate the spatial resolution of elastic waves with the reservoir identification capability of electromagnetic waves, enabling the seismoelectric coupling effect to find applications in diverse fields such as oil-gas exploration [35], earthquake early warning [26, 27], environmental protection [20], water conservancy exploration [15], and other areas of geophysics [3, 34]. Due to its importance in applications, much attention has already been paid to electroporoelasticity equations, cf. [11, 12, 13, 16, 18, 19, 29, 30].

Numerical methods are important tools to explore the multi-physics nature of electroporoelasticity equations. Hu and Meir [12] proposed a numerical scheme using the standard finite element method (FEM) and analyzed its error estimates.

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Liu et al. [16] investigated the well-posedness and applied splitting technique to set up a finite element approximation to improve the computational efficiency. In recent years, a multi-time stepping technique has been developed to accelerate the numerical computations for solving partial differential equations. Shan et al. [31] constructed a decoupled scheme with different time steps for a nonstationary Stokes-Darcy model and verified its stability and convergence. Rybak and Magiera [28] developed a mass conservative multi-time stepping method for solving coupled free flow and porous medium flow problems, proved its long time stability and performed error eatimates. Shevchenko et al. [32] presented a multi-time stepping integration method for the ultrasound heating problem and showed its efficiency and robustness. Zhang et al. [38] studied a finite element approximation to the Stokes-Darcy-Transport system with different time steps on different physical variables. In these studies, the large discrepancy in time scales of partial differential equations leads to that the equations can be solved with different time scales. The key idea is to use a small time step to discretize the temporal variable in the equation that changes rapidly, while using a large time step to solve the equation that changes slowly. For more references, we refer to [8, 14, 24, 33, 39] and the literature therein.

Electroporoelasticity equations form a complex system of coupled, multiphysics, multi-component, and multiscale models. It is common knowledge that electromagnetic waves propagate much faster than seismic waves. In this paper, we investigate a multi-time stepping algorithm to improve the efficiency of numerical approximations to three-dimensional quasi-static electroporoelasticity equations. The main idea consists of two steps. First, we decouple the electroporoelasticity equations into Maxwell's equations and Biot's equations. Then, we discretize Maxwell's equations with a smaller time step-size and approximate Biot's equations with a larger time step-size. We prove the stability and first-order convergence in temporal and spatial variables, respectively, of the multi-time stepping algorithm.

The rest of this paper is organized as follows. Section 2 introduces the quasi-static electroporoelasticity equations along with the finite element spaces. In Section 3, we propose a physics-based multi-time stepping algorithm and establish its stability. Error estimates for the numerical approximation are derived in Section 4. Section 5 presents numerical experiments that validate the theoretical analysis.

2. Preliminaries

In this section, we describe quasi-static electroporoelasticity equations and introduce finite element spaces used in this paper.

2.1. Electroporoelasticity equations. Let $[0,T]$ with $T>0$ be an interval and $\mathscr{D}\subset\mathbb{R}^3$ be an open bounded polyhedron with $\mathbf{n}$ the unit outward normal vector to boundary $\partial\mathscr{D}$. Consider three-dimensional quasi-static electroporoelasticity equations for $(t,\mathbf{x})\in[0,T]\times\mathscr{D}$

$$
\begin{aligned}
&\epsilon\frac{\partial}{\partial t}\mathbf{E}+\sigma\mathbf{E}-\nabla\times\mathbf{H}-L\nabla p=\mathbf{j},\\
&\mu\frac{\partial}{\partial t}\mathbf{H}+\nabla\times\mathbf{E}=\mathbf{0},\\
&-(\lambda+G)\nabla(\nabla\cdot\mathbf{u})-G\Delta\mathbf{u}+\alpha\nabla p=\mathbf{f},\\
&\frac{\partial}{\partial t}(c_0p+\alpha\nabla\cdot\mathbf{u})-\kappa\Delta p+L\nabla\cdot\mathbf{E}=g,
\end{aligned}
\tag{1}
$$

with initial value conditions for $\mathbf{x}\in\mathscr{D}$

$$
\mathbf{E}(0,\mathbf{x})=\mathbf{E}_0(\mathbf{x}),\quad \mathbf{H}(0,\mathbf{x})=\mathbf{H}_0(\mathbf{x}),\quad \mathbf{u}(0,\mathbf{x})=\mathbf{u}_0(\mathbf{x}),\quad p(0,\mathbf{x})=p_0(\mathbf{x}),
\tag{2}
$$

and boundary value conditions for $(t, \mathbf{x}) \in [0, T] \times \partial\mathscr{D}$

$$\mathbf{E}(t, \mathbf{x}) \times \mathbf{n} = \mathbf{0}, \quad p(t, \mathbf{x}) = 0, \quad \mathbf{u}(t, \mathbf{x}) = \mathbf{0}. \tag{3}$$

Here, $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, $\mathbf{u}$ is the displacement of solid matrix and p is the pressure in the fluid. The parameter ϵ is the electric permittivity, σ denotes the electrical conductivity, μ represents the magnetic permeability of the material, L is the frequency-dependent electrokinetic coupling coefficient, $\alpha \in (0, 1]$ represents the Biot-Willis coefficient [5], λ denotes the Lamé constant, G is the shear modulus of porous media, $c_0 = \frac{1}{M}$ represents the constrained specific storage coefficient with M the Biot modulus [4] and $\kappa = \frac{\gamma}{\eta}$ denotes the hydraulic conductivity with γ the permeability of porous media and η the shear viscosity of fluid. For simplifying analysis of error estimates, we assume that $\mathbf{f} = \mathbf{0}$ and parameters ϵ, σ, μ, L, α, λ, G, c_0 and κ are all fixed positive constants. Throughout this paper, we also denote $\mathbf{E}(t, \mathbf{x})$ by either $\mathbf{E}(t)$ or $\mathbf{E}$, etc. when no confusion occurs.

For any integer $\beta \geq 0$, denote by $H^\beta := H^\beta(\mathscr{D})$ the standard Sobolev space with norm $\|\cdot\|_\beta$ and $H_0^\beta = \{v \in H^\beta : D^\iota v|_{\partial\mathscr{D}} = 0, |\iota| < \beta\}$ [6, 36, 37]. Here, $H^0 = L^2(\mathscr{D})$ is the square integrable function space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Throughout this paper, we will use bold italic letters to represent spaces of three-dimensional vector fields ($\boldsymbol{H}^\beta = (H^\beta)^3$, etc.) and also utilize $\|\cdot\|_\beta$ to denote the norm in $\boldsymbol{H}^\beta$ when no confusion occurs. Let $C(\bar{\mathscr{D}})$ be a space consisting of continuous functions on $\bar{\mathscr{D}}$. Define three subspaces of $\boldsymbol{L}^2(\mathscr{D})$

$$\begin{aligned}
\boldsymbol{H}(div) &= \{\mathbf{v} : \mathbf{v} \in \boldsymbol{L}^2(\mathscr{D}),\ \nabla \cdot \mathbf{v} \in L^2(\mathscr{D})\}, \\
\boldsymbol{H}(\mathbf{curl}) &= \{\mathbf{v} : \mathbf{v} \in \boldsymbol{L}^2(\mathscr{D}),\ \nabla \times \mathbf{v} \in \boldsymbol{L}^2(\mathscr{D})\}, \\
\boldsymbol{H}_0(\mathbf{curl}) &= \{\mathbf{v} : \mathbf{v} \in \boldsymbol{H}(\mathbf{curl}),\ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\mathscr{D}\}.
\end{aligned}$$

For simplifying notations, denote by

$$\begin{aligned}
\mathbb{S} &:= \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl}) \times \boldsymbol{H}_0^1 \times H_0^1, \\
\mathbb{H} &:= \boldsymbol{L}^2(\mathscr{D}) \times \boldsymbol{L}^2(\mathscr{D}) \times \boldsymbol{L}^2(\mathscr{D}) \times L^2(\mathscr{D}).
\end{aligned}$$

The variational form of (1)–(3) is to seek $(\mathbf{E}, \mathbf{H}, \mathbf{u}, p) \in L^2(0, T; \mathbb{S}) \cap H^1(0, T; \mathbb{H})$ satisfying initial value condition (2) and

$$\begin{aligned}
&\big(\epsilon\frac{\partial}{\partial t}\mathbf{E}(t), \mathbf{D}\big) + \big(\sigma\mathbf{E}(t), \mathbf{D}\big) - \big(\mathbf{H}(t), \nabla \times \mathbf{D}\big) - \big(L\nabla p(t), \mathbf{D}\big) = \big(\mathbf{j}(t), \mathbf{D}\big), \\
&\big(\mu\frac{\partial}{\partial t}\mathbf{H}(t), \mathbf{B}\big) + \big(\nabla \times \mathbf{E}(t), \mathbf{B}\big) = 0, \\
&\big((\lambda + G)\nabla \cdot \mathbf{u}(t), \nabla \cdot \mathbf{v}\big) + \big(G\nabla\mathbf{u}(t), \nabla\mathbf{v}\big) - \big(p(t), \alpha\nabla \cdot \mathbf{v}\big) = 0, \\
&\big(c_0\frac{\partial}{\partial t}p(t), q\big) + \big(\alpha\nabla \cdot \frac{\partial}{\partial t}\mathbf{u}(t), q\big) + \big(\kappa\nabla p(t), \nabla q\big) - \big(L\mathbf{E}(t), \nabla q\big) = \big(g(t), q\big),
\end{aligned} \tag{4}$$

for all $t \in [0, T]$ and $(\mathbf{D}, \mathbf{B}, \mathbf{v}, q) \in \mathbb{S}$.

Now, we introduce some assumptions used in this paper.

(H1) $0 < L < \sqrt{\sigma\kappa}$.

(H2) The functions $\mathbf{E}_0$, $\mathbf{H}_0$, $\mathbf{u}_0$, p_0, $\mathbf{j}$ and g are sufficiently smooth.

Under these two assumptions, (4) has a unique solution $(\mathbf{E}, \mathbf{H}, \mathbf{u}, p) \in L^2(0, T; \mathbb{S}) \cap H^1(0, T; \mathbb{H})$, cf. [16].

Define a symmetric bilinear form

$$a(\mathbf{w}, \mathbf{v}) = ((\lambda + G)\nabla \cdot \mathbf{w}, \nabla \cdot \mathbf{v}) + (G\nabla\mathbf{w}, \nabla\mathbf{v}), \quad \forall\, \mathbf{w}, \mathbf{v} \in \boldsymbol{H}_0^1.$$

This leads to an equivalent norm $\|\cdot\|_a = \sqrt{a(\cdot, \cdot)}$ in $\boldsymbol{H}_0^1$, i.e., there exist two constants $0 < c_1 < c_2$ such that $c_1\|\cdot\|_1 \leq \|\cdot\|_a \leq c_2\|\cdot\|_1$.

2.2. Finite element spaces. Let $\mathcal{T}_h$ be a regular tetrahedral partition of $\mathscr{D}$. For each element $K \in \mathcal{T}_h$ with faces $\mathcal{F}_K$ and edges $\mathcal{E}_K$, let h_K be its diameter and $h = \max_{K\in\mathcal{T}_h} h_K$ be the mesh size of $\mathcal{T}_h$. For any integer $l \geq 0$, denote by $P_l(K)$ and $\tilde{P}_l(K)$ the spaces of polynomials with degree no more than l and homogeneous polynomials with degree of l, respectively. For any $l \geq 1$, define two spaces

$$\begin{aligned} \boldsymbol{S}_l(K) &= \{\mathbf{L} \in \tilde{\boldsymbol{P}}_l(K) : \mathbf{L}(\mathbf{x}) \cdot \mathbf{x} = 0,\ \forall \mathbf{x} \in K\}, \\ \boldsymbol{R}_l(K) &= \boldsymbol{P}_{l-1}(K) \oplus \boldsymbol{S}_l(K). \end{aligned}$$

Now, we construct four finite element spaces

$$\begin{aligned} \mathbb{E}_h &= \{\mathbf{E}_h \in \boldsymbol{H}_0(\mathbf{curl}) : \mathbf{E}_h|_K \in \boldsymbol{R}_1(K),\ \forall K \in \mathcal{T}_h\}, \\ \mathbb{H}_h &= \{\mathbf{H}_h \in \boldsymbol{L}^2(\mathscr{D}) : \mathbf{H}_h|_K \in \boldsymbol{P}_0(K),\ \forall K \in \mathcal{T}_h\}, \\ \mathbb{U}_h &= \{\mathbf{u}_h \in \boldsymbol{C}(\bar{\mathscr{D}}) : \mathbf{u}_h|_{\partial\mathscr{D}} = 0,\ \mathbf{u}_h|_K \in \boldsymbol{P}_1(K),\ \forall K \in \mathcal{T}_h\}, \\ \mathbb{P}_h &= \{p_h \in C(\bar{\mathscr{D}}) : p_h|_{\partial\mathscr{D}} = 0,\ p_h|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\}. \end{aligned}$$

Notice that $\mathbb{E}_h$ is the space consisting of Nédélec elements [23].

Let $\mathfrak{P}_h : \boldsymbol{L}^2(\mathscr{D}) \to \mathbb{E}_h$, $\mathcal{P}_h : \boldsymbol{L}^2(\mathscr{D}) \to \mathbb{H}_h$, $\mathscr{P}_h : \boldsymbol{L}^2(\mathscr{D}) \to \mathbb{U}_h$ and $P_h : L^2(\mathscr{D}) \to \mathbb{P}_h$ be L^2-orthogonal projection operators, respectively. The next lemma collects some approximation estimates of these operators.

Lemma 2.1. *([9, Lemma 11.18]) There exists a constant $C > 0$ independent of h such that for any $(\mathbf{E}, \mathbf{H}, \mathbf{u}, p) \in \boldsymbol{H}^2 \times \boldsymbol{H}^1 \times \boldsymbol{H}^2 \times H^2$,*

$$\begin{aligned} \|\mathbf{E} - \mathfrak{P}_h\mathbf{E}\| &\leq Ch^2\|\mathbf{E}\|_2, \\ \|\mathbf{H} - \mathcal{P}_h\mathbf{H}\| &\leq Ch\|\mathbf{H}\|_1, \\ \|\mathbf{u} - \mathscr{P}_h\mathbf{u}\| + h\|\mathbf{u} - \mathscr{P}_h\mathbf{u}\|_1 &\leq Ch^2\|\mathbf{u}\|_2, \\ \|p - P_hp\| + h\|p - P_hp\|_1 &\leq Ch^2\|p\|_2. \end{aligned}$$

Define three sets of moments for $\mathbf{w} \in \boldsymbol{H}^2$ on $K \in \mathcal{T}_h$ [21, 22, 23]

$$\begin{aligned} M_{e_K}(\mathbf{w}) &= \left\{\int_{e_K} (\mathbf{w} \cdot \boldsymbol{\tau}_{e_K})\chi \,\mathrm{d}e_K,\ \forall \chi \in P_{l-1}(e_K),\ e_K \in \mathcal{E}_K\right\}, \\ M_{f_K}(\mathbf{w}) &= \left\{\int_{f_K} (\mathbf{w} \times \mathbf{n}) \cdot \boldsymbol{\chi} \,\mathrm{d}f_K,\ \forall \boldsymbol{\chi} \in (P_{l-2}(f_K))^2,\ f_K \in \mathcal{F}_K\right\}, \\ M_K(\mathbf{w}) &= \left\{\int_K \mathbf{w} \cdot \boldsymbol{\chi} \,\mathrm{d}K,\ \forall \boldsymbol{\chi} \in \boldsymbol{P}_{l-3}(K)\right\}, \end{aligned}$$

where $\boldsymbol{\tau}_{e_K}$ denotes a unit vector parallel to edge e_K. Let $\mathcal{I}_h : \boldsymbol{H}^2 \to \mathbb{E}_h$ be an interpolation operator such that $(\mathcal{I}_h\mathbf{E})|_K$ is the unique polynomial in $\boldsymbol{R}_l(K)$ having the same moments as $\mathbf{E}|_K$, cf. [10, 21, 23].

Let $\mathcal{Q}_h : \boldsymbol{H}_0^1 \to \mathbb{U}_h$ and $\mathcal{R}_h : H_0^1 \to \mathbb{P}_h$ be a pair of Ritz projection operators such that for any $\mathbf{u} \in \boldsymbol{H}_0^1$ and $p \in H_0^1$,

$$a(\mathcal{Q}_h\mathbf{u}, \mathbf{v}_h) - (\mathcal{R}_hp, \alpha\nabla \cdot \mathbf{v}_h) = a(\mathbf{u}, \mathbf{v}_h) - (p, \alpha\nabla \cdot \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbb{U}_h, \tag{5}$$

$$(\kappa\nabla\mathcal{R}_hp, \nabla q_h) = (\kappa\nabla p, \nabla q_h), \quad \forall q_h \in \mathbb{P}_h. \tag{6}$$

Lemma 2.2. *([16, Lemma 14]) There exists a constant $C > 0$ independent of h such that for any $\mathbf{E} \in \boldsymbol{H}^2$, $\mathbf{u} \in \boldsymbol{H}^2 \cap \boldsymbol{H}_0^1$ and $p \in H^2 \cap H_0^1$,*

$$\begin{aligned} &\|\mathcal{I}_h\mathbf{E} - \mathbf{E}\| + \|\nabla \times (\mathcal{I}_h\mathbf{E} - \mathbf{E})\| \leq Ch\|\mathbf{E}\|_2, \\ &\|\mathcal{Q}_h\mathbf{u} - \mathbf{u}\|_1 \leq Ch\|\mathbf{u}\|_2 + Ch^2\|p\|_2, \\ &\|\mathcal{R}_hp - p\| + h\|\nabla(\mathcal{R}_hp - p)\| \leq Ch^2\|p\|_2. \end{aligned}$$

3. Physics-based multi-time stepping algorithm

Due to the multi-physics nature of electroporoelasticity equations, one may identify two distinct time scales which are for the electromagnetic waves and elastic waves. In this section, we propose a multi-time stepping numerical algorithm reflecting these two different time scales for Maxwell's equations and Biot's equations.

For any integers $N_M, N_K > 0$, let $\tau_M = T/N_M$ and $\tau_K = T/N_K$ be two different step-sizes. Denote by $t_m^M = m\tau_M$ $(0 \le m \le N_M)$ and $t_k^K = k\tau_K$ $(0 \le k \le N_K)$ the corresponding time partition nodes. In this paper, we always choose these two step-sizes such that the ratio $r = \tau_K/\tau_M$ is a fixed positive integer. Let $m_k = kr$, then $t_{m_k}^M = t_k^K$.

We will apply backward Euler method and FEM to construct a multi-time stepping scheme to approximate (4). In order to make the idea clear, assume we already have a numerical solution $(\mathbf{E}_h^{m_{k-1}}, \mathbf{H}_h^{m_{k-1}}, \mathbf{u}_h^{m_{k-1}}, p_h^{m_{k-1}})$ at $t_{m_{k-1}}^M$, we will find the next approximate solution $(\mathbf{E}_h^{m_k}, \mathbf{H}_h^{m_k}, \mathbf{u}_h^{m_k}, p_h^{m_k})$ at $t_{m_k}^M$ via two steps. First, we repeatedly solve Maxwell's equations r times to determine $(\mathbf{E}_h^{m_k}, \mathbf{H}_h^{m_k})$: successively seeking $(\mathbf{E}_h^m, \mathbf{H}_h^m) \in \mathbb{E}_h \times \mathbb{H}_h$ for $m = m_{k-1}+1, m_{k-1}+2, \dots, m_k$ such that

$$(7)\quad \begin{aligned}&(\epsilon\bar{\partial}_{t^M}\mathbf{E}_h^m, \mathbf{D}_h) + (\sigma\mathbf{E}_h^m, \mathbf{D}_h) - (\mathbf{H}_h^m, \nabla\times\mathbf{D}_h) = (L\nabla p_h^{m_{k-1}}, \mathbf{D}_h) + (\mathbf{j}(t_m^M), \mathbf{D}_h),\\ &(\mu\bar{\partial}_{t^M}\mathbf{H}_h^m, \mathbf{B}_h) + (\nabla\times\mathbf{E}_h^m, \mathbf{B}_h) = 0,\end{aligned}$$

for all $(\mathbf{D}_h, \mathbf{B}_h) \in \mathbb{E}_h \times \mathbb{H}_h$, where $\bar{\partial}_{t^M}\mathbf{E}_h^m = (\mathbf{E}_h^m - \mathbf{E}_h^{m-1})/\tau_M$, etc. Then, we solve Biot's equations to find $(\mathbf{u}_h^{m_k}, p_h^{m_k})$: searching $(\mathbf{u}_h^{m_k}, p_h^{m_k}) \in \mathbb{U}_h \times \mathbb{P}_h$ such that

$$(8)\quad \begin{aligned}&a(\mathbf{u}_h^{m_k}, \mathbf{v}_h) - (p_h^{m_k}, \alpha\nabla\cdot\mathbf{v}_h) = 0,\\ &(c_0\bar{\partial}_{t^K}p_h^{m_k}, q_h) + (\alpha\nabla\cdot\bar{\partial}_{t^K}\mathbf{u}_h^{m_k}, q_h) + (\kappa\nabla p_h^{m_k}, \nabla q_h)\\ &\qquad\qquad = \big(L\frac{1}{r}\sum_{i=m_{k-1}+1}^{m_k}\mathbf{E}_h^i, \nabla q_h\big) + (g(t_{m_k}^M), q_h),\end{aligned}$$

for all $(\mathbf{v}_h, q_h) \in \mathbb{U}_h \times \mathbb{P}_h$, where $\bar{\partial}_{t^K}p_h^{m_k} = (p_h^{m_k} - p_h^{m_{k-1}})/\tau_K$, etc.

The concrete computational procedure can be described as in Algorithm 1.

Algorithm 1 Multi-time stepping Algorithm

Take

$$(9)\quad (\mathbf{E}_h^0, \mathbf{H}_h^0, \mathbf{u}_h^0, p_h^0) = (\mathfrak{P}_h\mathbf{E}_0, \mathcal{P}_h\mathbf{H}_0, \mathscr{P}_h\mathbf{u}_0, P_hp_0).$$

for $k = 1, 2, \dots, N_K$ **do**
  for $m = m_{k-1}+1, m_{k-1}+2, \dots, m_k$ **do**
    Solve (7).
  end for
  Solve (8).
end for

Now, we investigate the stability of Algorithm 1.

Theorem 3.1. *Assume (H1) and (H2). Suppose* $(\mathbf{E}_h^m, \mathbf{H}_h^m)$ $(1 \le m \le N_M)$ *and* $(\mathbf{u}_h^{m_k}, p_h^{m_k})$ $(1 \le k \le N_K)$ *are obtained from Algorithm 1. Then there exists a constant* $C > 0$ *independent of* τ_M, τ_K *and* h *such that*

$$\|\mathbf{E}_h^m\|^2 + \|\mathbf{H}_h^m\|^2 + \|\mathbf{u}_h^{m_k}\|_1^2 + \|p_h^{m_k}\|^2 \le C.$$

Proof. Taking $(\mathbf{D}_h, \mathbf{B}_h) = (2\tau_M \mathbf{E}_h^m, 2\tau_M \mathbf{H}_h^m)$ in (7) and summing up, we get

$$\begin{aligned}(10)\qquad &(\epsilon\bar{\partial}_{t^M}\mathbf{E}_h^m, 2\tau_M\mathbf{E}_h^m) + (\mu\bar{\partial}_{t^M}\mathbf{H}_h^m, 2\tau_M\mathbf{H}_h^m) + 2\sigma\tau_M\|\mathbf{E}_h^m\|^2 \\ &= (L\nabla p_h^{m_{k-1}}, 2\tau_M\mathbf{E}_h^m) + (\mathbf{j}(t_m^M), 2\tau_M\mathbf{E}_h^m).\end{aligned}$$

Obviously, we have

$$\begin{aligned}(11)\qquad (\bar{\partial}_{t^M}\mathbf{E}_h^m, 2\tau_M\mathbf{E}_h^m) &= \|\mathbf{E}_h^m\|^2 - \|\mathbf{E}_h^{m-1}\|^2 + \|\mathbf{E}_h^m - \mathbf{E}_h^{m-1}\|^2 \\ &\geq \|\mathbf{E}_h^m\|^2 - \|\mathbf{E}_h^{m-1}\|^2.\end{aligned}$$

Similarly, we get

$$(12)\qquad (\bar{\partial}_{t^M}\mathbf{H}_h^m, 2\tau_M\mathbf{H}_h^m) \geq \|\mathbf{H}_h^m\|^2 - \|\mathbf{H}_h^{m-1}\|^2.$$

From (10)–(12), it follows that

$$\begin{aligned}(13)\qquad \epsilon\|\mathbf{E}_h^m\|^2 + \mu\|\mathbf{H}_h^m\|^2 &\leq \epsilon\|\mathbf{E}_h^{m-1}\|^2 + \mu\|\mathbf{H}_h^{m-1}\|^2 - 2\sigma\tau_M\|\mathbf{E}_h^m\|^2 \\ &\quad + (L\nabla p_h^{m_{k-1}}, 2\tau_M\mathbf{E}_h^m) + (\mathbf{j}(t_m^M), 2\tau_M\mathbf{E}_h^m).\end{aligned}$$

Inductively for $m = m_{k-1}+1, \ldots, m_k$ in (13), and then applying Cauchy–Schwarz and Young's inequalities, we obtain

$$\begin{aligned}(14)\qquad &\epsilon\|\mathbf{E}_h^{m_k}\|^2 + \mu\|\mathbf{H}_h^{m_k}\|^2 \\ &\leq \epsilon\|\mathbf{E}_h^{m_{k-1}}\|^2 + \mu\|\mathbf{H}_h^{m_{k-1}}\|^2 - 2\sigma\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\mathbf{E}_h^i\|^2 \\ &\quad + \sum_{i=m_{k-1}+1}^{m_k} (L\nabla p_h^{m_{k-1}}, 2\tau_M\mathbf{E}_h^i) + \sum_{i=m_{k-1}+1}^{m_k} (\mathbf{j}(t_i^M), 2\tau_M\mathbf{E}_h^i) \\ &\leq \epsilon\|\mathbf{E}_h^{m_{k-1}}\|^2 + \mu\|\mathbf{H}_h^{m_{k-1}}\|^2 - 2\sigma\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\mathbf{E}_h^i\|^2 \\ &\quad + \sum_{i=m_{k-1}+1}^{m_k} \big(\kappa\tau_M\|\nabla p_h^{m_{k-1}}\|^2 + \frac{L^2}{\kappa}\tau_M\|\mathbf{E}_h^i\|^2\big) \\ &\quad + \sum_{i=m_{k-1}+1}^{m_k} \big(\frac{1}{2\sigma}\tau_M\|\mathbf{j}(t_i^M)\|^2 + 2\sigma\tau_M\|\mathbf{E}_h^i\|^2\big) \\ &= \epsilon\|\mathbf{E}_h^{m_{k-1}}\|^2 + \mu\|\mathbf{H}_h^{m_{k-1}}\|^2 + \kappa\tau_K\|\nabla p_h^{m_{k-1}}\|^2 \\ &\quad + \frac{L^2}{\kappa}\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\mathbf{E}_h^i\|^2 + \frac{1}{2\sigma}\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\mathbf{j}(t_i^M)\|^2.\end{aligned}$$

Inductively for $k \geq 1$ in above inequality, we have

$$\begin{aligned}(15)\qquad &\epsilon\|\mathbf{E}_h^{m_k}\|^2 + \mu\|\mathbf{H}_h^{m_k}\|^2 \leq \epsilon\|\mathbf{E}_h^0\|^2 + \mu\|\mathbf{H}_h^0\|^2 + \kappa\tau_K \sum_{\ell=1}^{k} \|\nabla p_h^{m_{\ell-1}}\|^2 \\ &\quad + \frac{L^2}{\kappa}\tau_M \sum_{\ell=1}^{k} \sum_{i=m_{\ell-1}+1}^{m_\ell} \|\mathbf{E}_h^i\|^2 + \frac{1}{2\sigma}\tau_M \sum_{\ell=1}^{k} \sum_{i=m_{\ell-1}+1}^{m_\ell} \|\mathbf{j}(t_i^M)\|^2 \\ &= \epsilon\|\mathbf{E}_h^0\|^2 + \mu\|\mathbf{H}_h^0\|^2 + \kappa\tau_K \sum_{\ell=1}^{k} \|\nabla p_h^{m_{\ell-1}}\|^2 \\ &\quad + \frac{L^2}{\kappa}\tau_M \sum_{i=1}^{m_k} \|\mathbf{E}_h^i\|^2 + \frac{1}{2\sigma}\tau_M \sum_{i=1}^{m_k} \|\mathbf{j}(t_i^M)\|^2.\end{aligned}$$

For any $1 \le s \le r-1$, in a similar way as proving (14), inductively for $m = m_{k-1}+1, \dots, m_{k-1}+s$ in (13), and then utilizing (15) with k replaced by $k-1$, we get

$$
\begin{aligned}
&\epsilon\|\mathbf{E}_h^{m_{k-1}+s}\|^2 + \mu\|\mathbf{H}_h^{m_{k-1}+s}\|^2 \\
\le\ & \epsilon\|\mathbf{E}_h^{m_{k-1}}\|^2 + \mu\|\mathbf{H}_h^{m_{k-1}}\|^2 + \kappa\tau_K\|\nabla p_h^{m_{k-1}}\|^2 \\
& + \frac{L^2}{\kappa}\tau_M \sum_{i=m_{k-1}+1}^{m_{k-1}+s} \|\mathbf{E}_h^i\|^2 + \frac{1}{2\sigma}\tau_M \sum_{i=m_{k-1}+1}^{m_{k-1}+s} \|\mathbf{j}(t_i^M)\|^2 \\
\le\ & \epsilon\|\mathbf{E}_h^0\|^2 + \mu\|\mathbf{H}_h^0\|^2 + \kappa\tau_K \sum_{\ell=1}^{k} \|\nabla p_h^{m_{\ell-1}}\|^2 \\
& + \frac{L^2}{\kappa}\tau_M \sum_{i=1}^{m_{k-1}+s} \|\mathbf{E}_h^i\|^2 + \frac{1}{2\sigma}\tau_M \sum_{i=1}^{m_{k-1}+s} \|\mathbf{j}(t_i^M)\|^2.
\end{aligned} \tag{16}
$$

Taking $(\mathbf{v}_h, q_h) = (2\tau_K\bar{\partial}_{t^K}\mathbf{u}_h^{m_k}, 2\tau_K p_h^{m_k})$ in (8) and summing up lead to

$$
\begin{aligned}
& a(\mathbf{u}_h^{m_k}, 2\tau_K\bar{\partial}_{t^K}\mathbf{u}_h^{m_k}) + (c_0\bar{\partial}_{t^K}p_h^{m_k}, 2\tau_K p_h^{m_k}) + 2\kappa\tau_K\|\nabla p_h^{m_k}\|^2 \\
=\ & \big(L\frac{1}{r}\sum_{i=m_{k-1}+1}^{m_k} \mathbf{E}_h^i, 2\tau_K\nabla p_h^{m_k}\big) + (g(t_{m_k}^M), 2\tau_K p_h^{m_k}).
\end{aligned} \tag{17}
$$

In a similar way as proving (11), we have

$$
\begin{aligned}
a(\mathbf{u}_h^{m_k}, 2\tau_K\bar{\partial}_{t^K}\mathbf{u}_h^{m_k}) &\ge \|\mathbf{u}_h^{m_k}\|_a^2 - \|\mathbf{u}_h^{m_{k-1}}\|_a^2, \\
(\bar{\partial}_{t^K}p_h^{m_k}, 2\tau_K p_h^{m_k}) &\ge \|p_h^{m_k}\|^2 - \|p_h^{m_{k-1}}\|^2.
\end{aligned}
$$

These together with (17), Cauchy–Schwarz and Young's inequalities yield

$$
\begin{aligned}
& \|\mathbf{u}_h^{m_k}\|_a^2 + c_0\|p_h^{m_k}\|^2 + 2\kappa\tau_K\|\nabla p_h^{m_k}\|^2 \\
\le & \|\mathbf{u}_h^{m_{k-1}}\|_a^2 + c_0\|p_h^{m_{k-1}}\|^2 \\
& + \frac{L^2}{\kappa}\tau_K\frac{1}{r}\sum_{i=m_{k-1}+1}^{m_k} \|\mathbf{E}_h^i\|^2 + \kappa\tau_K\|\nabla p_h^{m_k}\|^2 + \tau_K\|g(t_{m_k}^M)\|^2 + \tau_K\|p_h^{m_k}\|^2.
\end{aligned}
$$

Inductively for $k \ge 1$ in above inequality, we get

$$
\begin{aligned}
& \|\mathbf{u}_h^{m_k}\|_a^2 + c_0\|p_h^{m_k}\|^2 + \kappa\tau_K\sum_{\ell=1}^{k}\|\nabla p_h^{m_\ell}\|^2 \\
\le & \|\mathbf{u}_h^0\|_a^2 + c_0\|p_h^0\|^2 + \frac{L^2}{\kappa}\tau_M\sum_{\ell=1}^{k}\sum_{i=m_{\ell-1}+1}^{m_\ell}\|\mathbf{E}_h^i\|^2 \\
& + \tau_K\sum_{\ell=1}^{k}\|g(t_{m_\ell}^M)\|^2 + \tau_K\sum_{\ell=1}^{k}\|p_h^{m_\ell}\|^2 \\
=\ & \|\mathbf{u}_h^0\|_a^2 + c_0\|p_h^0\|^2 + \tau_K\sum_{\ell=1}^{k}\|g(t_{m_\ell}^M)\|^2 + \frac{L^2}{\kappa}\tau_M\sum_{i=1}^{m_k}\|\mathbf{E}_h^i\|^2 + \tau_K\sum_{\ell=1}^{k}\|p_h^{m_\ell}\|^2.
\end{aligned} \tag{18}
$$

By virtue of (15) and (18), we obtain estimates at nodes $t^M_{m_k} = t^K_k$ $(1 \le k \le N_K)$,

$$\begin{aligned}
&\epsilon\|\mathbf{E}_h^{m_k}\|^2 + \mu\|\mathbf{H}_h^{m_k}\|^2 + \|\mathbf{u}_h^{m_k}\|_a^2 + c_0\|p_h^{m_k}\|^2 + \kappa\tau_K\|\nabla p_h^{m_k}\|^2 \\
&\le \epsilon\|\mathbf{E}_h^{0}\|^2 + \mu\|\mathbf{H}_h^{0}\|^2 + \|\mathbf{u}_h^{0}\|_a^2 + c_0\|p_h^{0}\|^2 + \kappa\tau_K\|\nabla p_h^{0}\|^2 \\
&\quad + \frac{1}{2\sigma}\tau_M \sum_{i=1}^{m_k} \|\mathbf{j}(t_i^M)\|^2 + \tau_K \sum_{\ell=1}^{k} \|g(t^M_{m_\ell})\|^2 \\
&\quad + \frac{2L^2}{\kappa}\tau_M \sum_{i=1}^{m_k} \|\mathbf{E}_h^i\|^2 + \tau_K \sum_{\ell=1}^{k} \|p_h^{m_\ell}\|^2.
\end{aligned} \tag{19}$$

In order to set up the boundedness of numerical solutions at nodes $t^M_{m_{k-1}+s}$ for $1 \le k \le N_K$ and $1 \le s \le r-1$, we supplement the definition of the approximation to $\mathbf{u}$ and p at nodes $t^M_{m_{k-1}+s}$ by $\mathbf{u}_h^{m_{k-1}+s} = \mathbf{u}_h^{m_{k-1}}$ and $p_h^{m_{k-1}+s} = p_h^{m_{k-1}}$, respectively. Then, form (16) and (18) with k replaced by $k-1$, it follows that

$$\begin{aligned}
&\epsilon\|\mathbf{E}_h^{m_{k-1}+s}\|^2 + \mu\|\mathbf{H}_h^{m_{k-1}+s}\|^2 + \|\mathbf{u}_h^{m_{k-1}+s}\|_a^2 + c_0\|p_h^{m_{k-1}+s}\|^2 \\
&= \epsilon\|\mathbf{E}_h^{m_{k-1}+s}\|^2 + \mu\|\mathbf{H}_h^{m_{k-1}+s}\|^2 + \|\mathbf{u}_h^{m_{k-1}}\|_a^2 + c_0\|p_h^{m_{k-1}}\|^2 \\
&\le \epsilon\|\mathbf{E}_h^{0}\|^2 + \mu\|\mathbf{H}_h^{0}\|^2 + \|\mathbf{u}_h^{0}\|_a^2 + c_0\|p_h^{0}\|^2 + \kappa\tau_K\|\nabla p_h^{0}\|^2 \\
&\quad + \frac{1}{2\sigma}\tau_M \sum_{i=1}^{m_{k-1}+s} \|\mathbf{j}(t_i^M)\|^2 + \tau_K \sum_{\ell=1}^{k-1} \|g(t^M_{m_\ell})\|^2 \\
&\quad + \frac{2L^2}{\kappa}\tau_K \sum_{i=1}^{m_{k-1}+s} \|\mathbf{E}_h^i\|^2 + \tau_K \sum_{\ell=1}^{k-1} \|p_h^{m_\ell}\|^2.
\end{aligned} \tag{20}$$

According to (19) and (20), we obtain for all $1 \le m \le N_M$ and $1 \le k \le N_K$,

$$\begin{aligned}
&\epsilon\|\mathbf{E}_h^{m}\|^2 + \mu\|\mathbf{H}_h^{m}\|^2 + \|\mathbf{u}_h^{m}\|_a^2 + c_0\|p_h^{m}\|^2 \\
&\le \epsilon\|\mathbf{E}_h^{0}\|^2 + \mu\|\mathbf{H}_h^{0}\|^2 + \|\mathbf{u}_h^{0}\|_a^2 + c_0\|p_h^{0}\|^2 + \kappa\tau_K\|\nabla p_h^{0}\|^2 \\
&\quad + \frac{1}{2\sigma}\tau_M \sum_{i=1}^{m} \|\mathbf{j}(t_i^M)\|^2 + \tau_K \sum_{\ell=1}^{k} \|g(t^M_{m_\ell})\|^2 \\
&\quad + \frac{2L^2}{\kappa}\tau_K \sum_{i=1}^{m} \|\mathbf{E}_h^i\|^2 + \tau_K \sum_{i=1}^{m} \|p_h^{i}\|^2.
\end{aligned} \tag{21}$$

By virtue of (9), we get

$$\begin{aligned}
\|\mathbf{E}_h^0\|^2 &= \|\mathfrak{P}_h\mathbf{E}_0\|^2 \le \|\mathbf{E}_0\|^2, \\
\|\mathbf{H}_h^0\|^2 &= \|\mathcal{P}_h\mathbf{H}_0\|^2 \le \|\mathbf{H}_0\|^2, \\
\|\mathbf{u}_h^0\|_a^2 &= \|\mathscr{P}_h\mathbf{u}_0\|_a^2 \le c_2^2\|\mathscr{P}_h\mathbf{u}_0\|_1^2 \le c_2^2\|\mathbf{u}_0\|_1^2, \\
\|p_h^0\|^2 &= \|P_h p_0\|^2 \le \|p_0\|^2, \\
\|\nabla p_h^0\|^2 &= \|\nabla P_h p_0\|^2 \le \|P_h p_0\|_1^2 \le \|p_0\|_1^2.
\end{aligned}$$

Applying discrete Gronwall's lemma to (21), we obtain that for all $1 \le m \le N_M$,

$$\|\mathbf{E}_h^m\|^2 + \|\mathbf{H}_h^m\|^2 + \|\mathbf{u}_h^m\|_a^2 + \|p_h^m\|^2 \le Ce^{C\sum_{i=1}^m \tau_K} \le Ce^{CrT} \le C,$$

where $C > 0$ is a constant independent of τ_M, τ_K and h. $\square$

4. Error estimates

In order to investigate the convergence order of numerical approximation, we need additional regularity of the exact solution to (4). Thus, we assume

(H3) $\mathbf{E} \in H^1(0,T;\boldsymbol{H}^2) \cap H^2(0,T;\boldsymbol{L}^2(\mathscr{D}))$,
$\mathbf{H} \in H^1(0,T;\boldsymbol{H}^1) \cap H^2(0,T;\boldsymbol{L}^2(\mathscr{D}))$,
$\mathbf{u} \in H^1(0,T;\boldsymbol{H}^2) \cap H^2(0,T;\boldsymbol{H}^1)$,
$p \in H^1(0,T;H^2) \cap H^2(0,T;L^2(\mathscr{D}))$.

Note, the errors can be decomposed as follows: for all $1 \le m \le N_M$ and $1 \le k \le N_K$,

$$\begin{aligned}
\mathbf{E}(t_m^M) - \mathbf{E}_h^m &= (\mathbf{E}(t_m^M) - \mathcal{I}_h\mathbf{E}(t_m^M)) + (\mathcal{I}_h\mathbf{E}(t_m^M) - \mathbf{E}_h^m) =: \Xi(t_m^M) + \Phi_h^m,\\
\mathbf{H}(t_m^M) - \mathbf{H}_h^m &= (\mathbf{H}(t_m^M) - \mathcal{P}_h\mathbf{H}(t_m^M)) + (\mathcal{P}_h\mathbf{H}(t_m^M) - \mathbf{H}_h^m) =: \Psi(t_m^M) + \Theta_h^m,\\
\mathbf{u}(t_{m_k}^M) - \mathbf{u}_h^{m_k} &= (\mathbf{u}(t_{m_k}^M) - \mathcal{Q}_h\mathbf{u}(t_{m_k}^M)) + (\mathcal{Q}_h\mathbf{u}(t_{m_k}^M) - \mathbf{u}_h^{m_k}) =: \boldsymbol{\phi}(t_{m_k}^M) + \boldsymbol{\varphi}_h^{m_k},\\
p(t_{m_k}^M) - p_h^{m_k} &= (p(t_{m_k}^M) - \mathcal{R}_h p(t_{m_k}^M)) + (\mathcal{R}_h p(t_{m_k}^M) - p_h^{m_k}) =: \xi(t_{m_k}^M) + \psi_h^{m_k}.
\end{aligned}$$

The error estimates for the projections and interpolation in the above equalities are easily obtained from (H3), Lemmas 2.1 and 2.2. In particular, there exists a constant $C > 0$ independent of τ_M, τ_K and h such that for all $1 \le m \le N_M$ and $1 \le k \le N_K$,

$$\begin{aligned}
\|\Xi(t_m^M)\| &= \|\mathbf{E}(t_m^M) - \mathcal{I}_h\mathbf{E}(t_m^M)\| \le Ch\|\mathbf{E}(t_m^M)\|_2 \le Ch,\\
\|\Psi(t_m^M)\| &= \|\mathbf{H}(t_m^M) - \mathcal{P}_h\mathbf{H}(t_m^M)\| \le Ch\|\mathbf{H}(t_m^M)\|_1 \le Ch,\\
\|\boldsymbol{\phi}(t_{m_k}^M)\|_1 &= \|\mathbf{u}(t_{m_k}^M) - \mathcal{Q}_h\mathbf{u}(t_{m_k}^M)\|_1\\
&\le Ch\|\mathbf{u}(t_{m_k}^M)\|_2 + Ch^2\|p(t_{m_k}^M)\|_2 \le Ch,\\
\|\xi(t_{m_k}^M)\| &= \|p(t_{m_k}^M) - \mathcal{R}_h p(t_{m_k}^M)\| \le Ch^2\|p(t_{m_k}^M)\|_2 \le Ch^2.
\end{aligned} \tag{22}$$

Now, we are ready to derive the error estimates between the exact and numerical solutions to (4).

Theorem 4.1. *Assume (H1)–(H3). Let* $(\mathbf{E},\mathbf{H},\mathbf{u},p)$ *be the exact solution to* (4) *and* $(\mathbf{E}_h^m,\mathbf{H}_h^m)$ $(1 \le m \le N_M)$ *and* $(\mathbf{u}_h^{m_k}, p_h^{m_k})$ $(1 \le k \le N_K)$ *be obtained from Algorithm 1. Then there exist three positive constants* $C_1 = C_1(\frac{\partial^2\mathbf{E}}{\partial t^2}, \frac{\partial^2\mathbf{H}}{\partial t^2})$, $C_2 = C_2(\frac{\partial p}{\partial t}, \frac{\partial^2 p}{\partial t^2}, \frac{\partial^2\mathbf{u}}{\partial t^2}, \frac{\partial\mathbf{E}}{\partial t})$ *and* C *independent of* τ_M, τ_K *and* h *such that*

$$\begin{aligned}
&\|\mathbf{E}(t_m^M) - \mathbf{E}_h^m\|^2 + \|\mathbf{H}(t_m^M) - \mathbf{H}_h^m\|^2 + \|\mathbf{u}(t_{m_k}^M) - \mathbf{u}_h^{m_k}\|_1^2 + \|p(t_{m_k}^M) - p_h^{m_k}\|^2\\
&\le C_1\tau_M^2 + C_2\tau_K^2 + Ch^2.
\end{aligned}$$

Proof. Taking $\mathbf{D} = \mathbf{D}_h \in \mathbb{E}_h$ in the first equation of (4) and noticing the first equation of (7), we get

$$\begin{aligned}
&(\epsilon\bar{\partial}_{t^M}\Phi_h^m, \mathbf{D}_h) + (\sigma\Phi_h^m, \mathbf{D}_h) - (\Theta_h^m, \nabla\times\mathbf{D}_h)\\
&= (\epsilon\bar{\partial}_{t^M}(\mathcal{I}_h\mathbf{E}(t_m^M) - \mathbf{E}_h^m), \mathbf{D}_h) + (\sigma(\mathcal{I}_h\mathbf{E}(t_m^M) - \mathbf{E}_h^m), \mathbf{D}_h) - (\mathcal{P}_h\mathbf{H}(t_m^M) - \mathbf{H}_h^m, \nabla\times\mathbf{D}_h)\\
&= (\epsilon(\mathcal{I}_h\bar{\partial}_{t^M}\mathbf{E}(t_m^M) - \bar{\partial}_{t^M}\mathbf{E}(t_m^M)), \mathbf{D}_h) + (\epsilon\bar{\partial}_{t^M}\mathbf{E}(t_m^M), \mathbf{D}_h) - (\epsilon\bar{\partial}_{t^M}\mathbf{E}_h^m, \mathbf{D}_h)\\
&\quad - (\sigma\Xi(t_m^M), \mathbf{D}_h) + (\sigma\mathbf{E}(t_m^M), \mathbf{D}_h) - (\sigma\mathbf{E}_h^m, \mathbf{D}_h) + (\Psi(t_m^M), \nabla\times\mathbf{D}_h)\\
&\quad - (\mathbf{H}(t_m^M), \nabla\times\mathbf{D}_h) + (\mathbf{H}_h^m, \nabla\times\mathbf{D}_h)\\
&= (\epsilon(\mathcal{I}_h\bar{\partial}_{t^M}\mathbf{E}(t_m^M) - \bar{\partial}_{t^M}\mathbf{E}(t_m^M)), \mathbf{D}_h) + (\epsilon\bar{\partial}_{t^M}\mathbf{E}(t_m^M), \mathbf{D}_h) - (\sigma\Xi(t_m^M), \mathbf{D}_h)\\
&\quad + (\sigma\mathbf{E}(t_m^M), \mathbf{D}_h) - (\mathbf{H}(t_m^M), \nabla\times\mathbf{D}_h) - (\mathbf{j}(t_m^M), \mathbf{D}_h) - (L\nabla p_h^{m_{k-1}}, \mathbf{D}_h)
\end{aligned}$$

$$
\begin{aligned}
&= (\epsilon(\mathcal{I}_h\bar{\partial}_{t^M}\mathbf{E}(t_m^M)-\bar{\partial}_{t^M}\mathbf{E}(t_m^M)),\mathbf{D}_h)+\big(\epsilon(\bar{\partial}_{t^M}\mathbf{E}(t_m^M)-\frac{\partial}{\partial t}\mathbf{E}(t_m^M)),\mathbf{D}_h\big)-(\sigma\Xi(t_m^M),\mathbf{D}_h)\\
&\quad + (L\nabla(p(t_m^M)-p(t_{m_{k-1}}^M)),\mathbf{D}_h) + (L\nabla\xi(t_{m_{k-1}}^M),\mathbf{D}_h) + (L\nabla\psi_h^{m_{k-1}},\mathbf{D}_h)\\
&=: (\epsilon\boldsymbol{\omega}_{M1}(t_m^M),\mathbf{D}_h) + (\epsilon\boldsymbol{\rho}_{M1}^m,\mathbf{D}_h) - (\sigma\Xi(t_m^M),\mathbf{D}_h)\\
&\quad + (L\nabla(p(t_m^M)-p(t_{m_{k-1}}^M)),\mathbf{D}_h) + (L\nabla\xi(t_{m_{k-1}}^M),\mathbf{D}_h) + (L\nabla\psi_h^{m_{k-1}},\mathbf{D}_h),
\end{aligned}
$$

where we have used the fact that $(\Psi(t_m^M),\nabla\times\mathbf{D}_h)=0$, which is due to $\nabla\times\mathbf{D}_h|_K\in\mathbb{H}_h|_K$. Similarly, by virtue of the second equations in (4) and in (7), we obtain for all $\mathbf{B}_h\in\mathbb{H}_h$,

$$
\begin{aligned}
&(\mu\bar{\partial}_{t^M}\Theta_h^m,\mathbf{B}_h) + (\nabla\times\Phi_h^m,\mathbf{B}_h)\\
&= (\mu\bar{\partial}_{t^M}(\mathcal{P}_h\mathbf{H}(t_m^M)-\mathbf{H}_h^m),\mathbf{B}_h) + (\nabla\times(\mathcal{I}_h\mathbf{E}(t_m^M)-\mathbf{E}_h^m),\mathbf{B}_h)\\
&= (\mu(\mathcal{P}_h\bar{\partial}_{t^M}\mathbf{H}(t_m^M)-\bar{\partial}_{t^M}\mathbf{H}(t_m^M)),\mathbf{B}_h) + (\mu\bar{\partial}_{t^M}\mathbf{H}(t_m^M),\mathbf{B}_h) - (\mu\bar{\partial}_{t^M}\mathbf{H}_h^m,\mathbf{B}_h)\\
&\quad - (\nabla\times\Xi(t_m^M),\mathbf{B}_h) + (\nabla\times\mathbf{E}(t_m^M),\mathbf{B}_h) - (\nabla\times\mathbf{E}_h^m,\mathbf{B}_h)\\
&= (\mu(\mathcal{P}_h\bar{\partial}_{t^M}\mathbf{H}(t_m^M)-\bar{\partial}_{t^M}\mathbf{H}(t_m^M)),\mathbf{B}_h) + \big(\mu(\bar{\partial}_{t^M}\mathbf{H}(t_m^M)-\frac{\partial}{\partial t}\mathbf{H}(t_m^M)),\mathbf{B}_h\big)\\
&\quad - (\nabla\times\Xi(t_m^M),\mathbf{B}_h)\\
&=: (\mu\boldsymbol{\omega}_{M2}(t_m^M),\mathbf{B}_h) + (\mu\boldsymbol{\rho}_{M2}^m,\mathbf{B}_h) - (\nabla\times\Xi(t_m^M),\mathbf{B}_h).
\end{aligned}
$$

Taking $\mathbf{D}_h=2\tau_M\Phi_h^m$ and $\mathbf{B}_h=2\tau_M\Theta_h^m$ in above two equations, respectively, and then summing up, we get an error equation

$$
\begin{aligned}
&(\epsilon\bar{\partial}_{t^M}\Phi_h^m,2\tau_M\Phi_h^m) + (\mu\bar{\partial}_{t^M}\Theta_h^m,2\tau_M\Theta_h^m) + 2\sigma\tau_M\|\Phi_h^m\|^2\\
&= (\epsilon\boldsymbol{\omega}_{M1}(t_m^M),2\tau_M\Phi_h^m) + (\epsilon\boldsymbol{\rho}_{M1}^m,2\tau_M\Phi_h^m) - (\sigma\Xi(t_m^M),2\tau_M\Phi_h^m)\\
&\quad + (L\nabla(p(t_m^M)-p(t_{m_{k-1}}^M)),2\tau_M\Phi_h^m) + (L\nabla\xi(t_{m_{k-1}}^M),2\tau_M\Phi_h^m)\\
&\quad + (L\nabla\psi_h^{m_{k-1}},2\tau_M\Phi_h^m) + (\mu\boldsymbol{\omega}_{M2}(t_m^M),2\tau_M\Theta_h^m) + (\mu\boldsymbol{\rho}_{M2}^m,2\tau_M\Theta_h^m)\\
&\quad - (\nabla\times\Xi(t_m^M),2\tau_M\Theta_h^m).
\end{aligned}
\tag{23}
$$

In a similar way as proving (11), we have

$$
\begin{aligned}
(\bar{\partial}_{t^M}\Phi_h^m,2\tau_M\Phi_h^m) &\ge \|\Phi_h^m\|^2-\|\Phi_h^{m-1}\|^2,\\
(\bar{\partial}_{t^M}\Theta_h^m,2\tau_M\Theta_h^m) &\ge \|\Theta_h^m\|^2-\|\Theta_h^{m-1}\|^2.
\end{aligned}
\tag{24}
$$

Inductively for $m=m_{k-1}+1,\ldots,m_k$ in (23), and then applying (24), Cauchy–Schwarz and Young's inequalities, we obtain

(25)

$$
\begin{aligned}
&\epsilon\|\Phi_h^{m_k}\|^2+\mu\|\Theta_h^{m_k}\|^2\\
&\le \epsilon\|\Phi_h^{m_{k-1}}\|^2+\mu\|\Theta_h^{m_{k-1}}\|^2+\epsilon\tau_M\sum_{i=m_{k-1}+1}^{m_k}\|\boldsymbol{\omega}_{M1}(t_i^M)\|^2+\epsilon\tau_M\sum_{i=m_{k-1}+1}^{m_k}\|\Phi_h^i\|^2\\
&\quad+\epsilon\tau_M\sum_{i=m_{k-1}+1}^{m_k}\|\boldsymbol{\rho}_{M1}^i\|^2+\epsilon\tau_M\sum_{i=m_{k-1}+1}^{m_k}\|\Phi_h^i\|^2+\sigma\tau_M\sum_{i=m_{k-1}+1}^{m_k}\|\Xi(t_i^M)\|^2\\
&\quad+\sigma\tau_M\sum_{i=m_{k-1}+1}^{m_k}\|\Phi_h^i\|^2+L\tau_M\sum_{i=m_{k-1}+1}^{m_k}\|\nabla(p(t_i^M)-p(t_{m_{k-1}}^M))\|^2\\
&\quad+L\tau_M\sum_{i=m_{k-1}+1}^{m_k}\|\Phi_h^i\|^2+L\tau_M\sum_{i=m_{k-1}+1}^{m_k}\|\nabla\xi(t_{m_{k-1}}^M)\|^2+L\tau_M\sum_{i=m_{k-1}+1}^{m_k}\|\Phi_h^i\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \kappa\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\nabla\psi_h^{m_{k-1}}\|^2 + \frac{L^2}{\kappa}\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\Phi_h^i\|^2 \\
&+\mu\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\boldsymbol{\omega}_{M2}(t_i^M)\|^2 + \mu\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\Theta_h^i\|^2 + \mu\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\boldsymbol{\rho}_{M2}^i\|^2 \\
&+\mu\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\Theta_h^i\|^2 + \tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\nabla\times\Xi(t_i^M)\|^2 + \tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\Theta_h^i\|^2 \\
\le\ & \epsilon\|\Phi_h^{m_{k-1}}\|^2 + \mu\|\Theta_h^{m_{k-1}}\|^2 + \kappa\tau_K\|\nabla\psi_h^{m_{k-1}}\|^2 + \epsilon\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\boldsymbol{\omega}_{M1}(t_i^M)\|^2 \\
&+ \epsilon\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\boldsymbol{\rho}_{M1}^i\|^2 + \sigma\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\Xi(t_i^M)\|^2 \\
&+ L\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\nabla(p(t_i^M) - p(t_{m_{k-1}}^M))\|^2 + L\tau_K\|\nabla\xi(t_{m_{k-1}}^M)\|^2 \\
&+ \mu\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\boldsymbol{\omega}_{M2}(t_i^M)\|^2 + \mu\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\boldsymbol{\rho}_{M2}^i\|^2 \\
&+ \tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\nabla\times\Xi(t_i^M)\|^2 + (2\epsilon+\sigma+2L+\frac{L^2}{\kappa})\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\Phi_h^i\|^2 \\
&+ (2\mu+1)\tau_M \sum_{i=m_{k-1}+1}^{m_k} \|\Theta_h^i\|^2.
\end{aligned}
$$

Inductively for $k \ge 1$ in above inequality, we have

$$
\begin{aligned}
&\epsilon\|\Phi_h^{m_k}\|^2 + \mu\|\Theta_h^{m_k}\|^2 \\
\le\ & \epsilon\|\Phi_h^0\|^2 + \mu\|\Theta_h^0\|^2 + \kappa\tau_K \sum_{\ell=1}^{k} \|\nabla\psi_h^{m_{\ell-1}}\|^2 + \epsilon\tau_M \sum_{i=1}^{m_k} \|\boldsymbol{\omega}_{M1}(t_i^M)\|^2 \\
&+ \epsilon\tau_M \sum_{i=1}^{m_k} \|\boldsymbol{\rho}_{M1}^i\|^2 + \sigma\tau_M \sum_{i=1}^{m_k} \|\Xi(t_i^M)\|^2 \\
&+ L\tau_M \sum_{\ell=1}^{k} \sum_{i=m_{\ell-1}+1}^{m_\ell} \|\nabla(p(t_i^M) - p(t_{m_{\ell-1}}^M))\|^2 + L\tau_K \sum_{\ell=1}^{k} \|\nabla\xi(t_{m_{\ell-1}}^M)\|^2 \\
&+ \mu\tau_M \sum_{i=1}^{m_k} \|\boldsymbol{\omega}_{M2}(t_i^M)\|^2 + \mu\tau_M \sum_{i=1}^{m_k} \|\boldsymbol{\rho}_{M2}^i\|^2 + \tau_M \sum_{i=1}^{m_k} \|\nabla\times\Xi(t_i^M)\|^2 \\
&+ (2\epsilon+\sigma+2L+\frac{L^2}{\kappa})\tau_M \sum_{i=1}^{m_k} \|\Phi_h^i\|^2 + (2\mu+1)\tau_M \sum_{i=1}^{m_k} \|\Theta_h^i\|^2.
\end{aligned} \tag{26}
$$

For any $1 \le s \le r-1$, in a similar way as proving (25), inductively for $m = m_{k-1}+1, \ldots, m_{k-1}+s$ in (23), and then utilizing (26) with k replaced by $k-1$,

we get
(27)

$$
\begin{aligned}
&\epsilon\|\Phi_h^{m_{k-1}+s}\|^2 + \mu\|\Theta_h^{m_{k-1}+s}\|^2 \\
\le\ & \epsilon\|\Phi_h^{m_{k-1}}\|^2 + \mu\|\Theta_h^{m_{k-1}}\|^2 + \kappa\tau_K\|\nabla\psi_h^{m_{k-1}}\|^2 + \epsilon\tau_M \sum_{i=m_{k-1}+1}^{m_{k-1}+s} \|\boldsymbol{\omega}_{M1}(t_i^M)\|^2 \\
&+ \epsilon\tau_M \sum_{i=m_{k-1}+1}^{m_{k-1}+s} \|\boldsymbol{\rho}_{M1}^i\|^2 + \sigma\tau_M \sum_{i=m_{k-1}+1}^{m_{k-1}+s} \|\Xi(t_i^M)\|^2 \\
&+ L\tau_M \sum_{i=m_{k-1}+1}^{m_{k-1}+s} \|\nabla(p(t_i^M) - p(t_{m_{k-1}}^M))\|^2 + L\tau_K\|\nabla\xi(t_{m_{k-1}}^M)\|^2 \\
&+ \mu\tau_M \sum_{i=m_{k-1}+1}^{m_{k-1}+s} \|\boldsymbol{\omega}_{M2}(t_i^M)\|^2 + \mu\tau_M \sum_{i=m_{k-1}+1}^{m_{k-1}+s} \|\boldsymbol{\rho}_{M2}^i\|^2 \\
&+ \tau_M \sum_{i=m_{k-1}+1}^{m_{k-1}+s} \|\nabla\times\Xi(t_i^M)\|^2 \\
&+ (2\epsilon + \sigma + 2L + \frac{L^2}{\kappa})\tau_M \sum_{i=m_{k-1}+1}^{m_{k-1}+s} \|\Phi_h^i\|^2 + (2\mu+1)\tau_M \sum_{i=m_{k-1}+1}^{m_{k-1}+s} \|\Theta_h^i\|^2 \\
\le\ & \epsilon\|\Phi_h^0\|^2 + \mu\|\Theta_h^0\|^2 + \kappa\tau_K \sum_{\ell=1}^{k} \|\nabla\psi_h^{m_{\ell-1}}\|^2 \\
&+ \epsilon\tau_M \sum_{i=1}^{m_{k-1}+s} \|\boldsymbol{\omega}_{M1}(t_i^M)\|^2 + \epsilon\tau_M \sum_{i=1}^{m_{k-1}+s} \|\boldsymbol{\rho}_{M1}^i\|^2 + \sigma\tau_M \sum_{i=1}^{m_{k-1}+s} \|\Xi(t_i^M)\|^2 \\
&+ L\tau_M \sum_{\ell=1}^{k-1} \sum_{i=m_{\ell-1}+1}^{m_\ell} \|\nabla(p(t_i^M) - p(t_{m_{\ell-1}}^M))\|^2 \\
&+ L\tau_M \sum_{i=m_{k-1}+1}^{m_{k-1}+s} \|\nabla(p(t_i^M) - p(t_{m_{k-1}}^M))\|^2 + L\tau_K \sum_{\ell=1}^{k} \|\nabla\xi(t_{m_{\ell-1}}^M)\|^2 \\
&+ \mu\tau_M \sum_{i=1}^{m_{k-1}+s} \|\boldsymbol{\omega}_{M2}(t_i^M)\|^2 + \mu\tau_M \sum_{i=1}^{m_{k-1}+s} \|\boldsymbol{\rho}_{M2}^i\|^2 + \tau_M \sum_{i=1}^{m_{k-1}+s} \|\nabla\times\Xi(t_i^M)\|^2 \\
&+ (2\epsilon + \sigma + 2L + \frac{L^2}{\kappa})\tau_M \sum_{i=1}^{m_{k-1}+s} \|\Phi_h^i\|^2 + (2\mu+1)\tau_M \sum_{i=1}^{m_{k-1}+s} \|\Theta_h^i\|^2.
\end{aligned}
$$

According to (5), the third equation in (4) and the first equation in (8), we get for all $\mathbf{v}_h \in \mathbb{U}_h$,

$$
\begin{aligned}
&a(\boldsymbol{\varphi}_h^{m_k}, \mathbf{v}_h) - (\psi_h^{m_k}, \alpha\nabla\cdot\mathbf{v}_h) \\
=\ & a(\mathcal{Q}_h\mathbf{u}(t_{m_k}^M) - \mathbf{u}_h^{m_k}, \mathbf{v}_h) - (\mathcal{R}_h p(t_{m_k}^M) - p_h^{m_k}, \alpha\nabla\cdot\mathbf{v}_h) \\
=\ & a(\mathbf{u}(t_{m_k}^M), \mathbf{v}_h) - (p(t_{m_k}^M), \alpha\nabla\cdot\mathbf{v}_h) - \big(a(\mathbf{u}_h^{m_k}, \mathbf{v}_h) - (p_h^{m_k}, \alpha\nabla\cdot\mathbf{v}_h)\big) \\
=\ & 0.
\end{aligned}
$$

By (6), the last equation in (4) and the second equation in (8), we obtain for all $q_h \in \mathbb{P}_h$,

$$\begin{aligned}
&(c_0\bar{\partial}_{t^K}\psi_h^{m_k}, q_h) + (\alpha\nabla\cdot\bar{\partial}_{t^K}\boldsymbol{\varphi}_h^{m_k}, q_h) + (\kappa\nabla\psi_h^{m_k}, \nabla q_h) \\
= {}& (c_0\bar{\partial}_{t^K}(\mathcal{R}_h p(t_{m_k}^M) - p_h^{m_k}), q_h) + (\alpha\nabla\cdot\bar{\partial}_{t^K}(\mathcal{Q}_h\mathbf{u}(t_{m_k}^M) - \mathbf{u}_h^{m_k}), q_h) \\
&+ (\kappa\nabla(\mathcal{R}_h p(t_{m_k}^M) - p_h^{m_k}), \nabla q_h) \\
= {}& (c_0(\mathcal{R}_h\bar{\partial}_{t^K}p(t_{m_k}^M) - \bar{\partial}_{t^K}p(t_{m_k}^M)), q_h) + (c_0\bar{\partial}_{t^K}p(t_{m_k}^M), q_h) - (c_0\bar{\partial}_{t^K}p_h^{m_k}, q_h) \\
&+ (\alpha\nabla\cdot(\mathcal{Q}_h\bar{\partial}_{t^K}\mathbf{u}(t_{m_k}^M) - \bar{\partial}_{t^K}\mathbf{u}(t_{m_k}^M)), q_h) + (\alpha\nabla\cdot\bar{\partial}_{t^K}\mathbf{u}(t_{m_k}^M), q_h) \\
&- (\alpha\nabla\cdot\bar{\partial}_{t^K}\mathbf{u}_h^{m_k}, q_h) + (\kappa\nabla p(t_{m_k}^M), \nabla q_h) - (\kappa\nabla p_h^{m_k}, \nabla q_h) \\
= {}& (c_0(\mathcal{R}_h\bar{\partial}_{t^K}p(t_{m_k}^M) - \bar{\partial}_{t^K}p(t_{m_k}^M)), q_h) + (c_0\bar{\partial}_{t^K}p(t_{m_k}^M), q_h) \\
&+ (\alpha\nabla\cdot(\mathcal{Q}_h\bar{\partial}_{t^K}\mathbf{u}(t_{m_k}^M) - \bar{\partial}_{t^K}\mathbf{u}(t_{m_k}^M)), q_h) + (\alpha\nabla\cdot\bar{\partial}_{t^K}\mathbf{u}(t_{m_k}^M), q_h) \\
&+ (\kappa\nabla p(t_{m_k}^M), \nabla q_h) - (g(t_{m_k}^M), q_h) - \big(L\frac{1}{r}\sum_{i=m_{k-1}+1}^{m_k}\mathbf{E}_h^i, \nabla q_h\big) \\
= {}& (c_0(\mathcal{R}_h\bar{\partial}_{t^K}p(t_{m_k}^M) - \bar{\partial}_{t^K}p(t_{m_k}^M)), q_h) + \big(c_0(\bar{\partial}_{t^K}p(t_{m_k}^M) - \frac{\partial}{\partial t}p(t_{m_k}^M)), q_h\big) \\
&+ (\alpha\nabla\cdot(\mathcal{Q}_h\bar{\partial}_{t^K}\mathbf{u}(t_{m_k}^M) - \bar{\partial}_{t^K}\mathbf{u}(t_{m_k}^M)), q_h) + \big(\alpha\nabla\cdot(\bar{\partial}_{t^K}\mathbf{u}(t_{m_k}^M) - \frac{\partial}{\partial t}\mathbf{u}(t_{m_k}^M)), q_h\big) \\
&+ \big(L\frac{1}{r}\sum_{i=m_{k-1}+1}^{m_k}(\mathbf{E}(t_{m_k}^M) - \mathbf{E}(t_i^M)), \nabla q_h\big) + \big(L\frac{1}{r}\sum_{i=m_{k-1}+1}^{m_k}\Xi(t_i^M), \nabla q_h\big) \\
&+ \big(L\frac{1}{r}\sum_{i=m_{k-1}+1}^{m_k}\Phi_h^i, \nabla q_h\big) \\
=: {}& (c_0\omega_{K3}(t_k^K), q_h) + (c_0\rho_{K3}^k, q_h) + (\alpha\nabla\cdot\boldsymbol{\omega}_{K4}(t_k^K), q_h) + (\alpha\nabla\cdot\boldsymbol{\rho}_{K4}^k, q_h) \\
&+ \big(L\frac{1}{r}\sum_{i=m_{k-1}+1}^{m_k}(\mathbf{E}(t_{m_k}^M) - \mathbf{E}(t_i^M)), \nabla q_h\big) + \big(L\frac{1}{r}\sum_{i=m_{k-1}+1}^{m_k}\Xi(t_i^M), \nabla q_h\big) \\
&+ \big(L\frac{1}{r}\sum_{i=m_{k-1}+1}^{m_k}\Phi_h^i, \nabla q_h\big).
\end{aligned}$$

Taking $\mathbf{v}_h = 2\tau_K\bar{\partial}_{t^K}\boldsymbol{\varphi}_h^{m_k}$ and $q_h = 2\tau_K\psi_h^{m_k}$ in above two equations, respectively, and then summing up, we get an error equation

$$\begin{aligned}
&a(\boldsymbol{\varphi}_h^{m_k}, 2\tau_K\bar{\partial}_{t^K}\boldsymbol{\varphi}_h^{m_k}) + (c_0\bar{\partial}_{t^K}\psi_h^{m_k}, 2\tau_K\psi_h^{m_k}) + 2\kappa\tau_K\|\nabla\psi_h^{m_k}\|^2 \\
= {}& (c_0\omega_{K3}(t_k^K), 2\tau_K\psi_h^{m_k}) + (c_0\rho_{K3}^k, 2\tau_K\psi_h^{m_k}) + (\alpha\nabla\cdot\boldsymbol{\omega}_{K4}(t_k^K), 2\tau_K\psi_h^{m_k}) \\
&+ (\alpha\nabla\cdot\boldsymbol{\rho}_{K4}^k, 2\tau_K\psi_h^{m_k}) + \big(L\frac{1}{r}\sum_{i=m_{k-1}+1}^{m_k}(\mathbf{E}(t_{m_k}^M) - \mathbf{E}(t_i^M)), 2\tau_K\nabla\psi_h^{m_k}\big) \\
&+ \big(L\frac{1}{r}\sum_{i=m_{k-1}+1}^{m_k}\Xi(t_i^M), 2\tau_K\nabla\psi_h^{m_k}\big) + \big(L\frac{1}{r}\sum_{i=m_{k-1}+1}^{m_k}\Phi_h^i, 2\tau_K\nabla\psi_h^{m_k}\big).
\end{aligned} \tag{28}$$

In a similar way as proving (11), we have

$$\begin{aligned}
a(\boldsymbol{\varphi}_h^{m_k}, 2\tau_K\bar{\partial}_{t^K}\boldsymbol{\varphi}_h^{m_k}) &\geq \|\boldsymbol{\varphi}_h^{m_k}\|_a^2 - \|\boldsymbol{\varphi}_h^{m_{k-1}}\|_a^2, \\
(\bar{\partial}_{t^K}\psi_h^{m_k}, 2\tau_K\psi_h^{m_k}) &\geq \|\psi_h^{m_k}\|^2 - \|\psi_h^{m_{k-1}}\|^2.
\end{aligned}$$

These together with (28), Cauchy–Schwarz and Young's inequalities lead to

$$
\begin{aligned}
&\|\boldsymbol{\varphi}_h^{m_k}\|_a^2 + c_0\|\psi_h^{m_k}\|^2 + 2\kappa\tau_K\|\nabla\psi_h^{m_k}\|^2 \\
\le\ & \|\boldsymbol{\varphi}_h^{m_{k-1}}\|_a^2 + c_0\|\psi_h^{m_{k-1}}\|^2 + c_0\tau_K\|\omega_{K3}(t_k^K)\|^2 + c_0\tau_K\|\psi_h^{m_k}\|^2 + c_0\tau_K\|\rho_{K3}^k\|^2 \\
&+ c_0\tau_K\|\psi_h^{m_k}\|^2 + \alpha\tau_K\|\nabla\cdot\boldsymbol{\omega}_{K4}(t_k^K)\|^2 + \alpha\tau_K\|\psi_h^{m_k}\|^2 + \alpha\tau_K\|\nabla\cdot\boldsymbol{\rho}_{K4}^k\|^2 \\
&+ \alpha\tau_K\|\psi_h^{m_k}\|^2 + \frac{3L^2}{\kappa}\frac{1}{r^2}\tau_K\Big\|\sum_{i=m_{k-1}+1}^{m_k}(\mathbf{E}(t_{m_k}^M) - \mathbf{E}(t_i^M))\Big\|^2 \\
&+ \frac{\kappa}{3}\tau_K\|\nabla\psi_h^{m_k}\|^2 + \frac{3L^2}{\kappa}\frac{1}{r^2}\tau_K\Big\|\sum_{i=m_{k-1}+1}^{m_k}\Xi(t_i^M)\Big\|^2 + \frac{\kappa}{3}\tau_K\|\nabla\psi_h^{m_k}\|^2 \\
&+ \frac{3L^2}{\kappa}\frac{1}{r^2}\tau_K\Big\|\sum_{i=m_{k-1}+1}^{m_k}\Phi_h^i\Big\|^2 + \frac{\kappa}{3}\tau_K\|\nabla\psi_h^{m_k}\|^2 \\
\le\ & \|\boldsymbol{\varphi}_h^{m_{k-1}}\|_a^2 + c_0\|\psi_h^{m_{k-1}}\|^2 + c_0\tau_K\|\omega_{K3}(t_k^K)\|^2 + c_0\tau_K\|\rho_{K3}^k\|^2 \\
&+ \alpha\tau_K\|\nabla\cdot\boldsymbol{\omega}_{K4}(t_k^K)\|^2 + \alpha\tau_K\|\nabla\cdot\boldsymbol{\rho}_{K4}^k\|^2 \\
&+ \frac{3L^2}{\kappa}\tau_M\sum_{i=m_{k-1}+1}^{m_k}\|\mathbf{E}(t_{m_k}^M) - \mathbf{E}(t_i^M)\|^2 + \frac{3L^2}{\kappa}\tau_M\sum_{i=m_{k-1}+1}^{m_k}\|\Xi(t_i^M)\|^2 \\
&+ \frac{3L^2}{\kappa}\tau_M\sum_{i=m_{k-1}+1}^{m_k}\|\Phi_h^i\|^2 + (2c_0+2\alpha)\tau_K\|\psi_h^{m_k}\|^2 + \kappa\tau_K\|\nabla\psi_h^{m_k}\|^2.
\end{aligned}
$$

Inductively for $k \ge 1$ in above inequality, we get

$$
\begin{aligned}
&\|\boldsymbol{\varphi}_h^{m_k}\|_a^2 + c_0\|\psi_h^{m_k}\|^2 + \kappa\tau_K\sum_{\ell=1}^{k}\|\nabla\psi_h^{m_\ell}\|^2 \\
\le\ & \|\boldsymbol{\varphi}_h^0\|_a^2 + c_0\|\psi_h^0\|^2 + c_0\tau_K\sum_{\ell=1}^{k}\|\omega_{K3}(t_\ell^K)\|^2 + c_0\tau_K\sum_{\ell=1}^{k}\|\rho_{K3}^\ell\|^2 \\
&+ \alpha\tau_K\sum_{\ell=1}^{k}\|\nabla\cdot\boldsymbol{\omega}_{K4}(t_\ell^K)\|^2 + \alpha\tau_K\sum_{\ell=1}^{k}\|\nabla\cdot\boldsymbol{\rho}_{K4}^\ell\|^2 \\
&+ \frac{3L^2}{\kappa}\tau_M\sum_{\ell=1}^{k}\sum_{i=m_{\ell-1}+1}^{m_\ell}\|\mathbf{E}(t_{m_\ell}^M) - \mathbf{E}(t_i^M)\|^2 + \frac{3L^2}{\kappa}\tau_M\sum_{i=1}^{m_k}\|\Xi(t_i^M)\|^2 \\
&+ \frac{3L^2}{\kappa}\tau_M\sum_{i=1}^{m_k}\|\Phi_h^i\|^2 + (2c_0+2\alpha)\tau_K\sum_{\ell=1}^{k}\|\psi_h^{m_\ell}\|^2.
\end{aligned}
\tag{29}
$$

According to (26) and (29), we obtain error estimates at nodes $t_{m_k}^M = t_k^K$ $(1 \le k \le N_K)$,

$$
\begin{aligned}
&\epsilon\|\Phi_h^{m_k}\|^2 + \mu\|\Theta_h^{m_k}\|^2 + \|\boldsymbol{\varphi}_h^{m_k}\|_a^2 + c_0\|\psi_h^{m_k}\|^2 + \kappa\tau_K\|\nabla\psi_h^{m_k}\|^2 \\
\le\ & \epsilon\|\Phi_h^0\|^2 + \mu\|\Theta_h^0\|^2 + \|\boldsymbol{\varphi}_h^0\|_a^2 + c_0\|\psi_h^0\|^2 + \kappa\tau_K\|\nabla\psi_h^0\|^2 \\
&+ \epsilon\tau_M\sum_{i=1}^{m_k}\|\boldsymbol{\omega}_{M1}(t_i^M)\|^2 + \epsilon\tau_M\sum_{i=1}^{m_k}\|\boldsymbol{\rho}_{M1}^i\|^2 + (\sigma + \frac{3L^2}{\kappa})\tau_M\sum_{i=1}^{m_k}\|\Xi(t_i^M)\|^2 \\
&+ L\tau_M\sum_{\ell=1}^{k}\sum_{i=m_{\ell-1}+1}^{m_\ell}\|\nabla(p(t_i^M) - p(t_{m_{\ell-1}}^M))\|^2 + L\tau_K\sum_{\ell=1}^{k}\|\nabla\xi(t_{m_{\ell-1}}^M)\|^2
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&+\mu\tau_M\sum_{i=1}^{m_k}\|\boldsymbol{\omega}_{M2}(t_i^M)\|^2+\mu\tau_M\sum_{i=1}^{m_k}\|\boldsymbol{\rho}_{M2}^i\|^2+\tau_M\sum_{i=1}^{m_k}\|\nabla\times\Xi(t_i^M)\|^2\\
&+c_0\tau_K\sum_{\ell=1}^{k}\|\omega_{K3}(t_\ell^K)\|^2+c_0\tau_K\sum_{\ell=1}^{k}\|\rho_{K3}^\ell\|^2+\alpha\tau_K\sum_{\ell=1}^{k}\|\nabla\cdot\boldsymbol{\omega}_{K4}(t_\ell^K)\|^2\\
&+\alpha\tau_K\sum_{\ell=1}^{k}\|\nabla\cdot\boldsymbol{\rho}_{K4}^\ell\|^2+\frac{3L^2}{\kappa}\tau_M\sum_{\ell=1}^{k}\sum_{i=m_{\ell-1}+1}^{m_\ell}\|\mathbf{E}(t_{m_\ell}^M)-\mathbf{E}(t_i^M)\|^2\\
&+(2\epsilon+\sigma+2L+\frac{4L^2}{\kappa})\tau_M\sum_{i=1}^{m_k}\|\Phi_h^i\|^2+(2\mu+1)\tau_M\sum_{i=1}^{m_k}\|\Theta_h^i\|^2\\
&+(2c_0+2\alpha)\tau_K\sum_{\ell=1}^{k}\|\psi_h^{m_\ell}\|^2.
\end{aligned}
$$

In order to analyze error estimates for numerical solutions at nodes $t^M_{m_{k-1}+s}$ for $1\le k\le N_K$ and $1\le s\le r-1$, we assume $\boldsymbol{\varphi}_h^{m_{k-1}+s}=\boldsymbol{\varphi}_h^{m_{k-1}}$ and $\psi_h^{m_{k-1}+s}=\psi_h^{m_{k-1}}$, respectively. Then, from (27) and (29) with k replaced by $k-1$, it follows that

(31)

$$
\begin{aligned}
&\epsilon\|\Phi_h^{m_{k-1}+s}\|^2+\mu\|\Theta_h^{m_{k-1}+s}\|^2+\|\boldsymbol{\varphi}_h^{m_{k-1}+s}\|_a^2+c_0\|\psi_h^{m_{k-1}+s}\|^2\\
=&\ \epsilon\|\Phi_h^{m_{k-1}+s}\|^2+\mu\|\Theta_h^{m_{k-1}+s}\|^2+\|\boldsymbol{\varphi}_h^{m_{k-1}}\|_a^2+c_0\|\psi_h^{m_{k-1}}\|^2\\
\le&\epsilon\|\Phi_h^0\|^2+\mu\|\Theta_h^0\|^2+\|\boldsymbol{\varphi}_h^0\|_a^2+c_0\|\psi_h^0\|^2+\kappa\tau_K\|\nabla\psi_h^0\|^2\\
&+\epsilon\tau_M\sum_{i=1}^{m_{k-1}+s}\|\boldsymbol{\omega}_{M1}(t_i^M)\|^2+\epsilon\tau_M\sum_{i=1}^{m_{k-1}+s}\|\boldsymbol{\rho}_{M1}^i\|^2\\
&+(\sigma+\frac{3L^2}{\kappa})\tau_M\sum_{i=1}^{m_{k-1}+s}\|\Xi(t_i^M)\|^2+L\tau_M\sum_{\ell=1}^{k-1}\sum_{i=m_{\ell-1}+1}^{m_\ell}\|\nabla(p(t_i^M)-p(t_{m_{\ell-1}}^M))\|^2\\
&+L\tau_M\sum_{i=m_{k-1}+1}^{m_{k-1}+s}\|\nabla(p(t_i^M)-p(t_{m_{k-1}}^M))\|^2+L\tau_K\sum_{\ell=1}^{k}\|\nabla\xi(t_{m_{\ell-1}}^M)\|^2\\
&+\mu\tau_M\sum_{i=1}^{m_{k-1}+s}\|\boldsymbol{\omega}_{M2}(t_i^M)\|^2+\mu\tau_M\sum_{i=1}^{m_{k-1}+s}\|\boldsymbol{\rho}_{M2}^i\|^2+\tau_M\sum_{i=1}^{m_{k-1}+s}\|\nabla\times\Xi(t_i^M)\|^2\\
&+c_0\tau_K\sum_{\ell=1}^{k-1}\|\omega_{K3}(t_\ell^K)\|^2+c_0\tau_K\sum_{\ell=1}^{k-1}\|\rho_{K3}^\ell\|^2+\alpha\tau_K\sum_{\ell=1}^{k-1}\|\nabla\cdot\boldsymbol{\omega}_{K4}(t_\ell^K)\|^2\\
&+\alpha\tau_K\sum_{\ell=1}^{k-1}\|\nabla\cdot\boldsymbol{\rho}_{K4}^\ell\|^2+\frac{3L^2}{\kappa}\tau_M\sum_{\ell=1}^{k-1}\sum_{i=m_{\ell-1}+1}^{m_\ell}\|\mathbf{E}(t_{m_\ell}^M)-\mathbf{E}(t_i^M)\|^2\\
&+(2\epsilon+\sigma+2L+\frac{4L^2}{\kappa})\tau_M\sum_{i=1}^{m_{k-1}+s}\|\Phi_h^i\|^2+(2\mu+1)\tau_M\sum_{i=1}^{m_{k-1}+s}\|\Theta_h^i\|^2\\
&+(2c_0+2\alpha)\tau_K\sum_{\ell=1}^{k-1}\|\psi_h^{m_\ell}\|^2.
\end{aligned}
$$

According to (30) and (31), we get for all $1 \le m \le N_M$ and $1 \le k \le N_K$,

$$
\begin{aligned}
&\epsilon\|\Phi_h^m\|^2 + \mu\|\Theta_h^m\|^2 + \|\boldsymbol{\varphi}_h^m\|_a^2 + c_0\|\psi_h^m\|^2 \\
\le&\epsilon\|\Phi_h^0\|^2 + \mu\|\Theta_h^0\|^2 + \|\boldsymbol{\varphi}_h^0\|_a^2 + c_0\|\psi_h^0\|^2 + \kappa\tau_K\|\nabla\psi_h^0\|^2 + \epsilon\tau_M\sum_{i=1}^m\|\boldsymbol{\omega}_{M1}(t_i^M)\|^2 \\
&+\epsilon\tau_M\sum_{i=1}^m\|\boldsymbol{\rho}_{M1}^i\|^2 + (\sigma+\frac{3L^2}{\kappa})\tau_M\sum_{i=1}^m\|\Xi(t_i^M)\|^2 \\
&+L\tau_M\sum_{\ell=1}^k\sum_{i=m_{\ell-1}+1}^{m_\ell}\|\nabla(p(t_i^M)-p(t_{m_{\ell-1}}^M))\|^2 + L\tau_K\sum_{\ell=1}^k\|\nabla\xi(t_{m_{\ell-1}}^M)\|^2 \\
&+\mu\tau_M\sum_{i=1}^m\|\boldsymbol{\omega}_{M2}(t_i^M)\|^2 + \mu\tau_M\sum_{i=1}^m\|\boldsymbol{\rho}_{M2}^i\|^2 + \tau_M\sum_{i=1}^m\|\nabla\times\Xi(t_i^M)\|^2 \\
&+c_0\tau_K\sum_{\ell=1}^k\|\omega_{K3}(t_\ell^K)\|^2 + c_0\tau_K\sum_{\ell=1}^k\|\rho_{K3}^\ell\|^2 + \alpha\tau_K\sum_{\ell=1}^k\|\nabla\cdot\boldsymbol{\omega}_{K4}(t_\ell^K)\|^2 \\
&+\alpha\tau_K\sum_{\ell=1}^k\|\nabla\cdot\boldsymbol{\rho}_{K4}^\ell\|^2 + \frac{3L^2}{\kappa}\tau_M\sum_{\ell=1}^k\sum_{i=m_{\ell-1}+1}^{m_\ell}\|\mathbf{E}(t_{m_\ell}^M)-\mathbf{E}(t_i^M)\|^2 \\
&+(2\epsilon+\sigma+2L+\frac{4L^2}{\kappa})\tau_K\sum_{i=1}^m\|\Phi_h^i\|^2 + (2\mu+1)\tau_K\sum_{i=1}^m\|\Theta_h^i\|^2 \\
&+(2c_0+2\alpha)\tau_K\sum_{i=1}^m\|\psi_h^i\|^2.
\end{aligned}
\tag{32}
$$

From Lemmas 2.1 and 2.2, it follows that

$$
\begin{aligned}
\|\Phi_h^0\| &= \|\mathcal{I}_h\mathbf{E}_0 - \mathbf{E}_h^0\| \le \|\mathcal{I}_h\mathbf{E}_0 - \mathbf{E}_0\| + \|\mathbf{E}_0 - \mathfrak{P}_h\mathbf{E}_0\| \\
&\le Ch\|\mathbf{E}_0\|_2 + Ch^2\|\mathbf{E}_0\|_2 \le Ch\|\mathbf{E}_0\|_2, \\
\|\Theta_h^0\| &= \|\mathcal{P}_h\mathbf{H}_0 - \mathbf{H}_h^0\| = 0, \\
\|\boldsymbol{\varphi}_h^0\|_a &= \|\mathcal{Q}_h\mathbf{u}_0 - \mathbf{u}_h^0\|_a \le c_2\|\mathcal{Q}_h\mathbf{u}_0 - \mathbf{u}_0\|_1 + c_2\|\mathbf{u}_0 - \mathscr{P}_h\mathbf{u}_0\|_1 \\
&\le Ch\|\mathbf{u}_0\|_2 + Ch^2\|p_0\|_2 + Ch\|\mathbf{u}_0\|_2 \le Ch\|\mathbf{u}_0\|_2 + Ch^2\|p_0\|_2, \\
\|\psi_h^0\| &= \|\mathcal{R}_hp_0 - p_h^0\| \le \|\mathcal{R}_hp_0 - p_0\| + \|p_0 - P_hp_0\| \\
&\le Ch^2\|p_0\|_2 + Ch^2\|p_0\|_2 \le Ch^2\|p_0\|_2, \\
\|\nabla\psi_h^0\| &= \|\nabla(\mathcal{R}_hp_0 - p_h^0)\| \le \|\nabla(\mathcal{R}_hp_0 - p_0)\| + \|\nabla(p_0 - P_hp_0)\| \\
&\le Ch\|p_0\|_2 + Ch\|p_0\|_2 \le Ch\|p_0\|_2,
\end{aligned}
\tag{33}
$$

where $C > 0$ is a constant independent of τ_M, τ_K and h. Direct computation yields

$$
\begin{aligned}
\boldsymbol{\omega}_{M1}(t_i^M) =& \mathcal{I}_h\bar{\partial}_{t^M}\mathbf{E}(t_i^M) - \bar{\partial}_{t^M}\mathbf{E}(t_i^M) \\
=& \frac{1}{\tau_M}\int_{t_{i-1}^M}^{t_i^M}\big(\mathcal{I}_h\frac{\partial}{\partial t}\mathbf{E}(\theta) - \frac{\partial}{\partial t}\mathbf{E}(\theta)\big)\,\mathrm{d}\theta.
\end{aligned}
$$

Utilizing Cauchy–Schwarz inequality and Lemma 2.2, we have

$$
\begin{aligned}
\|\boldsymbol{\omega}_{M1}(t_i^M)\|^2 &\le \Big(\frac{1}{\tau_M}\int_{t_{i-1}^M}^{t_i^M} \|\mathcal{I}_h\frac{\partial}{\partial t}\mathbf{E}(\theta)-\frac{\partial}{\partial t}\mathbf{E}(\theta)\|\,\mathrm{d}\theta\Big)^2\\
&\le \frac{1}{\tau_M^2}\int_{t_{i-1}^M}^{t_i^M} 1^2\,\mathrm{d}\theta\int_{t_{i-1}^M}^{t_i^M} \|\mathcal{I}_h\frac{\partial}{\partial t}\mathbf{E}(\theta)-\frac{\partial}{\partial t}\mathbf{E}(\theta)\|^2\,\mathrm{d}\theta \qquad (34)\\
&\le C\frac{1}{\tau_M}h^2\int_{t_{i-1}^M}^{t_i^M} \|\frac{\partial}{\partial t}\mathbf{E}(\theta)\|_2^2\,\mathrm{d}\theta.
\end{aligned}
$$

Noticing Lemma 2.1 and in a similar way, we get

$$
\begin{aligned}
\|\boldsymbol{\omega}_{M2}(t_i^M)\|^2 &\le \frac{1}{\tau_M}\int_{t_{i-1}^M}^{t_i^M} \|\mathcal{P}_h\frac{\partial}{\partial t}\mathbf{H}(\theta)-\frac{\partial}{\partial t}\mathbf{H}(\theta)\|^2\,\mathrm{d}\theta\\
&\le C\frac{1}{\tau_M}h^2\int_{t_{i-1}^M}^{t_i^M} \|\frac{\partial}{\partial t}\mathbf{H}(\theta)\|_1^2\,\mathrm{d}\theta,\\
\|\omega_{K3}(t_\ell^K)\|^2 &\le \frac{1}{\tau_K}\int_{t_{\ell-1}^K}^{t_\ell^K} \|\mathcal{R}_h\frac{\partial}{\partial t}p(\theta)-\frac{\partial}{\partial t}p(\theta)\|^2\,\mathrm{d}\theta\\
&\le C\frac{1}{\tau_K}h^4\int_{t_{\ell-1}^K}^{t_\ell^K} \|\frac{\partial}{\partial t}p(\theta)\|_2^2\,\mathrm{d}\theta, \qquad (35)\\
\|\nabla\cdot\boldsymbol{\omega}_{K4}(t_\ell^K)\|^2 &\le \frac{1}{\tau_K}\int_{t_{\ell-1}^K}^{t_\ell^K} \|\nabla\cdot\big(\mathcal{Q}_h\frac{\partial}{\partial t}\mathbf{u}(\theta)-\frac{\partial}{\partial t}\mathbf{u}(\theta)\big)\|^2\,\mathrm{d}\theta\\
&\le \frac{1}{\tau_K}\int_{t_{\ell-1}^K}^{t_\ell^K} \|\mathcal{Q}_h\frac{\partial}{\partial t}\mathbf{u}(\theta)-\frac{\partial}{\partial t}\mathbf{u}(\theta)\|_1^2\,\mathrm{d}\theta\\
&\le C\frac{1}{\tau_K}h^2\int_{t_{\ell-1}^K}^{t_\ell^K} \|\frac{\partial}{\partial t}\mathbf{u}(\theta)\|_2^2\,\mathrm{d}\theta + C\frac{1}{\tau_K}h^4\int_{t_{\ell-1}^K}^{t_\ell^K} \|\frac{\partial}{\partial t}p(\theta)\|_2^2\,\mathrm{d}\theta.
\end{aligned}
$$

Applying Taylor expansion yields

$$
\boldsymbol{\rho}_{M1}^i=\bar{\partial}_{t^M}\mathbf{E}(t_i^M)-\frac{\partial}{\partial t}\mathbf{E}(t_i^M)=\frac{1}{\tau_M}\int_{t_{i-1}^M}^{t_i^M}(t_{i-1}^M-\theta)\frac{\partial^2}{\partial t^2}\mathbf{E}(\theta)\,\mathrm{d}\theta.
$$

By virtue of Cauchy–Schwarz inequality, we obtain

$$
\|\boldsymbol{\rho}_{M1}^i\|^2\le\Big(\int_{t_{i-1}^M}^{t_i^M}\|\frac{\partial^2}{\partial t^2}\mathbf{E}(\theta)\|\,\mathrm{d}\theta\Big)^2\le\tau_M\int_{t_{i-1}^M}^{t_i^M}\|\frac{\partial^2}{\partial t^2}\mathbf{E}(\theta)\|^2\,\mathrm{d}\theta. \qquad (36)
$$

Similarly, we get

$$
\begin{aligned}
\|\boldsymbol{\rho}_{M2}^i\|^2 &\le \tau_M\int_{t_{i-1}^M}^{t_i^M}\|\frac{\partial^2}{\partial t^2}\mathbf{H}(\theta)\|^2\,\mathrm{d}\theta,\\
\|\rho_{K3}^\ell\|^2 &\le \tau_K\int_{t_{\ell-1}^K}^{t_\ell^K}\|\frac{\partial^2}{\partial t^2}p(\theta)\|^2\,\mathrm{d}\theta, \qquad (37)\\
\|\nabla\cdot\boldsymbol{\rho}_{K4}^\ell\|^2 &\le \|\boldsymbol{\rho}_{K4}^\ell\|_1^2\le\tau_K\int_{t_{\ell-1}^K}^{t_\ell^K}\|\frac{\partial^2}{\partial t^2}\mathbf{u}(\theta)\|_1^2\,\mathrm{d}\theta.
\end{aligned}
$$

From Lemma 2.2, it follows that

$$
\begin{aligned}
\|\Xi(t_i^M)\|^2 &= \|\mathbf{E}(t_i^M) - \mathcal{I}_h\mathbf{E}(t_i^M)\|^2 \le Ch^2\|\mathbf{E}(t_i^M)\|_2^2,\\
\|\nabla\xi(t_{m_{\ell-1}}^M)\|^2 &= \|\nabla(p(t_{\ell-1}^K) - \mathcal{R}_h p(t_{\ell-1}^K))\|^2 \le Ch^2\|p(t_{\ell-1}^K)\|_2^2,\\
\|\nabla\times\Xi(t_i^M)\|^2 &= \|\nabla\times(\mathbf{E}(t_i^M) - \mathcal{I}_h\mathbf{E}(t_i^M))\|^2 \le Ch^2\|\mathbf{E}(t_i^M)\|_2^2.
\end{aligned}
\tag{38}
$$

For any $m_{\ell-1}+1 \le i \le m_\ell$, applying Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
\|\nabla(p(t_i^M) - p(t_{m_{\ell-1}}^M))\|^2 &= \|\int_{t_{m_{\ell-1}}^M}^{t_i^M} \nabla\frac{\partial}{\partial t}p(\theta)\,\mathrm{d}\theta\|^2\\
\le \Big(\int_{t_{\ell-1}^K}^{t_\ell^K} \|\frac{\partial}{\partial t}p(\theta)\|_1\,\mathrm{d}\theta\Big)^2 &\le \tau_K\int_{t_{\ell-1}^K}^{t_\ell^K}\|\frac{\partial}{\partial t}p(\theta)\|_1^2\,\mathrm{d}\theta.
\end{aligned}
\tag{39}
$$

Similarly, we have for $m_{\ell-1}+1 \le i \le m_\ell$,

$$
\|\mathbf{E}(t_{m_\ell}^M) - \mathbf{E}(t_i^M)\|^2 \le \tau_K\int_{t_{\ell-1}^K}^{t_\ell^K}\|\frac{\partial}{\partial t}\mathbf{E}(\theta)\|^2\,\mathrm{d}\theta.
\tag{40}
$$

According to (32)–(40) and (H3), we obtain that for all $1 \le m \le N_M$ and $1 \le k \le N_K$,

$$
\begin{aligned}
&\epsilon\|\Phi_h^m\|^2 + \mu\|\Theta_h^m\|^2 + \|\boldsymbol{\varphi}_h^m\|_a^2 + c_0\|\psi_h^m\|^2\\
&\le \epsilon Ch^2\|\mathbf{E}_0\|_2^2 + Ch^2\|\mathbf{u}_0\|_2^2 + Ch^4\|p_0\|_2^2 + c_0Ch^4\|p_0\|_2^2 + \kappa Ch^2\|p_0\|_2^2\\
&\quad + \epsilon Ch^2\int_0^{t_m^M}\|\frac{\partial}{\partial t}\mathbf{E}(\theta)\|_2^2\,\mathrm{d}\theta + \epsilon\tau_M^2\int_0^{t_m^M}\|\frac{\partial^2}{\partial t^2}\mathbf{E}(\theta)\|^2\,\mathrm{d}\theta\\
&\quad + (\sigma + \frac{3L^2}{\kappa})Ch^2\sum_{i=1}^m\tau_M\|\mathbf{E}(t_i^M)\|_2^2 + L\tau_K^2\int_0^{t_k^K}\|\frac{\partial}{\partial t}p(\theta)\|_1^2\,\mathrm{d}\theta\\
&\quad + LCh^2\sum_{\ell=1}^k\tau_K\|p(t_{\ell-1}^K)\|_2^2 + \mu Ch^2\int_0^{t_m^M}\|\frac{\partial}{\partial t}\mathbf{H}(\theta)\|_1^2\,\mathrm{d}\theta\\
&\quad + \mu\tau_M^2\int_0^{t_m^M}\|\frac{\partial^2}{\partial t^2}\mathbf{H}(\theta)\|^2\,\mathrm{d}\theta + Ch^2\sum_{i=1}^m\tau_M\|\mathbf{E}(t_i^M)\|_2^2\\
&\quad + c_0Ch^4\int_0^{t_k^K}\|\frac{\partial}{\partial t}p(\theta)\|_2^2\,\mathrm{d}\theta + c_0\tau_K^2\int_0^{t_k^K}\|\frac{\partial^2}{\partial t^2}p(\theta)\|^2\,\mathrm{d}\theta\\
&\quad + \alpha Ch^2\int_0^{t_k^K}\|\frac{\partial}{\partial t}\mathbf{u}(\theta)\|_2^2\,\mathrm{d}\theta + \alpha Ch^4\int_0^{t_k^K}\|\frac{\partial}{\partial t}p(\theta)\|_2^2\,\mathrm{d}\theta\\
&\quad + \alpha\tau_K^2\int_0^{t_k^K}\|\frac{\partial^2}{\partial t^2}\mathbf{u}(\theta)\|_1^2\,\mathrm{d}\theta + \frac{3L^2}{\kappa}\tau_K^2\int_0^{t_k^K}\|\frac{\partial}{\partial t}\mathbf{E}(\theta)\|^2\,\mathrm{d}\theta\\
&\quad + (2\epsilon + \sigma + 2L + \frac{4L^2}{\kappa})\sum_{i=1}^m\tau_K\|\Phi_h^i\|^2 + (2\mu+1)\sum_{i=1}^m\tau_K\|\Theta_h^i\|^2\\
&\quad + (2c_0 + 2\alpha)\sum_{i=1}^m\tau_K\|\psi_h^i\|^2\\
&\le C_1\tau_M^2 + C_2\tau_K^2 + Ch^2 + C\sum_{i=1}^m\tau_K\big(\|\Phi_h^i\|^2 + \|\Theta_h^i\|^2 + \|\psi_h^i\|^2\big),
\end{aligned}
$$

where $C_1 = C_1(\frac{\partial^2 \mathbf{E}}{\partial t^2}, \frac{\partial^2 \mathbf{H}}{\partial t^2})$, $C_2 = C_2(\frac{\partial p}{\partial t}, \frac{\partial^2 p}{\partial t^2}, \frac{\partial^2 \mathbf{u}}{\partial t^2}, \frac{\partial \mathbf{E}}{\partial t})$ and C are positive constants independent of τ_M, τ_K and h. Applying discrete Gronwall's lemma to above inequality, we get for all $1 \le m \le N_M$,

$$\begin{aligned}&\|\Phi_h^m\|^2 + \|\Psi_h^m\|^2 + \|\boldsymbol{\varphi}_h^m\|_a^2 + \|\psi_h^m\|^2 \\ &\le (C_1\tau_M^2 + C_2\tau_K^2 + Ch^2)e^{C\sum_{i=1}^m \tau_K} \le C_1\tau_M^2 + C_2\tau_K^2 + Ch^2.\end{aligned}$$

This leads that for all $1 \le m \le N_M$ and $1 \le k \le N_K$,

$$\|\Phi_h^m\|^2 + \|\Psi_h^m\|^2 + \|\boldsymbol{\varphi}_h^{m_k}\|_1^2 + \|\psi_h^{m_k}\|^2 \le C_1\tau_M^2 + C_2\tau_K^2 + Ch^2,$$

which together with (22) completes the proof. $\square$

Remark 4.1. *We would mention that the electromagnetic waves* $(\mathbf{E}, \mathbf{H})$ *vary much rapidly than the elastic waves* $(\mathbf{u}, p)$ *in physical world. Therefore, the constant* C_1 *is reasonably larger than* C_2*. In order to construct a fast numerical algorithm and keep the accuracy, we will choose the step-sizes such that* $C_1\tau_M^2 = C_2\tau_K^2$ *which indicates to choose* $r\tau_M = \tau_K$ *for some fixed integer* $r > 0$*.*

TABLE 1. Errors and convergence orders of Algorithm 1 ($r = 4$).

h	$\left\|\mathbf{E}(T) - \mathbf{E}_h^{N_M}\right\|$	Order	$\left\|\mathbf{H}(T) - \mathbf{H}_h^{N_M}\right\|$	Order
1/4	0.09156409	-	0.18640165	-
1/8	0.05003390	0.8719	0.09339199	0.9970
1/12	0.03339432	0.9972	0.06224391	1.0007
1/16	0.02507113	0.9965	0.04671507	0.9976
h	$\left\|\mathbf{u}(T) - \mathbf{u}_h^{N_M}\right\|_1$	Order	$\left\|p(T) - p_h^{N_M}\right\|$	Order
1/4	1.44226916	-	0.08017291	-
1/8	0.75396900	0.9358	0.02233873	1.8436
1/12	0.50650102	0.9812	0.01012946	1.9505
1/16	0.38087188	0.9909	0.00573148	1.9795

TABLE 2. Errors and convergence orders of Algorithm 1 ($r = 3$).

h	$\left\|\mathbf{E}(T) - \mathbf{E}_h^{N_M}\right\|$	Order	$\left\|\mathbf{H}(T) - \mathbf{H}_h^{N_M}\right\|$	Order
1/4	0.09154328	-	0.18640474	-
1/8	0.05002734	0.8717	0.09339180	0.9971
1/12	0.03339072	0.9971	0.06224386	1.0007
1/16	0.02506857	0.9965	0.04671505	0.9976
h	$\left\|\mathbf{u}(T) - \mathbf{u}_h^{N_M}\right\|_1$	Order	$\left\|p(T) - p_h^{N_M}\right\|$	Order
1/4	1.44226894	-	0.08019893	-
1/8	0.75396866	0.9358	0.02235443	1.8430
1/12	0.50650076	0.9812	0.01013972	1.9498
1/16	0.38087167	0.9909	0.00573940	1.9782

Table 3. Errors and convergence orders of Algorithm 1 ($r = 2$).

h	$\left\|\mathbf{E}(T) - \mathbf{E}_h^{N_M}\right\|$	Order	$\left\|\mathbf{H}(T) - \mathbf{H}_h^{N_M}\right\|$	Order
1/4	0.09152293	-	0.18640783	-
1/8	0.05002099	0.8716	0.09339160	0.9971
1/12	0.03338728	0.9970	0.06224381	1.0007
1/16	0.02506616	0.9964	0.04671503	0.9976
h	$\left\|\mathbf{u}(T) - \mathbf{u}_h^{N_M}\right\|_1$	Order	$\left\|p(T) - p_h^{N_M}\right\|$	Order
1/4	1.44226872	-	0.08022432	-
1/8	0.75396833	0.9358	0.02236992	1.8425
1/12	0.50650050	0.9812	0.01014992	1.9490
1/16	0.38087146	0.9909	0.00574732	1.9769

Table 4. Errors and convergence orders of splitting FEM.

h	$\left\|\mathbf{E}(T) - \mathbf{E}_h^{N_M}\right\|$	Order	$\left\|\mathbf{H}(T) - \mathbf{H}_h^{N_M}\right\|$	Order
1/4	0.09150305	-	0.18641093	-
1/8	0.05001483	0.8715	0.09339140	0.9971
1/12	0.03338399	0.9970	0.06224376	1.0007
1/16	0.02506390	0.9964	0.04671501	0.9976
h	$\left\|\mathbf{u}(T) - \mathbf{u}_h^{N_M}\right\|_1$	Order	$\left\|p(T) - p_h^{N_M}\right\|$	Order
1/4	1.44226849	-	0.08024908	-
1/8	0.75396800	0.9358	0.02238519	1.8419
1/12	0.50650023	0.9812	0.01016005	1.9482
1/16	0.38087126	0.9909	0.00575525	1.9756

5. Numerical experiments

Let $T = 0.1$, $\mathscr{D} = [0,1]^3$ and set physical parameters $\epsilon = 1$, $\sigma = 2$, $L = 1$, $\mu = 1$, $\lambda = 1$, $G = 1$, $\alpha = 1$, $c_0 = 1$ and $\kappa = 2$. Take uniform tetrahedral partition for $\mathscr{D}$ with mesh size h. The numerical experiments are conducted on the FEniCS computing platform [1, 17].

Example 5.1. *The initial value* $(\mathbf{E}_0, \mathbf{H}_0, \mathbf{u}_0, p_0)$ *and right-hand side functions* $\mathbf{j}$, $\mathbf{f}$, g *are chosen such that the exact solution to* (1)–(3) *reads*

$$\mathbf{E} = \begin{pmatrix} \sin(\pi t)\sin(\pi x)\sin(\pi y)\sin(\pi z) \\ \sin(\pi t)\sin(\pi x)\sin(\pi y)\sin(\pi z) \\ \sin(\pi t)\sin(\pi x)\sin(\pi y)\sin(\pi z) \end{pmatrix},$$

$$\mathbf{H} = \begin{pmatrix} \cos(\pi t)(\sin(\pi x)\cos(\pi y)\sin(\pi z) - \sin(\pi x)\sin(\pi y)\cos(\pi z)) \\ \cos(\pi t)(\sin(\pi x)\sin(\pi y)\cos(\pi z) - \cos(\pi x)\sin(\pi y)\sin(\pi z)) \\ \cos(\pi t)(\cos(\pi x)\sin(\pi y)\sin(\pi z) - \sin(\pi x)\cos(\pi y)\sin(\pi z)) \end{pmatrix},$$

$$\mathbf{u} = \begin{pmatrix} e^{-t}\sin(\pi x)\sin(\pi y)\sin(\pi z) \\ e^{-t}\sin(\pi x)\sin(\pi y)\sin(\pi z) \\ e^{-t}\sin(\pi x)\sin(\pi y)\sin(\pi z) \end{pmatrix},$$

$$p = e^{-t}\sin(\pi x)\sin(\pi y)\sin(\pi z).$$

TABLE 5. Errors and convergence orders of standard FEM.

h	$\left\Vert \mathbf{E}(T) - \mathbf{E}_h^{N_M} \right\Vert$	Order	$\left\Vert \mathbf{H}(T) - \mathbf{H}_h^{N_M} \right\Vert$	Order
1/4	0.09148317	-	0.18641405	-
1/8	0.05000852	0.8713	0.09339119	0.9972
1/12	0.03338053	0.9969	0.06224371	1.0007
1/16	0.02506151	0.9964	0.04671499	0.9976
h	$\left\Vert \mathbf{u}(T) - \mathbf{u}_h^{N_M} \right\Vert_1$	Order	$\left\Vert p(T) - p_h^{N_M} \right\Vert$	Order
1/4	1.44226850	-	0.08026594	-
1/8	0.75396801	0.9358	0.02239562	1.8416
1/12	0.50650024	0.9812	0.01016832	1.9474
1/16	0.38087126	0.9909	0.00576269	1.9740

TABLE 6. CPU running time.

Time(s) \ h	1/4	1/8	1/12	1/16
Algorithm 1 (r=4)	23	164	691	2669
Algorithm 1 (r=3)	27	174	790	3416
Algorithm 1 (r=2)	36	239	941	3602
Splitting FEM	40	306	1255	6121
Standard FEM	43	435	3948	23906

We take $\tau_M = 1/1800$ and $\tau_K = r\tau_M$ $(r = 4, 3, 2)$, respectively, and then carry out Algorithm 1 with different mesh sizes $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \frac{1}{16}$, respectively. The corresponding errors and convergence orders are shown in Tables 1–3. Meanwhile, we also perform numerical computations using splitting FEM [16] and standard FEM [12] with time step-size $\tau = 1/1800$ and present the corresponding errors and convergence orders in Tables 4 and 5, respectively. All the numerical experiments indicate that the three methods possess the same accuracy. We also record the CPU running time in Table 6 to show the efficiency of our method.

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Conflict of interest

The authors declare that there is no conflict of interest.

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