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NUMERICAL ANALYSIS OF AUGMENTED FVM FOR NONLINEAR TIME FRACTIONAL DEGENERATE PARABOLIC EQUATION

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Abstract. Utilizing the nonlinear Time Fractional Degenerate Parabolic Equation (TFDPE) in modeling provides a comprehensive approach to studying phenomena exhibiting both fractional order dynamics and degenerate parabolic behavior, facilitating accurate predictions and insights across diverse scientific domains. However, the numerical solution of TFDPE is a challenging task and traditional numerical methods cannot solve this equation because of the spatial singularity influence. In this paper, we find the numerical solution of nonlinear TFDPE with both strongly and weakly degenerate cases using the higher order augmented finite volume method on uniform grids. To handle the singularity of TFDPE, we choose an intermediate point near the singular point and split the whole domain into singular and regular subdomains. Then, we find the solution on singular subdomain using the Puiseux series while on the regular subdomain we find the solution by finite volume schemes. The main idea is to recover the Puiseux series on singular subdomain using the Picard iteration methods which is also a challenging because of the time fractional derivative in the original equation. The solution on the singular subdomain is in the form of Puiseux series, which has multiple undetermined augmented variables and these variables play a role in organically combining the singular and regular subdomains. To approximate the time fractional derivative, we use the second order weighted and shifted Grünwald difference (WSGD) scheme and give the comparison of our results with L_1-scheme. We use the discrete energy method to prove that the schemes has temporal second order while spatial second and fourth-order on the whole domain and for the augmented variables in discrete L_2-norm. Finally, we give some numerical examples to confirm the accuracy and order of convergence of the proposed schemes for the whole domain and the augmented variables. We also give an interesting example with coefficient blow-up at the degenerate point and show the schemes are working the same as the other cases.

Key words. Time fractional degenerate parabolic equations, finite volume method, Puiseux series.

1. Introduction

In this paper, we establish second and fourth-order augmented finite volume schemes based on the Puiseux series for the following TFDPE

$$ {}_0^\beta D_t^C u + u_t - \left(x^\alpha u_x\right)_x = f\left(x,t,u\right), \quad (x,t) \in P, \tag{1}$$

where $0 < \alpha < 2$, $0 < \beta < 1$, $P = (0,b) \times (0,T)$ and ${}_0^\beta D_t^C$ is well known Caputo derivative [1,2]. The function $f(x,t,u)$ is a Lipchitz continuous at u, and it may have singularity at $x = 0$. The degenerate problem is where the coefficient of equation is nonnegative and the equation degenerates when the coefficient vanishes. It is obvious that the coefficient x^α of equation (1) vanishes when $x = 0$. Thus equation (1) is degenerate at part $\{0\} \times (0,T)$ of the lateral boundary. These mathematically results in a loss of uniform ellipticity, which in the real world would be represented such as tsunami wave velocity vanishing at the coastline, subsonic-sonic flow degenerating at the sonic state, local density of traffic flow at zero, etc.

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Thus, because of this degeneracy the traditional numerical methods cannot solve equation (1). To overcome this singularity, we use the spatial domain separation strategy. In this way, we choose an intermediate point near the singular point which is 0 for the equation (1) and convert into two subdomains namely singular subdomain P_s (left part) and regular subdomain P_r (right part). Then, we use the Puiseux series expansion on the singular subdomain to find the solution while on regular subdomain we use finite volume method on uniform grids without creating mesh points inside the singular subdomain (see, Figure 1). We also combined the both subdomains by the unknown variables of time involving in the singular subdomain solution.

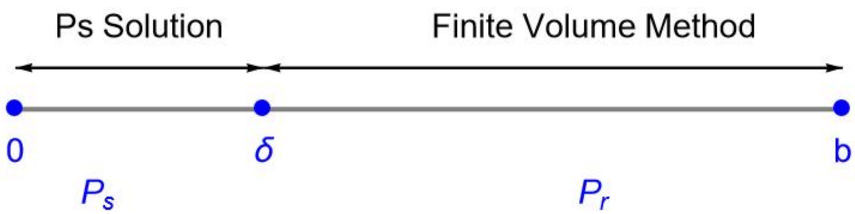


FIGURE 1. Domain separation strategy, where Ps solution means the Puisuex series solution.

The purpose of the term u_t in equation (1) is that we can approximate the time-fractional derivative directly by any second-order fractional approximation without modifying the original equation, and we can attain the numerical analysis of the schemes. Secondly, the equation (1) represents the models having both short and long-term memory effects because many systems have both short and long-term effects, such as the fractional Zener model and fractional Kelvin-Voigt model. Since the time-fractional derivative is for the long-term memory effects and the ordinary time derivative is for the short-term memory effects, we use both terms to get both kinds of memory effects from the single equation. On the basis of α, the equation (1) is classified as weakly and strongly degenerate. In [3], the authors solved equation (1) with classical time derivative only for $0 < \alpha < 1$, which is known as a weakly degenerate case, and they also created a mesh point in the singular subdomain, which may impact the accuracy and convergence of the scheme. It is also noted that the convergence analysis of the numerical schemes given in [3] is not provided. In this paper, we use the time derivative in the sense of fractional as well as solve the equation (1) for weakly degenerate ($0 < \alpha < 1$), strongly degenerate ($1 \leq \alpha < 2$) cases and ($\alpha < 0$) without creating mesh points inside the singular subdomain, due to which there is no impact on the accuracy and convergence of the schemes. The proposed method for this paper has no restriction in splitting the whole domain into the singular and regular subdomains, and we can choose the intermediate point independently. We also give the convergence analysis of the numerical schemes in L_2-norm using the energy method over the whole domain, which can support the convergence analysis for the schemes given in [3]. It is important to mention here that the schemes are also working well if we set $u_t = 0$ in equation (1), shown by examples. For equation (1), we have the following initial and boundary conditions:

$$u(0,t) = u(b,t) = 0 \text{ for } 0 < \alpha < 1, \tag{2}$$

$$(x^{\alpha} u_x)(0,t) = u(b,t) = 0 \text{ for } 1 \leq \alpha < 2, \tag{3}$$

$$u(x,0) = 0 \text{ for } 0 < \alpha < 2. \tag{4}$$

Using time fractional derivatives instead of conventional time derivatives have multiple advantages. One of the most significant benefits is that time fractional derivatives offer a more accurate representation of systems with memory effects. Such systems rely not only on the current input but also on the past inputs and states. Additionally, time fractional derivatives provide better time-frequency localization than traditional derivatives, effectively allowing us to analyze the behavior of the system at different time scales. This enables us to examine short-term and long-term behaviours of the system, providing a more thorough understanding of its dynamics. There are several numerical methods established for solving time fractional partial differential equations. For instance, one can consult [4–8].

In many practical problems, degenerate partial differential equations are appeared. For example, in the motion of liquids and gases in the porous medium, in plasma, climatology, physics, engineering, and many others [9–12]. The nonlinear degenerate wave equation helps describe the movement of long waves, such as those generated by tsunamis, from their source to the shore. This equation is particular to the beach area, where the speed of the waves is zero at the boundary, causing the equation to degenerate. This paper focuses on studying nonlinear TFDPE. There are many numerical methods established to solve the degenerate partial differential equations with classical derivatives involving spectral method [13], Adomian decomposition method [14], finite volume element methods [15, 16], finite difference method [17] and many others. Moreover, a high-order asymptotic augmented numerical method was established in [18] to solve two-point boundary-value problems. Later, a parabolic equation was solved with this new method in [3]. Motivated by the above literature, we are interested in finding the solution of the nonlinear TFDPE using higher-order finite volume schemes.

In this paper, we explore the degeneracy of a function using Puiseux series expansion [20]. Unlike the power series, the Puiseux series can include negative and fractional exponential and logarithmic terms. The function has a local approximation at the degenerate point, but the approximation error increases gradually as we move away from it. To deal with this, we introduce an intermediate point and divide the domain into singular and regular subdomains. First, we transform the original equation using the Green function into an equivalent Fredholm integral equation. Then, through the Picard iteration and symbolic calculation, we obtain the Puiseux series expansion of the solution on the singular subdomain, which contains unknown functions related to time that need to be determined.

It is important to understand that retrieving the Puiseux series can be difficult due to the presence of the time-fractional derivative in nonlinear TFDPE. The symbolic calculations needed to obtain the Puiseux series require us to compute fractional derivatives for the product of augmented variables, which are unknown functions of t and other known functions of t. This complication arises because of the time-fractional derivative. Nevertheless, we implement a variety of strategies and achieve the Puiseux series by utilizing the Picard iteration method and symbolic calculations with Algorithm 1. It is observed from the literature that during the numerical solution of degenerate partial differential equations, some mesh points were created in the singular subdomain near the intermediate point to find the augmented variables that impact the accuracy as well as the convergence of the schemes because, near the singular point, schemes are not working well as traditional numerical schemes. Also, the choice of the intermediate point is difficult. We find the numerical solution without creating mesh points inside the singular subdomain near the intermediate point, which gives a high accuracy, convergence, and independent choice of the intermediate point in the schemes. The singular subdomain solution which is in the form of Puisuex series has many unknown variables called augmented variables related to the singularity and these variables help us in combining singular and regular subdomains. However, calculating these variables for the Puiseux series can be difficult due to their multiple forms. To simplify this process, we utilized formulas based on numerical integration methods and converted all augmented variables into a single one. This conversion allows for higher accuracy without using mesh points in a singular subdomain or constraints on mesh points in a regular subdomain. Then, we calculate this single augmented variable which representing many augmented variables by using the finite volume schemes on the regular subdomain and also an evidence of combining both subdomains. This method is called augmented finite volume method in the literatures because with the help of this method we can solve the singular equation (1) on the whole domain. While introducing augmented variables into series expansion presents some challenges for proving convergence analysis, we also overcome this difficulty and prove convergence analysis for the augmented variables.

The organization of the remaining paper is as follows: We explain the method to find the solution on the singular subdomain by expanding the Puiseux series at $x = 0$ in Section 2. Section 3 gives second and fourth-order schemes based on the finite volume method. Section 4 uses the discrete energy method to prove the convergence analysis of the second and fourth-order schemes in discrete L_2-norm over the whole domain. To show the efficiency of schemes, we give numerical examples for both cases of degeneracy and coefficient with blow-up at the degenerate point in Section 5. Finally, we give some concluding remarks on the whole progress.

2. Puiseux Series Expansion

In this section, we give an expansion of the Puiseux series about the singular point $x = 0$, and the series coefficients will be the functions of t

$$u(x,t) = \sum_{j=0}^{\infty} c_j(t)\,\varphi_j(x). \tag{5}$$

We can write the equation (1) in the form of Fredholm integral equation [3, 18], which has the following form:

$$u(x,t) = \int_0^b G(x,\sigma)\left[u_t(\sigma,t) + {}_0^{\beta}D_t^C u(\sigma,t) - f(\sigma,t,u(\sigma,t))\right] d\sigma\,, \tag{6}$$

where the Green function $G(x,\sigma)$ is given as:

$$G(x,\sigma) = \begin{cases} \frac{\sigma^{1-\alpha}}{1-\alpha}\left(x^{1-\alpha}b^{1-\alpha} - 1\right), & 0 \le \sigma \le x, \\ \frac{x^{1-\alpha}}{1-\alpha}\left(\sigma^{1-\alpha}b^{1-\alpha} - 1\right), & x \le \sigma \le b. \end{cases} \tag{7}$$

From the Green function, the solution of (1) is in the following form of Puiseux series at the singular point $x = 0$

$$u(x,t) = A(t) + a_0(t)\,x^{1-\alpha} + \sum_{k=1}^{\infty} a_k(t)\,x^{\alpha_k},\ 1-\alpha < \alpha_1 < \alpha_2 < \cdots \to \infty, \tag{8}$$

and

$$u_t(x,t) + {}_0^{\beta}D_t^C u(x,t) - f(x,t,u(x,t)) = \sum_{k=1}^{\infty} c_k(t)\,x^{\gamma_k},\ \gamma_1 < \gamma_2 < \gamma_3 < \cdots \to \infty. \tag{9}$$

By using (8) and (9) into (6), we have

$$a_0(t)\,x^{1-\alpha}+\sum_{k=1}^{\infty}a_k(t)\,x^{\alpha_k}=\int_0^b G(x,\sigma)\sum_{k=1}^{\infty}c_k(\sigma)\,x^{\gamma_k}d\sigma. \tag{10}$$

Hence, from (7), we have

$$a_0(t)=-\sum_{k=1}^{\infty}\frac{c_k(t)\,b^{\gamma_k+1}}{(\gamma_k+1)(2-\alpha+\gamma_k)},\ \alpha_k=2-\alpha-\gamma_k,\ a_k(t)=\frac{c_k(t)}{(\gamma_k+1)\,\alpha_k},\quad k=1,2,3,\cdots. \tag{11}$$

Similarly, for strongly degenerate case, we have

$$A(t)=-\sum_{k=1}^{\infty}\frac{c_k(t)\,b^{2-\alpha+\gamma_k}}{(\gamma_k+1)(2-\alpha+\gamma_k)},\ \alpha_k=2-\alpha-\gamma_k,\ a_k(t)=\frac{c_k(t)}{(\gamma_k+1)\,\alpha_k},\quad k=1,2,3,\cdots. \tag{12}$$

In the expansion of the Puiseux series for $u_t(x,t)+{}_0^{\beta}D_t^{C}u(x,t)-f(x,t,u(x,t))$ about $x=0$, the objective is to determine coefficients $c_k(t)$ and powers γ_k to recover the Puiseux series expansion of $u(x,t)$ at $x=0$. Particularly, the following algorithm can be used to recover the Puiseux series for weakly degenerate case. Similarly, we can design an algorithm for strongly degenerate case with $u_0(x)=A(t)$.

Algorithm 1. *The Algorithm to obtain the Puiseux series*

1. *Let* $u_0(x)=a_0(t)\,x^{1-\alpha}$ *be the initial function;*
2. ***for*** $k=1$ *to* Q *(suitable number)* ***do***
3. *By the Puiseux series* $\frac{\partial u_{k-1}(x,t)}{\partial t}+{}_0^{\beta}D_t^{C}u_{k-1}(x,t)-f(x,t,u_{k-1}(x,t))=\sum_{j=1}^{k}c_j(t)\,x^{\gamma_j}$ *with remainder* $R_k(x,t)$, *compute* $c_k(t)$ *and* γ_k;
4. *Calculate* α_k *and* $a_k(t)$ *from (12);*
5. *Use* $u_k(x,t)=u_{k-1}(x,t)+a_k(t)\,x^{\alpha_k}$;
6. ***end***

So, the solution of equation (1) on singular subdomain has the following form:

$$u_Q(x,t)=a_0(t)\,x^{1-\alpha}+\sum_{k=1}^{Q}a_k(t)\,x^{\alpha_k}. \tag{13}$$

As we know, during the expansion of the Puiseux series in Algorithm 1, we compute the time-fractional and time-integer derivatives at every iteration. Because of these derivatives, the Puiseux series has $a_0(t)$ or $A(t)$ and both kind of derivatives of the $a_0(t)$ or $A(t)$. Thus, the Puiseux series has the following form:

$$\begin{aligned} &u_Q\left(x,t,A_0(t),A_0^{(\beta)}(t),A_0'(t),A_0^{(\beta+1)}(t),...,A_0^{(q\beta)}(t)\right)\\ &=\begin{cases} u_Q\left(x,t,a_0(t),a_0^{(\beta)}(t),a_0'(t),a_0^{(\beta+1)}(t),...,a_0^{(q\beta)}(t)\right),\\ u_Q\left(x,t,A(t),A^{(\beta)}(t),A'(t),A^{(\beta+1)}(t),...,A^{(q\beta)}(t)\right).\end{cases}\end{aligned} \tag{14}$$

At the initial time layer $u_Q\left(x,t,A_0(t),A_0^{(\beta)}(t),A_0'(t),A_0^{(\beta+1)}(t),...,A_0^{(q\beta)}(t)\right)$ can be calculated from the initial condition $u(x,0)=0$. In the next section, we calculate these augmented variables and $u(x,t)$ numerically by using the finite volume schemes.

3. Second and Fourth Order Augmented Numerical Schemes

This section derives the second and fourth-order augmented finite volume schemes in space and second order in time. So, for the sake of brevity we call second and fourth order schemes that means second order in time while second and fourth order in space. In order to find the solution of the proposed singular equation, we choose an intermediate point ξ to split the whole domain into the singular subdomain $P_s = (0, \xi] \times (0, T)$ and regular subdomain $P_r = (\xi, b) \times (0, T)$. The intermediate point is vital in achieving good error and convergence order in both subdomains. This is because the Puiseux series cannot be relied upon to solve for the entire domain, as its error increases as one moves away from the singular point. Additionally, the proposed schemes have high errors near the singular point, which is very close to the intermediate point.

As we already computed the solution of the proposed equation on P_s in the last section by expansion of the Puiseuex series, we solve the following TFDPE to find the numerical solution on P_r:

(15)

$$\begin{cases} u_t + {}_0^{\beta}D_t^C u - (x^{\alpha} u_x)_x = f(x,t,u), \quad (x,t) \in P_r, \\ u(\xi,t) \approx u_Q\left(\xi, t, A_0(t), A_0^{(\beta)}(t), A_0'(t), A_0^{(\beta+1)}(t), ..., A_0^{(q\beta)}(t)\right), \ t \in (0,T), \\ u(b,t) = 0, \ t \in (0,T), \ u(x,0) = 0, \quad x \in P_r. \end{cases}$$

The important point to note here is that the equation (15) cannot be solved due to the left boundary condition, which contains unknown augmented variables in the form of $A_0(t), A_0^{(\beta)}(t), \ A_0'(t), A_0^{(\beta+1)}(t), ..., A_0^{(q\beta)}(t)$. To make the equation solvable, we need another condition which makes the equation (15) solvable. To handle this difficulty, we can use the solution of the singular subdomain, which we found through the expansion of the Puiseux series. Thus, we introduce the condition that $\frac{\partial u}{\partial x}(\xi,t) = \frac{\partial u_Q}{\partial x}(\xi, t, A_0(t), A_0^{(\beta)}(t), A_0'(t), A_0^{(\beta+1)}(t), ..., A_0^{(q\beta)}(t))$. This additional condition makes equation (15) solvable.

To determine the spatial step size, we use $h = (b - \xi)/N$, and for the temporal step size, we use $\Delta t = T/M$. Here, M and N are positive integers. We can achieve a uniform partitions by setting $x_i = \xi + ih$ for $i = 0, 1, 2, ...N$, and $t_k = k\Delta t$ for $k = 0, 1, 2, ...M$. We also denote the control volumes as $\Omega_0 = [x_0, x_{1/2}]$, $\Omega_i = [x_{i-1/2}, x_{i+1/2}]$, and $\Omega_N = [x_{N-1/2}, x_N]$, where $x_{i-1/2}$ is the midpoint of the interval $[x_{i-1}, x_i]$, similarly, we let $t_{k-1/2}$ be the midpoint of the interval $[t_{k-1}, t_k]$ for $i = 1, 2, ...N-1$ and $k = 1, 2, ...M-1$. For $0 \le i \le N$ and $0 \le k \le M$, we use the notations u_i^k for the uniform grids in the P_r and we also denote u^k as $u(x, t_k)$. For the sake of brevity, we will use the following notations frequently in the next part of the paper:

$$\delta_t^{\beta} u^{k-\frac{1}{2}} = (\Delta t)^{-\beta} \sum_{j=0}^{k} w_j u^{k-j}, \ \delta_t u^{k-\frac{1}{2}} = \frac{u^k - u^{k-1}}{\Delta t}, \ u^{k-\frac{1}{2}} = \frac{u^k + u^{k-1}}{2}, \tag{16}$$

where $w_0 = \frac{1+\beta}{2} g_0$, $w_j = \frac{1+\beta}{2} g_j + \frac{1-\beta}{2} g_{j-1}$, $j \ge 1$ and $g_0 = 1$, $g_1 = -\beta$ and $g_j = \left(1 - \frac{1+\beta}{j}\right) g_{j-1}$, $j \ge 1$. The second order approximation to discretize ${}_0^{\beta}D_t^C u$ is:

$$^{L}\delta_t^{\beta} u^{k-\frac{1}{2}} = (\Delta t)^{-\beta} \sum_{j=0}^{k} w_j u^{k-j} + O\left(\Delta t^2\right). \tag{17}$$

To discretize the Caputo derivative, we can use the second-order approximation, which is denoted by ${}^{L}\delta_t^{\beta}$. However, for the sake of brevity, we will use δ_t^{β} instead of ${}^{L}\delta_t^{\beta}$. The fractional approximation (17) is a consequence of the following approximation for the left and right fractional derivatives with $(p,q)=\left(\frac{1}{2},-\frac{1}{2}\right)$, which is provided in [6].

$$\begin{aligned}
{}^{L}\delta_t^{\beta}u(x) &= \frac{\lambda_1}{\Delta t^{\beta}}\sum_{j=0}^{\left[\frac{t}{\Delta t}\right]+p} g_j u\left(t-(j-p)\Delta t\right)+\frac{\lambda_2}{\Delta t^{\beta}}\sum_{j=0}^{\left[\frac{t}{\Delta t}\right]+q} g_j u\left(t-(j-q)\Delta t\right),\\
{}^{R}\delta_t^{\beta}u(x) &= \frac{\lambda_1}{\Delta t^{\beta}}\sum_{j=0}^{\left[\frac{T-t}{\Delta t}\right]+p} g_j u\left(t+(j-p)\Delta t\right)+\frac{\lambda_2}{\Delta t^{\beta}}\sum_{j=0}^{\left[\frac{T-t}{\Delta t}\right]+q} g_j u\left(t+(j-q)\Delta t\right),
\end{aligned}$$

where $\lambda_1=\frac{\beta-2q}{2(p-q)}$, $\lambda_2=\frac{2p-\beta}{2(p-q)}$. From (15), we have

$$\delta_t u^{k-\frac{1}{2}}+\delta_t^{\beta}u^{k-\frac{1}{2}}+\left(x^{\alpha}u_x^{k-\frac{1}{2}}\right)_x=f\left(x,t_{k-\frac{1}{2}},u^{k-\frac{1}{2}}\right). \tag{18}$$

Remark 1. *The above approximations can attain the accurate second order of convergence if u satisfies the conditions of [6, Theorem 2.4]. Further explanations about the order of convergence and error are given in numerical examples sections.*

In order to prevent the creation of meshes within the singular subdomain near the intermediate point, it is necessary to convert multiple augmented variables involved in the singular subdomain solution to a single one. If the mesh point is created within the singular subdomain, it can affect the accuracy and convergence order of the schemes and the selection of the intermediate point. To address this issue, we utilize the methods outlined in [3], which are based on numerical integration methods to convert multiple augmented variables to a single one. Thus, we have

$$u_0^k\approx u_Q\left(\xi,t_k,A_0\left(t_{k-1}\right),A_0^{(\beta)}\left(t_{k-1}\right),A_0'\left(t_{k-1}\right),A_0^{(\beta+1)}\left(t_{k-1}\right),\ldots,A_0^{(q\beta)}\left(t_k\right)\right).$$

To make it shorter, we use the notation $\hat{g}_p^{k-\frac{1}{2}}\left(A_0^{k-\frac{1}{2}}\right)=\frac{\hat{g}_p^{k-1}\left(A_0^{k-1}\right)+\hat{g}_p^{k}\left(A_0^{k}\right)}{2}$. Here, $\hat{g}_p^{k}\left(A_0^{k}\right)=\frac{\partial u_Q}{\partial x}(x,t_k,\ A_{0h}\left(t_{k-1}\right),A_{0h}^{(\beta)}\left(t_{k-1}\right),A_{0h}^{(\beta)}\left(t_{k-1}\right),A_{0h}^{(\beta+1)}\left(t_{k-1}\right),\ldots,$ $A_{0h}^{(q\beta)}\left(t_k\right))$ and A_{0h} is used as the approximate value of augmented variables.

Now, we give the derivation of second and fourth-order schemes. We only give the detailed derivation of the second-order scheme and the fourth-order scheme can be derived with similar lines. By integrating (18) over the control volumes, we have

$$\begin{cases}
\int_{\Omega_0}\left(\delta_t u^{k-\frac{1}{2}}(x)+\delta_t^{\beta}u^{k-\frac{1}{2}}(x)\right)dx+\xi^{\alpha}\hat{g}_p^{k-\frac{1}{2}}\left(A_0^{k-\frac{1}{2}}\right)-x_{\frac{1}{2}}^{\alpha}u_x^{k-\frac{1}{2}}\left(x_{\frac{1}{2}}\right)\\
=\int_{\Omega_0}f\left(x,t_{k-\frac{1}{2}},u^{k-\frac{1}{2}}\right)dx, \qquad 1\le k\le M,\\
\int_{\Omega_i}\left(\delta_t u^{k-\frac{1}{2}}(x)+\delta_t^{\beta}u^{k-\frac{1}{2}}(x)\right)dx+x_{i-\frac{1}{2}}^{\alpha}u_x^{k-\frac{1}{2}}\left(x_{i-\frac{1}{2}}\right)-x_{i+\frac{1}{2}}^{\alpha}u_x^{k-\frac{1}{2}}\left(x_{i+\frac{1}{2}}\right)\\
=\int_{\Omega_i}f\left(x,t_{k-\frac{1}{2}},u^{k-\frac{1}{2}}\right)dx,\ 1\le i\le N-1,\ 1\le k\le M.
\end{cases} \tag{19}$$

Through Taylor's expansion, we have

$$u_x^{k-\frac{1}{2}}\left(x_{i-\frac{1}{2}}\right)=\frac{u_i^{k-\frac{1}{2}}-u_{i-1}^{k-\frac{1}{2}}}{h}-R_{2,i-\frac{1}{2}}^{k-1},\ 1\le i\le N \tag{20}$$

where,

$$R_{2,i-\frac{1}{2}}^{k-1} = -\frac{1}{2h}\left[\int_{x_{i-\frac{1}{2}}}^{x_i} u_{xxx}^{k-\frac{1}{2}}(x)\,(x_i - x)^2\,dx + \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} u_{xxx}^{k-\frac{1}{2}}(x)\,(x_{i-1} - x)^2\,dx\right].$$

Now we discretize the integrals in (19) with the following two integral formulas:

$$\begin{aligned} \int_{\Omega_0} F(x)\,dx &= \frac{h}{2}F(x_0) + R_{3,0}^F, \\ \int_{\Omega_i} F(x)\,dx &= hF(x_i) + R_{3,i}^F,\ 1 \le i \le N-1, \end{aligned} \tag{21}$$

where

$$\begin{aligned} R_{3,0}^F &= \int_{x_0}^{x_{\frac{1}{2}}} F_x(x)\left(x_{\frac{1}{2}} - x\right)dx \\ R_{3,i}^F &= \frac{h}{2}\left[\int_{x_i}^{x_{i-\frac{1}{2}}} F_{xx}(x)\left(x_{i-\frac{1}{2}} - x\right)dx + \int_{x_i}^{x_{i+\frac{1}{2}}} F_{xx}(x)\left(x_{i+\frac{1}{2}} - x\right)dx\right] \\ &\quad + \frac{1}{2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} F_{xx}(x)\left(x_{i-\frac{1}{2}} - x\right)\left(x_{i+\frac{1}{2}} - x\right)dx. \end{aligned}$$

From (19), (20) and (21), we have the following second order scheme:

$$\begin{aligned} &\delta_t u_0^{k-\frac{1}{2}} + \delta_t^\beta u_0^{k-\frac{1}{2}} + \tfrac{2\xi^\alpha}{h}\hat{g}_p^{k-\frac{1}{2}}\left(A_0^{k-\frac{1}{2}}\right) - \tfrac{2x_{1/2}^\alpha}{h^2}\left(u_1^{k-\frac{1}{2}} - u_0^{k-\frac{1}{2}}\right) = f_0^{k-\frac{1}{2}}, \\ &\quad 1 \le k \le M \\ &\delta_t u_i^{k-\frac{1}{2}} + \delta_t^\beta u_i^{k-\frac{1}{2}} + \tfrac{x_{i-1/2}^\alpha}{h^2}\left(u_i^{k-\frac{1}{2}} - u_{i-1}^{k-\frac{1}{2}}\right) - \tfrac{x_{i+1/2}^\alpha}{h^2}\left(u_{i+1}^{k-\frac{1}{2}} - u_i^{k-\frac{1}{2}}\right) \\ &= f_i^{k-\frac{1}{2}},\ 1 \le k \le M,\ 1 \le i \le N-1. \end{aligned} \tag{22}$$

And, we have the following fourth-order compact finite volume schemes:

$$\begin{aligned} &\left(\tfrac{h}{12} - \mu_1\right)\delta_t^\beta u_1^{k-\frac{1}{2}} + \left(\tfrac{5h}{12} - \mu_1\right)\delta_t^\beta u_0^{k-\frac{1}{2}} + \left(\tfrac{h}{12} - \mu_1\right)\delta_t u_1^{k-\frac{1}{2}} + \left(\tfrac{5h}{12} - \mu_1\right)\delta_t u_0^{k-\frac{1}{2}} \\ &-\omega_1\left[u_1^{k-\frac{1}{2}} - u_0^{k-\frac{1}{2}}\right] + \xi^\alpha \hat{g}_p^{k-\frac{1}{2}}\left(A_0^{k-\frac{1}{2}}\right) = \left(\tfrac{h}{12} - \mu_1\right)f_1^{k-\frac{1}{2}} + \left(\tfrac{5h}{12} - \mu_1\right)f_0^{k-\frac{1}{2}} \\ &+\tfrac{h^2}{12}\left[\mathcal{R}_0^{k-\frac{1}{2}} + \hat{g}_p^{k-\frac{1}{2}}\left(A_0^{k-\frac{1}{2}}\right)\mathcal{X}_0^{k-\frac{1}{2}}\right] - \tfrac{h^2}{12}\left[\hat{g}_{p_{tt}}^{k-\frac{1}{2}}\left(A_0^{k-\frac{1}{2}}\right) + \hat{g}_{p_{xt}}^{k-\frac{1}{2}}\left(A_0^{k-\frac{1}{2}}\right)\right], \\ &\left(\tfrac{h}{12} + \mu_i\right)\delta_t^\beta u_{i-1}^{k-\frac{1}{2}} + \left(\tfrac{5h}{6} + \mu_i - \mu_{i+1}\right)\delta_t^\beta u_i^{k-\frac{1}{2}} + \left(\tfrac{h}{12} + \mu_{i+1}\right)\delta_t^\beta u_{i+1}^{k-\frac{1}{2}} \\ &+\left(\tfrac{h}{12} + \mu_i\right)\delta_t u_{i-1}^{k-\frac{1}{2}} + \left(\tfrac{5h}{6} + \mu_i - \mu_{i+1}\right)\delta_t u_i^{k-\frac{1}{2}} + \left(\tfrac{h}{12} + \mu_{i+1}\right)\delta_t u_{i+1}^{k-\frac{1}{2}} \\ &-\omega_{i+1}\left[u_{i+1}^{k-\frac{1}{2}} - u_i^{k-\frac{1}{2}}\right] + \omega_i\left[u_i^{k-\frac{1}{2}} - u_{i-1}^{k-\frac{1}{2}}\right] = \left(\mu_i + \tfrac{h}{12}\right)f_{i-1}^{k-\frac{1}{2}} \\ &+\left(\mu_i - \mu_{i+1} + \tfrac{5h}{6}\right)f_i^{k-\frac{1}{2}} + \left(\mu_{i+1} + \tfrac{h}{12}\right)f_{i+1}^{k-\frac{1}{2}},\ 1 \le k \le M,\ 1 \le i \le N-1, \end{aligned} \tag{23}$$

where for $1 \le i \le N-1$

$$
\begin{aligned}
&\omega_i = \frac{x^{\alpha}_{i-\frac{1}{2}}}{h} - \frac{h}{24}\left(\alpha+\alpha^2\right) x^{\alpha-2}_{i-\frac{1}{2}},\ \mu_i = \frac{\alpha h^2}{24 x_{i-\frac{1}{2}}},\\
&\mathcal{R}\left(x,t,u\right) = \frac{\partial}{\partial x} f\left(x,t,u\right),\ \mathcal{X}\left(x,t,u\right) = \frac{\partial}{\partial u} f\left(x,t,u\right),\\
&f_i^{k-\frac{1}{2}} = f\left(x_i, t_{k-\frac{1}{2}}, u_i^{k-\frac{1}{2}}\right),\\
&\hat{g}_{p_{tt}}^{k-\frac{1}{2}}\left(A_0^{k-\frac{1}{2}}\right) = \frac{\hat{g}^k_{p_{tt}}\left(A_0^k\right)+\hat{g}^{k-1}_{p_{tt}}\left(A_0^{k-1}\right)}{2} \approx \frac{\partial}{\partial x}\delta_t^{\beta} u^{k-\frac{1}{2}}\left(x_0\right),\\
&\hat{g}^k_{p_{tt}}\left(A_0^k\right) = \frac{\partial^{\beta+1} u_Q}{\partial t^{\beta}\partial x}\left(\xi, t, A_{0h}\left(t_{k-1}\right), A'_{0h}\left(t_{k-1}\right),\right.\\
&\quad \left. A_{0h}^{(\beta)}\left(t_{k-1}\right), A_{0h}^{(\beta+1)}\left(t_{k-1}\right), , ..., A_{0h}^{(q\beta)}\left(t_k\right)\right),\\
&\hat{g}_{p_{tx}}^{k-\frac{1}{2}}\left(A_0^{k-\frac{1}{2}}\right) = \frac{\hat{g}^k_{p_{tx}}\left(A_0^k\right)+\hat{g}^{k-1}_{p_{tx}}\left(A_0^{k-1}\right)}{2} \approx \frac{\partial}{\partial x}\delta_t u^{k-\frac{1}{2}}\left(x_0\right),\\
&\hat{g}^k_{p_{tx}}\left(A_0^k\right) = \frac{\partial^{2} u_Q}{\partial t\partial x}\left(\xi, t, A_{0h}\left(t_{k-1}\right), A'_{0h}\left(t_{k-1}\right),\right.\\
&\quad \left. A_{0h}^{(\beta)}\left(t_{k-1}\right), A_{0h}^{(\beta+1)}\left(t_{k-1}\right), , ..., A_{0h}^{(q\beta)}\left(t_k\right)\right).
\end{aligned}
$$

Thus the schemes (22) and (23) are solvable, there are exactly N unknowns and N equations. In these nonlinear systems, an augmented variable $A_{0h}^{(q\beta)}\left(t_k\right)$ and u_i^k for $1 \le k \le M$, $1 \le i \le N-1$ are unknowns. These systems can be solved by using the Newton's Raphson method. Particularly, we can solve them with the following algorithm for the second-order scheme, and similarly, we can design the algorithm for the fourth-order scheme.

Algorithm 2. (1) *The system of equation* $F_k\left(U_k\right) = 0$ *for* $1 \le k \le M$ *in (22) is considered with* $A_{0h}^{(q\beta)}\left(t_k\right)$ *and numerical solution* u_i^k, $1 \le i \le N-1$;
(2) *The initial conditions determine the initial values for* $A_{0h}^{(q\beta)}\left(t_k\right)$ *and* u_i^k;
(3) ***for*** $k = 1$ *to* M ***do;***
(4) *Assume* $U_k = \left[A_{0h}^{(q\beta)}\left(t_k\right), u_i^k, u_2^k, ..., u_{N-1}^k\right]$; *error* $= 10^{-4}$;
(5) ***while*** *error* $> 10^{-16}$;
(6) *Find* $F_k\left(U_k\right)$ *and* $F'_k\left(U_k\right)$;
(7) *Solve system of linear equations* $F'\left(U_k\right) V_k = -F_k\left(U_k\right)$;
(8) $U_k \leftarrow U_k + V_k$;
(9) *error* $= \left\|V_k\right\|_{\infty}$;
(10) ***endwhile;***
(11) $A_{0h}^{(q\beta-j)}\left(t_k\right) = A_{0h}^{(q\beta-j)}\left(A_{0h}^{(q\beta)}\left(t_k\right)\right)$, $j = -\beta, 1, ...q\beta$;
(12) ***endfor.***

4. Convergence Analysis

We give the convergence analysis of the second and fourth-order schemes in this section. Let $\eta_i^k = u_i^k - u\left(x_i, t_k\right)$, $\eta^k = \left[\eta_1^k, \eta_2^k, ..., \eta_{N-1}^k\right]$, $G_p^k = \hat{g}_p^k\left(\hat{A}^k\right) - \frac{\partial u}{\partial x}\left(\delta, t_k\right)$, $e^k = A_{0h}^{(q\beta)}\left(t_k\right) - A_0^{(q\beta)}\left(t_k\right)$, $M_i^k = f\left(x_i, t_k, u_i^k\right) - f\left(x_i, t_k, u\left(x_i, t_k\right)\right)$ and $M^k = \left[M_0^k, M_1^k, ..., M_{N-1}^k\right]$. We define the inner product and corresponding norms:

$$
\left(\eta^k, e^k\right) = \frac{h}{2}\eta_0^k e_0^k + \sum_{i=1}^{N-1} h\eta_i^k e_i^k,\ \left\|\eta^k\right\| = \left(\eta^k, \eta^k\right)^{\frac{1}{2}}. \tag{24}
$$

Since $\eta_N^k = 0$, therefore we also define the H^1-norm

$$
\left|\eta^k\right|_1 = \left(\sum_{i=1}^{N} h x^{\alpha}_{i-\frac{1}{2}}\left(\frac{\eta_i - \eta_{i-1}}{h}\right)^2\right)^{\frac{1}{2}}. \tag{25}
$$

Assumption 1. *Let $f(x,t,u)$ be the Lipschitz continuous with respect to u, then there exist C_f such that for $(x,t) \in P_r$ and real numbers u_1, u_2, we have*

$$|f(x,t,u_1) - f(x,t,u_2)| \le C_f |u_1 - u_2| .$$

Assumption 2. *We also give the following assumptions based on the Algorithm 1:*

A. *The function $u_Q\left(x,t,A_0(t),A_0^{(\beta)}(t),A_0'(t),A_0^{(\beta+1)}(t),...,A_0^{(q\beta)}(t)\right)$ is Lipschitz continuous with respect to the augmented variables.*

B. *The function $\frac{\partial u_Q\left(x,t,A_0(t),A_0^{(\beta)}(t),A_0'(t),A_0^{(\beta+1)}(t),...,A_0^{(q\beta)}(t)\right)}{\partial x}$ is Lipschitz continuous with respect to the augmented variables.*

C. *The function $\frac{\partial^2}{\partial t \partial x} u_Q\left(x,t,A_0(t),A_0^{(\beta)}(t),A_0'(t),A_0^{(\beta+1)}(t),...,A_0^{(q\beta)}(t)\right)$ is Lipschitz continuous with respect to the augmented variables.*

D. *The function $\frac{\partial^{\beta+1}}{\partial t^\beta \partial x} u_Q\left(x,t,A_0(t),A_0^{(\beta)}(t),A_0'(t),A_0^{(\beta+1)}(t),...,A_0^{(q\beta)}(t)\right)$ is Lipschitz continuous with respect to the augmented variables.*

Lemma 1. *The following relationship holds for $x \in [0,\delta]$ with $\delta \in (0,1)$*

$$\begin{aligned} & \left| u(x,t_k) - u_Q\left(x,t,A_0(t_k),A_0^{(\beta)}(t_k),A_0'(t_k),A_0^{(\beta+1)}(t_k),...,A_0^{(q\beta)}(t_k)\right)\right| \\ = & \left| \sum_{i=Q+1}^{\infty} a_i(t_k) x^{\alpha_i} \right| \le \sum_{i=Q+1}^{\infty} |a_i(t_k)| \delta^{\alpha_i} \le \varepsilon_0, \end{aligned}$$

where ε_0 is is very small real number when Q is suitabley large and δ is suitably small.

Lemma 2. *For real vector $u \in \mathbb{R}^k$ and $\{w_k\}_{k=0}^{\infty}$ given in (16), we have*

$$\sum_{n=0}^{k} \left(\sum_{p=0}^{n} w_p u_{n-p} \right) u_n \ge 0. \tag{26}$$

Proof. The proof is similar to the proof of [21, Lemma 3.2]. □

Lemma 3. *[19] For $1 \le k \le M$ and multiple augmented variables to a single one, we have the following estimate:*

$$A_0^{(q-i)}(t_k) = \begin{cases} A_0^{(q-i)}(t_{k-1}) + \frac{\Delta t}{2}\left[A_0^{(q)}(t_k) + A_0^{(q)}(t_{k-1})\right] + O\left(\Delta t^3\right), \ i=1, \\ A_0^{(q-i)}(t_{k-1}) + \frac{\Delta t^i}{2^i}\left[A_0^{(q)}(t_k) + A_0^{(q)}(t_{k-1})\right] + \sum\limits_{j=1}^{k-1} \frac{\Delta t^j}{2^{j-1}} A_0^{(q-k+j)}(t_k) \\ +O\left(\Delta t^3\right), \ 2 \le i \le q. \end{cases}$$

Lemma 4. *[19] We have the following estimate for $1 \le k \le M$, $1 \le i \le q$, $0 \le j \le q$ and $\left|A_0^{(q-j)}(t_0) - A_{0h}^{(q-j)}(t_0)\right| = 0$,*

$$\left|A_0^{(q-i)}(t_k) - A_{0h}^{(q-i)}(t_k)\right| \le C\left(\Delta t^3 + \Delta t \sum_{n=1}^{k} \left|A_0^{(q)}(t_n) - A_{0h}^{(q)}(t_n)\right|\right). \tag{27}$$

We have the following error equation for the hybrid scheme (22)

(28)
$$\begin{aligned}
&\tfrac{1}{2}\delta_t\eta_0^{k-\frac{1}{2}}+\tfrac{1}{2}\delta_t^\beta\eta_0^{k-\frac{1}{2}}+\tfrac{\delta^\alpha}{h}G_p^{k-\frac{1}{2}}-\tfrac{x_{1/2}^\alpha}{h^2}\left(\eta_1^{k-\frac{1}{2}}-\eta_0^{k-\frac{1}{2}}\right)-M_0^{k-\frac{1}{2}}\\
&\qquad=\tfrac{1}{2}R_{2,0}^{k-\frac{1}{2}}+\tfrac{1}{2}R_{3,0}^{k-\frac{1}{2}},\\
&\delta_t\eta_0^{k-\frac{1}{2}}+\delta_t^\beta\eta_i^{k-\frac{1}{2}}+\tfrac{x_{i-1/2}^\alpha}{h^2}\left(\eta_i^{k-\frac{1}{2}}-\eta_{i-1}^{k-\frac{1}{2}}\right)-\tfrac{x_{i+1/2}^\alpha}{h^2}\left(\eta_{i+1}^{k-\frac{1}{2}}-\eta_i^{k-\frac{1}{2}}\right)-M_i^{k-\frac{1}{2}}\\
&\qquad=R_{2,i}^{k-\frac{1}{2}}+R_{3,i}^{k-\frac{1}{2}},
\end{aligned}$$

where

$$\max\left\{\left\|R_{1,i}^{k-\frac{1}{2}}\right\|^2,\left\|R_{2,i}^{k-\frac{1}{2}}\right\|^2,\left\|R_{3,i}^{k-\frac{1}{2}}\right\|^2\right\}\le C\left(\Delta t^2+h^2\right)^2. \tag{29}$$

Now, we split the error η_0^k into two parts as follows:

$$\begin{aligned}
\eta_0^k=&u(\delta,t_k)-u_Q\left(\delta,t_k,A_{0h}(t_k),A_{0h}^{(\beta)}(t_k),A_{0h}'(t_k),A_{0h}^{(\beta+1)}(t_k),...,A_{0h}^{(q\beta)}(t_k)\right)\\
=&\boldsymbol{\eta}_{0p}^k+\boldsymbol{\eta}_0^k,
\end{aligned}$$

where $\boldsymbol{\eta}_{0p}^k=\ \boldsymbol{\eta}_{0p_1}^k+\ \boldsymbol{\eta}_{0p_2}^k$ and

$$\begin{aligned}
\boldsymbol{\eta}_{0p_1}^k\ =&\ u(\delta,t_k)-u_Q\left(\delta,t_k,A_0(t_k),A_0^{(\beta)}(t_k),A_0'(t_k),A_0^{(\beta+1)}(t_k),...,A_0^{(q\beta)}(t_k)\right),\\
\boldsymbol{\eta}_{0p_2}^k\ =&\ u_Q\left(\delta,t_k,A_0(t_k),A_0^{(\beta)}(t_k),A_0'(t_k),A_0^{(\beta+1)}(t_k),...,A_{0h}^{(q\beta)}(t_k)\right)\\
&-u_Q\left(\delta,t_k,A_{0h}(t_k),A_{0h}^{(\beta)}(t_k),A_{0h}'(t_k),A_{0h}^{(\beta+1)}(t_k),...,A_{0h}^{(q\beta)}(t_k)\right),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\eta}_0^k\ =&\ u_Q\left(\delta,t_k,A_0(t_k),A_0^{(\beta)}(t_k),A_0'(t_k),A_0^{(\beta+1)}(t_k),...,A_0^{(q\beta)}(t_k)\right)\\
&-u_Q\left(\delta,t_k,A_{0h}(t_k),A_{0h}^{(\beta)}(t_k),A_{0h}'(t_k),A_{0h}^{(\beta+1)}(t_k),...,A_{0h}^{(q\beta)}(t_k)\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
G_p^k=&u_x(\delta,t_k)-\frac{\partial u_Q}{\partial x}\left(\delta,t_k,A_{0h}(t_k),A_{0h}^{(\beta)}(t_k),A_{0h}'(t_k),A_{0h}^{(\beta+1)}(t_k),...,A_{0h}^{(q\beta)}(t_k)\right)\\
=&\mathcal{G}_p^k+\boldsymbol{G}_p^k,
\end{aligned}$$

where $\mathcal{G}_p^k=\mathcal{G}_{p1}^k+\mathcal{G}_{p2}^k$,

$$\begin{aligned}
\mathcal{G}_{p1}^k\ =&\ u_x(\delta,t_k)-\frac{\partial u_Q}{\partial x}\left(\delta,t_k,A_0(t_k),A_0^{(\beta)}(t_k),A_0'(t_k),A_0^{(\beta+1)}(t_k),...,A_0^{(q\beta)}(t_k)\right),\\
\mathcal{G}_{p2}^k\ =&\ \frac{\partial u_Q}{\partial x}\left(\delta,t_k,A_0(t_k),A_0^{(\beta)}(t_k),A_0'(t_k),A_0^{(\beta+1)}(t_k),...,A_{0h}^{(q\beta)}(t_k)\right)\\
&-\frac{\partial u_Q}{\partial x}\left(\delta,t_k,A_{0h}(t_k),A_{0h}^{(\beta)}(t_k),A_{0h}'(t_k),A_{0h}^{(\beta+1)}(t_k),...,A_{0h}^{(q\beta)}(t_k)\right),
\end{aligned}$$

$$\begin{aligned}
\mathbf{G}_p^k\ =&\ \frac{\partial u_Q}{\partial x}\left(\delta,t_k,A_0(t_k),A_0^{(\beta)}(t_k),A_0'(t_k),A_0^{(\beta+1)}(t_k),...,A_0^{(q\beta)}(t_k)\right)\\
&-\frac{\partial u_Q}{\partial x}\left(\delta,t_k,A_{0h}(t_k),A_{0h}^{(\beta)}(t_k),A_{0h}'(t_k),A_{0h}^{(\beta+1)}(t_k),...,A_{0h}^{(q\beta)}(t_k)\right).
\end{aligned}$$

We have the following lemma with the help of Lemmas 1, 3 and 4.

Lemma 5. *Under the Assumption 2, for suitable Q and any $1\le k\le M$, we have*

$$\left|\boldsymbol{\eta}_0^k\right|\ =\ \left|a^k\right|\left|e^k\right|\le C\left(\Delta t^3+\sum_{j=1}^k\Delta t\left|e^j\right|\right),\ \left|\boldsymbol{\eta}_{0p}^k\right|\le\varepsilon_0+C\left(\Delta t^3+\sum_{j=1}^k\Delta t\left|e^j\right|\right),$$

$$\left|\boldsymbol{G}_p^k\right| = \left|m^k\right|\left|e^k\right| \leq \mathbf{m}^k\left|a^k\right|\left|e^k\right|, \quad \left|\mathcal{G}_p^k\right| \leq \varepsilon_0 + C\left(\Delta t^3 + \sum_{j=1}^{k}\Delta t\left|e^j\right|\right),$$

where

$$a^k = \frac{\partial u_Q}{\partial A_0^{(q\beta)}(t_k)}\left(\delta, t_k, A_0(t_k), A_0^{(\beta)}(t_k), A_0'(t_k), A_0^{(\beta+1)}(t_k), ..., \rho_1^k\right),$$
$$m^k = \frac{\partial u_{Qx}}{\partial A_0^{(q\beta)}(t_k)}\left(\delta, t_k, A_0(t_k), A_0^{(\beta)}(t_k), A_0'(t_k), A_0^{(\beta+1)}(t_k), ..., \rho_2^k\right),$$
$$\mathbf{m}^k = \max_{0\leq k\leq M}\left|\frac{m^k}{a^k}\right|,$$

where the ρ_1^k and ρ_2^k are lies between $A_0(t_k)$ and $A_{0h}(t_k)$, ε_0 is very small real number when Q is appropriately large and δ is small.

Under the Assumptions 1 and 2, we have the following Theorem.

Theorem 1. *Let $\mathbf{u}$ be the numerical solution of the difference scheme (22) and $u(x,t)$ be the exact solution of TFDPE. Then, for very small h and Δt, we have:*

$$\left\|\eta^k\right\|, \left\|e^k\right\| \leq C\left(\varepsilon_0 + h^2 + \Delta t^2\right). \tag{30}$$

Proof. Multiplying second and third equations in (28) with $h\eta_0^{k-\frac{1}{2}}$ and $h\eta_i^{k-\frac{1}{2}}$, respectively. Then, summing up and using $\eta_N^k = 0$, we have

$$\begin{aligned}(31)\quad & \frac{1}{2\Delta t}\left(\left\|\eta^k\right\| - \left\|\eta^{k-1}\right\|\right) + \left|\eta^k\right|_1^2 - \left|\eta^{k-1}\right|_1^2 + \left(\delta_t^\beta\eta^{k-\frac{1}{2}}, \eta^{k-\frac{1}{2}}\right) \\ = \; & -G_p^{k-\frac{1}{2}}\eta_0^{k-\frac{1}{2}} + \left(M^{k-1}, \eta^{k-\frac{1}{2}}\right) + \frac{h}{2}R_{2,0}^{k-\frac{1}{2}}\eta_0^{k-\frac{1}{2}} \\ & + \sum_{i=1}^{N-1} hR_{2,i}^{k-\frac{1}{2}}\eta_i^{k-\frac{1}{2}} + \frac{h}{2}R_{3,0}^{k-\frac{1}{2}}\eta_0^{k-\frac{1}{2}} + \sum_{i=1}^{N-1} hR_{3,i}^{k-\frac{1}{2}}\eta_i^{k-\frac{1}{2}}.\end{aligned}$$

Now, adding $\delta^\alpha\eta_0^{k-\frac{1}{2}}\delta_t\eta_0^{k-\frac{1}{2}} = \delta^\alpha\frac{\left|a^ke^k\right|^2 - \left|a^{k-1}e^{k-1}\right|^2}{2\Delta t}$ on both sides of (31), we have

$$\begin{aligned}(32)\quad & \frac{1}{2\Delta t}\left(\left\|\eta^k\right\|^2 - \left\|\eta^{k-1}\right\|^2 + \delta^\alpha(\left|a^ke^k\right|^2 - \left|a^{k-1}e^{k-1}\right|^2)\right) + \left|\eta^k\right|_1^2 - \left|\eta^{k-1}\right|_1^2 \\ = \; & \delta^\alpha\left(\eta_0^{k-\frac{1}{2}} - G_p^{k-\frac{1}{2}}\right)\left(\delta_t\eta_0^{k-\frac{1}{2}}\right) - \delta^\alpha\eta_0^{k-\frac{1}{2}}\delta_t\left(\boldsymbol{\eta}_{0p}^{k-\frac{1}{2}}\right) - \left(\delta_t^\beta\eta^{k-\frac{1}{2}}, \eta^{k-\frac{1}{2}}\right) \\ & + \left(M^{k-1}, \eta^{k-\frac{1}{2}}\right) + \left(R_2^{k-\frac{1}{2}}, \eta^{k-\frac{1}{2}}\right) + \left(R_3^{k-\frac{1}{2}}, \eta^{k-\frac{1}{2}}\right).\end{aligned}$$

Now applying Assumption 1 and Cauchy-Schwartz inequality, we have

$$\begin{aligned}(33)\quad & \left(M^{k-1}, \eta^{k-\frac{1}{2}}\right) \\ = \; & \left(M^{k-1}, \eta^{k-\frac{1}{2}}\right) \\ \leq \; & \left(C^f\right)^2\left(\left\|\eta^k\right\|^2 + \left\|\eta^{k-1}\right\|^2\right) + \frac{1}{2}\left(\left\|\eta^k\right\|^2 + \left\|\eta^{k-1}\right\|^2\right).\end{aligned}$$

Again, using Young's inequality, we have

$$\begin{aligned}(34)\quad & \left(R_2^{k-\frac{1}{2}}, \eta^{k-\frac{1}{2}}\right) + \left(R_3^{k-\frac{1}{2}}, \eta^{k-\frac{1}{2}}\right) \\ \leq \; & \frac{1}{2}\left(\left\|R_2^{k-\frac{1}{2}}\right\| + \left\|R_3^{k-\frac{1}{2}}\right\|\right) + \frac{1}{2}\left(\left\|\eta^k\right\|^2 + \left\|\eta^{k-1}\right\|^2\right).\end{aligned}$$

Thus, by using (33) and (34) in (32), we have

$$(35)\quad \frac{1}{2\Delta t}\left(\left|\eta^k\right|_1^2 - \left|\eta^{k-1}\right|_1^2 + \left\|\eta^k\right\|^2 - \left\|\eta^{k-1}\right\|^2 + \delta^\alpha\left(\left|a^ke^k\right|^2 - \left|a^{k-1}e^{k-1}\right|^2\right)\right)$$

$$\leq \delta^{\alpha}\left(\eta_0^{k-\frac{1}{2}}-G_p^{k-\frac{1}{2}}\right)\left(\delta_t\eta_0^{k-\frac{1}{2}}\right)-\delta^{\alpha}\eta_0^{k-\frac{1}{2}}\delta_t\left(\boldsymbol{\eta}_{0p}^{k-\frac{1}{2}}\right)$$
$$+\left(C^f\right)^2\left(\left\|\eta^k\right\|^2+\left\|\eta^{k-1}\right\|^2\right)+\left\|\eta^k\right\|^2+\left\|\eta^{k-1}\right\|^2+R^{k-\frac{1}{2}}-\left(\delta_t^{\beta}\eta^{k-\frac{1}{2}},\eta^{k-\frac{1}{2}}\right),$$

where $R^{k-\frac{1}{2}}=\frac{1}{2}\left(\left\|R_2^{k-\frac{1}{2}}\right\|+\left\|R_3^{k-\frac{1}{2}}\right\|\right)\leq C\left(\Delta t^2+h^2\right)$. Multiplying both sides of (35) with $2\Delta t$ and summing up from $k=1$ to K and from Lemma 2, we have

$$\left|\eta^K\right|_1^2+\left\|\eta^K\right\|^2+\delta^{\alpha}\left|a^Ke^K\right|^2-\left|\eta^0\right|_1^2-\left\|\eta^0\right\|^2-\delta^{\alpha}\left|a^0e^0\right|^2$$
$$\leq 2\left(C^f\right)^2\Delta t\sum_{k=1}^{K}\left(\left\|\eta^k\right\|^2+\left\|\eta^{k-1}\right\|^2\right)+2\Delta t\sum_{k=1}^{K}\left[\left\|\eta^k\right\|^2+\left\|\eta^{k-1}\right\|^2+R^{k-\frac{1}{2}}\right]$$
$$+2\Delta t\sum_{k=1}^{K}\left[\delta^{\alpha}\left(\eta_0^{k-\frac{1}{2}}-G_p^{k-\frac{1}{2}}\right)\left(\delta_t\eta_0^{k-\frac{1}{2}}\right)-\delta^{\alpha}\eta_0^{k-\frac{1}{2}}\delta_t\left(\boldsymbol{\eta}_{0p}^{k-\frac{1}{2}}\right)\right].$$

Since $\eta_0^{k-\frac{1}{2}}=\boldsymbol{\eta}_{0p}^{k-\frac{1}{2}}+\boldsymbol{\eta}_0^{k-\frac{1}{2}}$ with $\boldsymbol{\eta}_{0p_1}^{k-\frac{1}{2}}\leq\varepsilon_0$, $\left|\delta_t\boldsymbol{\eta}_{0p_2}^{k-\frac{1}{2}}\right|\leq C\left(\Delta t^2+\Delta t\sum_{n=1}^{k}\left|e^n\right|+\left|e^k\right|\right.$ $\left.+\left|e^{k-1}\right|\right)$ and $\left|\delta_t\boldsymbol{\eta}_0^{k-\frac{1}{2}}\right|\leq C\left(\Delta t^2+\Delta t\sum_{n=1}^{k}\left|e^n\right|+\left|e^k\right|+\left|e^{k-1}\right|\right)$, therefore from Lemma 5, Cauchy-Schwartz inequality, and careful calculations, we have

$$\left|\eta^K\right|_1^2+\left\|\eta^K\right\|^2+\delta^{\alpha}\left|a^Ke^K\right|^2$$
$$\leq C\left(\left(\varepsilon_0+h^2+\Delta t^2\right)+\Delta t\sum_{k=1}^{K}\left(\left\|\eta^k\right\|^2+\delta^{\alpha}\left|a^ke^k\right|^2+\left|\eta^k\right|_1\right)\right),$$

with $\eta^0=e^0=0$. Finally, with the Gronwal inequality, we have

$$\left|\eta^K\right|_1^2+\left\|\eta^K\right\|^2+\delta^{\alpha}\left|a^Ke^K\right|^2\leq C\left(\varepsilon_0+\Delta t^2+h^2\right)^2$$

which implies that

$$\left\|\eta^K\right\|\leq C\left(\varepsilon_0+\Delta t^2+h^2\right),\quad \left|e^K\right|^2\leq C\left(\varepsilon_0+\Delta t^2+h^2\right).$$

Hence, the proof is completed. □

We give the following result under the Assumption 2.

Theorem 2. *Let* $u_Q\left(x,t_k,A_0\left(t_k\right),A_0^{(\beta)}\left(t_k\right),A_0'\left(t_k\right),A_0^{(\beta+1)}\left(t_k\right),...,A_0^{(q\beta)}\left(t_k\right)\right)$ *be the approximate solution of TFDPE over the singular subdomain at time level* k*. Then, we have*

(36)
$$\max_{x\in(0,\delta)}\left|u\left(x,t_k\right)-u_Q\left(x,t_k,A_0\left(t_k\right),A_0^{(\beta)}\left(t_k\right),A_0'\left(t_k\right),A_0^{(\beta+1)}\left(t_k\right),...,A_0^{(q\beta)}\left(t_k\right)\right)\right|$$
$$\leq C\left(\varepsilon_0+\Delta t^2+h^2\right).$$

Proof. By mean value theorem and Lemma 1, we obtain

$$\left|u\left(x,t_k\right)-u_Q\left(x,t_k,A_{0h}\left(t_k\right),A_{0h}^{(\beta)}\left(t_k\right),A_{0h}'\left(t_k\right),A_{0h}^{(\beta+1)}\left(t_k\right),...,A_{0h}^{(q\beta)}\left(t_k\right)\right)\right|$$
$$\leq \left|u\left(x,t_k\right)-u_Q\left(x,t_k,A_0\left(t_k\right),A_0^{(\beta)}\left(t_k\right),A_0'\left(t_k\right),A_0^{(\beta+1)}\left(t_k\right),...,A_0^{(q\beta)}\left(t_k\right)\right)\right|$$
$$+\left|u_Q\left(x,t_k,A_0\left(t_k\right),A_0^{(\beta)}\left(t_k\right),A_0'\left(t_k\right),A_0^{(\beta+1)}\left(t_k\right),...,A_0^{(q\beta)}\left(t_k\right)\right)\right.$$
$$\left.-u_Q\left(x,t_k,A_{0h}\left(t_k\right),A_{0h}^{(\beta)}\left(t_k\right),A_{0h}'\left(t_k\right),A_{0h}^{(\beta+1)}\left(t_k\right),...,A_{0h}^{(q\beta)}\left(t_k\right)\right)\right|$$

$$\leq \quad \varepsilon_0 + C\left|e^k\right|.$$

Now from Theorem 1, we have

$$\max_{x\in(0,\delta)}\left|u\left(x,t_k\right)-u_Q\left(x,t_k,A_{0h}\left(t_k\right),A_{0h}^{(\beta)}\left(t_k\right),A_{0h}'\left(t_k\right),A_{0h}^{(\beta+1)}\left(t_k\right),...,A_{0h}^{(q\beta)}\left(t_k\right)\right)\right|$$
$$\leq C\left(\varepsilon_0+\Delta t^2+h^2\right).$$

Thus, the proof is completed. □

We have the following direct estimates for the fourth order scheme. One can easily obtain by the notions of Theorems 1 and 2.

Theorem 3. *Let* **u** *be the numerical solution of the difference scheme (23) and* $u\left(x,t\right)$ *be the exact solution of TFDPE. Then, for very small* h *and* Δt*, we have:*

$$\left\|\eta^k\right\|,\left\|e^k\right\| \leq C\left(\varepsilon_0+h^4+\Delta t^2\right). \tag{37}$$

Theorem 4. *Let* $u_Q\left(x,t_k,A_0\left(t_k\right),A_0^{(\beta)}\left(t_k\right),A_0'\left(t_k\right),A_0^{(\beta+1)}\left(t_k\right),...,A_0^{(q\beta)}\left(t_k\right)\right)$ *be the approximate solution of TFDPE over the singular subdomain at time level* k*. Then, we have*

$$\max_{x\in(0,\delta)}\left|u\left(x,t_k\right)-u_Q\left(x,t_k,A_0\left(t_k\right),A_0^{(\beta)}\left(t_k\right),A_0'\left(t_k\right),A_0^{(\beta+1)}\left(t_k\right),...,A_0^{(q\beta)}\left(t_k\right)\right)\right|$$
$$\leq C\left(\varepsilon_0+\Delta t^2+h^4\right).$$

Remark 2. *It is clear from Theorems 1 and 2, the hybrid scheme (22) has second-order accuracy over the whole domain P. Similarly, from Theorems 3 and 4, the scheme (23) has fourth order accuracy over the whole domain P.*

Theorem 5. *The hybrid schemes (22) and (23) are unconditionally stable.*

Proof. From Theorems 1, 2, 3 and 4, the schemes are unconditionally stable. □

5. Numerical Examples

This section affirms the validation of the second and fourth-order schemes with different examples. We give three examples for the original equation (1) with both long and short-term memory effects and show that the schemes work well. Then, we consider $u_t = 0$ in the original equation and test three examples to show the efficiency of the schemes. We also observed that the schemes give almost the same results for the original equation and $u_t = 0$ in the original equation. We also show the worth of introducing the intermediate point by giving some graphs and showing that the schemes have high accuracy and good order of convergence. We have included Figure 2 for Example 1 to demonstrate the purpose of the intermediate point, the Puiseux series expansion, and the suggested numerical schemes when $T = 1$ and $\Delta t = h^2$. In Figure 2 (a), it is evident that the error increases as we move away from the singular point. This indicates that the Puiseux series is unsuitable for approximating the entire domain solution since the error would increase. On the other hand, Figure 2 (b) shows that the error is high near the singular point but gradually decreases as we move away from the intermediate point near the singular point. This illustrates the effectiveness of the proposed methods. To find the errors and order of the convergence for the augmented variables and the whole domain, we use the following formulas:

$$
\begin{aligned}
&E_s^{L_\infty} u = \max_{1\le k\le M} \sup_{0\le x\le \xi} \left|u_M\left(x,t_k\right) - u\left(x,t_k\right)\right|,\\
&E_r^{L_\infty} u = \max_{1\le k\le M} \max_{0\le i\le N} \left|u\left(x_i,t_k\right) - u_i^k\right|,\\
&E^{L_\infty} u = \max\left(E_s^{L_\infty} u, E_r^{L_\infty} u\right),\\
&E_r^{L_2} u = \max_{1\le k\le M} \left[\frac{h}{2}\left(u\left(x_0,t_k\right) - u_0^k\right)^2 + h\sum_{i=1}^{N}\left(u\left(x_i,t_k\right) - u_i^k\right)^2\right],\\
&E_{Aug}^{L_\infty} = \max_{1\le k\le M} \left|A_0^{(q\beta)}\left(t_k\right) - A_{0h}^{(q\beta)}\left(t_k\right)\right|, \text{ order} = \log_2 \frac{\|u-u_{2h}\|}{\|u-u_h\|},
\end{aligned}
$$

where h is the space step size and u_h is the numerical solution.

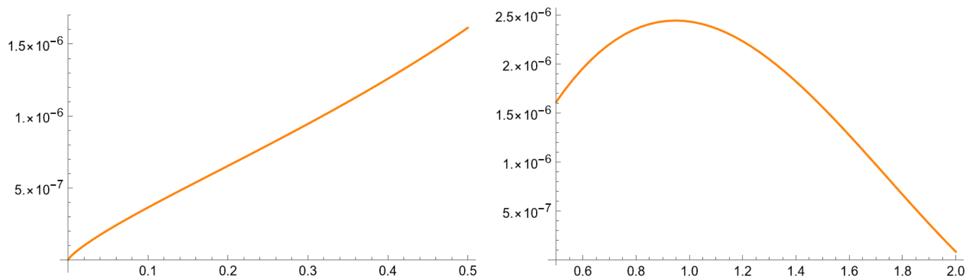
Example 1. *We consider weakly TFDPE (1) with $\alpha = 1/5$, $T = 1$ and $b = 2$ with exact solution $u\left(x,t\right) = t^6\left(2-x\right)\sin x$. First, we find the Puiseux series about the singular point $x = 0$ upto the 8th order using the Algorithm 1, then choose an intermediate point to split the whole domain into singular and regular subdomains. The Tables 1 and 2 are showing the efficiency of second and fourth-order schemes, respectively, while Tables 3 and 4 shows the efficiency of second and fourth-order schemes, respectively, when $u_t = 0$ in equation (1). In the same case if randomly we choose an intermediate point, then one can observe from Tables 5 and 6 that we cannot achieve the desired order of convergence. If an intermediate point is very near to the singular point, then the results for regular subdomain are not good while if an intermediate point is very away from the singular point, then the results on singular subdomain are not good. Therefore, the choice of an intermediate point is very crucial for getting good results.*

TABLE 1. Order of convergence and errors of $u(x,t)$ ($\Delta t = h$, $\delta = 0.8$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/10	8.16541e-03	*	1.1695e-02	*	1.1695e-02	*	9.40876e-03	*	3.1829e-03	*
	1/20	2.05698e-03	1.989	2.95477e-03	1.98478	2.95477e-03	1.98478	2.38061e-03	1.98268	7.96742e-04	1.99816
0.2	1/40	5.15539e-04	1.99637	7.41411e-04	1.9947	7.41411e-04	1.9947	5.97579e-04	1.99413	1.99375e-04	1.99863
	1/80	1.28724e-04	2.0018	1.85568e-04	1.99833	1.85568e-04	1.99833	1.49528e-04	1.99871	4.97318e-05	2.00324
	1/10	8.63167e-03	*	1.22937e-02	*	1.22937e-02	*	9.8937e-03	*	3.39608e-03	*
	1/20	2.1888e-03	1.9795	3.12668e-03	1.97522	3.12668e-03	1.97522	2.52041e-03	1.97285	8.57348e-04	1.98592
0.5	1/40	5.50567e-04	1.99115	7.87339e-04	1.98958	7.87339e-04	1.98958	6.34985e-04	1.98886	2.15535e-04	1.99196
	1/80	1.3776e-04	1.99876	1.97417e-04	1.99574	1.97417e-04	1.99574	1.59203e-04	1.99586	5.39058e-05	1.99941

TABLE 2. Order of convergence and errors of $u(x,t)$ ($\Delta t = h^2, \delta = 0.5$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/4	6.52203e-03	*	9.74239e-03	*	9.74239e-03	*	8.92038e-03	*	9.1622e-03	*
	1/8	4.17215e-04	3.96646	6.2187e-04	3.96959	6.2187e-04	3.96959	5.72432e-04	3.96193	5.87028e-04	3.96419
0.5	1/16	2.61845e-05	3.99401	3.92603e-05	3.98547	3.92603e-05	3.98547	3.59751e-05	3.99203	3.68531e-05	3.99357
	1/32	1.61243e-06	4.0214	2.44289e-06	4.00641	2.44289e-06	4.00641	2.23829e-06	4.00653	2.26461e-06	4.02445
	1/4	5.31124e-03	*	7.79704e-03	*	7.79704e-03	*	7.12034e-03	*	7.46019e-03	*
	1/8	3.37909e-04	3.97434	4.99582e-04	3.96413	4.99582e-04	3.96413	4.56371e-04	3.96367	4.75232e-04	3.97251
0.8	1/16	2.1181e-05	3.99579	3.14623e-05	3.98902	3.14623e-05	3.98902	2.86795e-05	3.99211	2.97952e-05	3.99548
	1/32	1.30244e-06	4.02349	1.95949e-06	4.00507	1.95949e-06	4.00507	1.78571e-06	4.00545	1.82674e-06	4.02773



(a) $x \in (0, 0.5)$ (b) $x \in (0.5, 2)$

FIGURE 2. $u(x,t)$ error in Example 1 for $\beta = 0.5$, $T = 1$ and $h = 1/32$.

TABLE 3. Order of convergence and errors of $u(x,t)$ ($u_t = 0$, $\Delta t = h$, $\delta = 0.8$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/10	1.19354e-02	*	1.52867e-02	*	1.52867e-02	*	1.24554e-02	*	8.70952e-03	*
	1/20	3.02685e-03	1.97936	3.87881e-03	1.97859	3.87881e-03	1.97859	3.1646e-03	1.97667	2.20792e-03	1.9799
0.2	1/40	7.60166e-04	1.99343	9.74621e-04	1.9927	9.74621e-04	1.9927	7.95463e-04	1.99216	5.54551e-04	1.9933
	1/80	1.89737e-04	2.00231	2.437e-04	1.99974	2.437e-04	1.99974	1.98923e-04	1.99958	1.38383e-04	2.00265
	1/10	1.2458e-02	*	1.58591e-02	*	1.58591e-02	*	1.29169e-02	*	9.1717e-03	*
	1/20	3.18763e-03	1.96651	4.05993e-03	1.96579	4.05993e-03	1.96579	3.31179e-03	1.96357	2.34942e-03	1.96489
0.5	1/40	8.04539e-04	1.98625	1.02511e-03	1.98568	1.02511e-03	1.98568	8.36616e-04	1.98498	5.93521e-04	1.98493
	1/80	2.0144e-04	1.997811	2.571e-04	1.99538	2.571e-04	1.99538	2.09846e-04	1.99523	1.48632e-04	1.99756

TABLE 4. Order of convergence and errors of $u(x,t)$ ($u_t = 0, \Delta t = h^2, \delta = 0.5$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/4	3.9814e-03	*	6.58842e-03	*	6.58842e-03	*	5.93956e-03	*	4.15117e-03	*
	1/8	2.51292e-04	3.98584	4.17374e-04	3.98052	4.17374e-04	3.98052	3.78528e-04	3.97188	2.62174e-04	3.98492
0.5	1/16	1.57215e-05	3.99855	2.63264e-05	3.98676	2.63264e-05	3.98676	2.37571e-05	3.99397	1.64055e-05	3.99827
	1/32	9.68476e-07	4.02088	1.64287e-06	4.00222	1.64287e-06	4.00222	1.48222e-06	4.00253	1.00734e-06	4.02555
	1/4	3.67143e-03	*	5.96531e-03	*	5.96531e-03	*	5.37618e-03	*	3.85732e-03	*
	1/8	2.30518e-04	3.9934	3.78798e-04	3.9771	3.78798e-04	3.9771	3.42433e-04	3.97269	2.42313e-04	3.99266
0.8	1/16	1.44023e-05	4.00051	2.38658e-05	3.98841	2.38658e-05	3.98841	2.1491e-05	3.99401	1.51415e-05	4.00029
	1/32	8.86185e-07	4.02254	1.48963e-06	4.00192	1.48963e-06	4.00192	1.34096e-06	4.00239	9.28379e-07	4.02765

TABLE 5. Order of convergence and errors of $u(x,t)$ ($\Delta t = h$, $\delta = 0.02$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/10	8.32956e-03	*	2.06393e-02	*	2.06393e-02	*	1.58321e-02	*	1.3741e-02	*
	1/20	2.58664e-03	1.68716	1.6691e-03	0.306325	1.6691e-03	0.306325	1.04357e-02	0.601324	4.26691e-02	1.68722
0.2	1/40	6.91172e-04	1.90396	4.28511e-03	1.96166	4.28511e-03	1.96166	2.01087e-03	2.37563	1.14018e-02	1.90393
	1/80	1.72873e-04	1.99933	4.4163e-03	-0.0435038	4.4163e-03	-0.0435038	2.07086e-03	-0.0424103	2.85187e-03	1.99928

Example 2. *We consider strongly TFDPE (1) with* $\alpha = 4/3$, $T = 1$ *and* $b = 2$ *with exact solution* $u\,(x,t) = t^5 \cos\left(\frac{3-x}{2}\pi\right)$. *First, we find the Puiseux series about the singular point* $x = 0$ *upto the* $11th$ *order using the Algorithm 1, then choose an intermediate point to split the whole domain into singular and regular domains. The Tables 7 and 8 are showing the efficiency of second and fourth-order schemes, respectively, while Tables 9 and 10 are showing efficiency of second and fourth order schemes, respectively, when* $u_t = 0$ *in equation (1).*

TABLE 6. Order of convergence and errors of $u(x,t)$ ($\Delta t = h$, $\delta = 1.5$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/10	1.61435e-04	*	2.2205e-04	*	2.2205e-04	*	1.02691e-04	*	8.37239e-05	*
	1/20	5.54823e-05	1.54086	5.54823e-05	2.00079	5.54823e-05	2.00079	2.44607e-05	2.06978	1.6267e-05	2.36369
0.2	1/40	2.6865e-05	1.0463	2.6865e-05	1.0463	2.6865e-05	1.0463	6.72705e-06	1.86242	1.94163e-06	3.06661
	1/80	2.33717e-05	0.200963	1.93433e-05	0.473896	2.33717e-05	0.200963	6.2507e-06	0.105957	4.92834e-06	-1.34383

TABLE 7. Order of convergence and errors of $u(x,t)$ ($\Delta t = h$, $\delta = 0.4$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/10	6.18764e-03	*	1.24938e-02	*	1.24938e-02	*	1.21092e-02	*	6.49444e-05	*
	1/20	1.56805e-03	1.98041	3.14971e-03	1.98792	3.14971e-03	1.98792	3.05452e-03	1.98709	1.5172e-05	2.09779
0.2	1/40	4.00076e-04	1.97063	7.91066e-04	1.99335	7.91066e-04	1.99335	7.6746e-04	1.99278	3.7722e-06	2.00794
	1/80	1.06696e-04	1.90676	1.98602e-04	1.99392	1.98602e-04	1.99392	1.93348e-04	1.98889	9.79529e-07	1.94524
	1/10	6.45139e-03	*	1.28234e-02	*	1.28234e-02	*	1.24472e-02	*	7.15119e-05	*
	1/20	1.64745e-03	1.96937	3.25495e-03	1.97807	3.25495e-03	1.97807	3.15994e-03	1.97785	1.70351e-05	2.06968
0.5	1/40	4.21506e-04	1.96661	8.20092e-04	1.98878	8.20092e-04	1.98878	7.9646e-04	1.98822	4.27198e-06	1.99553
	1/80	1.1207e-04	1.91115	2.06127e-04	1.99225	2.06127e-04	1.99225	2.0082e-04	1.9877	1.10882e-06	1.94588

TABLE 8. Order of convergence and errors of $u(x,t)$ ($\Delta t = h^2, \delta = 0.5$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/4	3.35332e-03	*	5.12628e-03	*	5.12628e-03	*	4.96607e-03	*	2.05828e-04	*
	1/8	2.13256e-04	3.97493	3.2842e-04	3.9643	3.2842e-04	3.9643	3.14735e-04	3.97989	1.29385e-05	3.9917
0.5	1/16	1.33928e-05	3.99306	2.05819e-05	3.99609	2.05819e-05	3.99609	1.97358e-05	3.99525	8.12246e-07	3.99361
	1/32	8.38069e-07	3.99824	1.28833e-06	3.9978	1.28833e-06	3.9978	1.23448e-06	3.99884	5.08307e-08	3.99814
	1/4	2.92083e-03	*	4.46236e-03	*	4.46236e-03	*	4.35544e-03	*	1.90689e-04	*
	1/8	1.84682e-04	3.98327	2.87678e-04	3.95528	2.87678e-04	3.95528	2.75611e-04	3.98211	1.1888e-05	4.00365
0.8	1/16	1.15847e-05	3.99475	1.80269e-05	3.99623	1.80269e-05	3.99623	1.72769e-05	3.99572	7.44645e-07	3.9968
	1/32	7.24731e-07	3.99863	1.1274e-06	3.99907	1.1274e-06	3.99907	1.08059e-06	3.99895	4.65743e-08	3.99895

TABLE 9. Order of convergence and errors of $u(x,t)$ ($u_t = 0$, $\Delta t = h$, $\delta = 0.4$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/10	1.51802e-02	*	2.2905e-02	*	2.2905e-02	*	2.24938e-02	*	1.22929e-03	*
	1/20	3.87087e-03	1.97146	5.81965e-03	1.97666	5.81965e-03	1.97666	5.70644e-03	1.97887	3.09371e-04	1.99041
0.2	1/40	9.89676e-04	1.96763	1.47092e-03	1.98421	1.47092e-03	1.98421	1.4433e-03	1.98322	7.85503e-05	1.97765
	1/80	2.63702e-04	1.90805	3.7675e-04	1.96504	3.7675e-04	1.96504	3.7054e-04	1.96167	2.04685e-05	1.94021
	1/10	1.34267e-02	*	2.06477e-02	*	2.06477e-02	*	2.02707e-02	*	1.10161e-03	*
	1/20	3.46332e-03	1.95487	5.28218e-03	1.96678	5.28218e-03	1.96678	5.18929e-03	1.96579	2.82688e-04	1.96234
0.5	1/40	8.89776e-04	1.96064	1.33978e-03	1.97914	1.33978e-03	1.97914	1.31647e-03	1.97886	7.25156e-05	1.96285
	1/80	2.3669e-04	1.91044	3.41199e-04	1.97331	3.41199e-04	1.97331	3.36255e-04	1.96905	1.89662e-05	1.93486

Example 3. *We consider an interesting example with the coefficients blow-up at the degenerate point of the nonlinear TFDPE with* $\alpha = -1/2$, $T = 1$, $b = 2$, *with exact solution* $u(x,t) = t^{3+\alpha}x(2-x)\exp(2x)$. *First, we find the Puiseux series about the singular point* $x = 0$ *upto the* 14*th order using the Algorithm 1, then choose an intermediate point to split the whole domain into singular and regular subdomains. The Tables 11 and 12 are showing the efficiency of second and fourth-order schemes, respectively, while Tables 13 and 14 are showing efficiency of second and fourth-order schemes, respectively, when* $u_t = 0$ *in equation (1).*

Now, we give an example to discuss the accuracy of temporal second order WS-GD scheme and L_1-scheme having $\Delta t^{2-\beta}$ order of convergence. We also give the comparison of both schemes that can be helpful for the new readers.

Example 4. *We consider weakly TFDPE (1) with* $\alpha = 1/5$, $T = 1$ *and* $b = 2$ *with exact solution* $u(x,t) = t^p(2-x)\sin x$, $(p \geq 0)$. *First, we find the Puiseux series about the singular point* $x = 0$ *upto the* 8*th order using the Algorithm 1, then choose an intermediate point to split the whole domain into singular and regular*

TABLE 10. Order of convergence and errors of $u(x,t)$ ($u_t = 0, \Delta t = h^2, \delta = 0.5$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/4	5.46967e-03	*	8.02144e-03	*	8.02144e-03	*	7.64099e-03	*	3.34709e-04	*
	1/8	3.52961e-04	3.95387	5.11984e-04	3.96969	5.11984e-04	3.96969	4.89174e-04	3.96534	2.18944e-05	3.93427
0.5	1/16	2.22332e-05	3.98872	3.21597e-05	3.99277	3.21597e-05	3.99277	3.07494e-05	3.99172	1.38705e-06	3.98047
	1/32	1.39225e-06	3.99723	2.01468e-06	3.99663	2.01468e-06	3.99663	1.92456e-06	3.99796	8.69942e-08	3.99495
	1/4	3.85067e-03	*	5.84063e-03	*	5.84063e-03	*	5.63453e-03	*	2.41097e-04	*
	1/8	2.47239e-04	3.96113	3.74961e-04	3.96131	3.74961e-04	3.96131	3.59583e-04	3.9699	1.55571e-05	3.95397
0.8	1/16	1.556e-05	3.98999	2.35734e-05	3.99151	2.35734e-05	3.99151	2.25886e-05	3.99266	9.82977e-07	3.98427
	1/32	9.74179e-07	3.99751	1.47576e-06	3.99763	1.47576e-06	3.99763	1.41356e-06	3.99818	6.16139e-08	3.99583

TABLE 11. Order of convergence and errors of $u(x,t)$ ($\Delta t = h$, $\delta = 0.5$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/10	1.42257e-02	*	1.59454e-01	*	1.59454e-01	*	1.26136e-01	*	4.01693e-02	*
	1/20	3.56089e-03	1.99819	4.04994e-02	1.97717	4.04994e-02	1.97717	3.17071e-02	1.99211	1.00523e-02	1.99857
0.2	1/40	8.91648e-04	1.99769	1.01392e-02	1.99795	1.01392e-02	1.99795	7.93774e-03	1.99801	2.51775e-03	1.99732
	1/80	2.24074e-04	1.9925	2.53579e-03	1.99944	2.53579e-03	1.99944	1.98531e-03	1.99936	6.33549e-04	1.99061
	1/10	1.31898e-02	*	1.55343e-01	*	1.55343e-01	*	1.22043e-01	*	3.72452e-02	*
	1/20	3.30061e-03	1.99862	3.93857e-02	1.97972	3.93857e-02	1.97972	3.06799e-02	1.99203	9.3169e-03	1.99913
0.5	1/40	8.26499e-04	1.99765	9.8606e-03	1.99793	9.8606e-03	1.99793	7.68068e-03	1.99799	2.33366e-03	1.99725
	1/80	2.07772e-04	1.99201	2.46647e-03	1.99923	2.46647e-03	1.99923	1.92102e-03	1.99936	5.87488e-04	1.98997

TABLE 12. Order of convergence and errors of $u(x,t)$ ($\Delta t = h^2, \delta = 0.5$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/4	2.03176e-03	*	1.65779e-02	*	1.65779e-02	*	1.38287e-02	*	6.0826e-03	*
	1/8	1.26462e-04	4.00596	1.07939e-03	3.94097	1.07939e-03	3.94097	8.65513e-04	3.99797	3.78961e-04	4.00457
0.5	1/16	7.90513e-06	3.99977	6.7428e-05	4.00073	6.7428e-05	4.00073	5.40844e-05	4.00027	2.36902e-05	3.99968
	1/32	4.94814e-07	3.99783	4.2139e-06	4.00012	4.2139e-06	4.00012	3.38027e-06	4.00000	1.4833e-06	3.99741
	1/4	1.81907e-03	*	1.59191e-02	*	1.59191e-02	*	1.32427e-02	*	5.45821e-03	*
	1/8	1.13413e-04	4.00355	1.04301e-03	3.93193	1.04301e-03	3.93193	8.28942e-04	3.99779	3.39847e-04	4.00547
0.8	1/16	7.09296e-06	3.99905	6.51497e-05	4.00086	6.51497e-05	4.00086	5.18056e-05	4.00009	2.12561e-05	3.99894
	1/32	4.44221e-07	3.99704	4.07166e-06	4.00007	4.07166e-06	4.00007	3.238e-06	3.99993	1.33173e-06	3.99651

TABLE 13. Order of convergence and errors of $u(x,t)$ ($u_t = 0$, $\Delta t = h$, $\delta = 0.5$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/10	2.16517e-02	*	1.84479e-01	*	1.84479e-01	*	1.49313e-01	*	2.80008e-02	*
	1/20	5.44494e-03	1.99149	4.70564e-02	1.97099	4.70564e-02	1.97099	3.80544e-02	1.9722	7.01051e-03	1.99788
0.2	1/40	1.36292e-03	1.99822	1.18794e-02	1.98592	1.18794e-02	1.98592	9.57307e-03	1.99101	1.75469e-03	1.99831
	1/80	3.41957e-04	1.99481	2.97446e-03	1.99777	2.97446e-03	1.99777	2.39613e-03	1.99828	4.40056e-04	1.99546
	1/10	2.00936e-02	*	1.82161e-01	*	1.82161e-01	*	1.46487e-01	*	2.58833e-02	*
	1/20	5.03629e-03	1.9963	4.59812e-02	1.98609	4.59812e-02	1.98609	3.68646e-02	1.99046	6.48296e-03	1.9973
0.5	1/40	1.26124e-03	1.99752	1.15179e-02	1.99717	1.15179e-02	1.99717	9.22903e-03	1.99798	1.62322e-03	1.99779
	1/80	3.16661e-04	1.99383	2.88106e-03	1.9992	2.88106e-03	1.9992	2.3083e-03	1.99935	4.07231e-04	1.99494

subdomains. Now we discuss the accuracy of both WSGD and L_1- schemes for our problem.

***Case 1**: Let $p = \beta$ $(0 < \beta < 1)$ in $u(x,t)$, then WSGD scheme and L_1 scheme behaviors are presented in Tables 15 and 16. From Tables 15 and 16, one can easily understand that the both schemes attained the order β while the order of WSGD scheme is little better than the L_1 scheme. Furthermore, the error of L_1 scheme is better than WSGD scheme when β is near to 0 and still order of WSGD scheme is better than the L_1 scheme while when β go away from the 0, the order and error of WSGD scheme are better.*

***Case 2**: Let $p = 1 + \beta$ $(0 < \beta < 1)$ in $u(x,t)$, then WSGD scheme and L_1 scheme behaviors are presented in tables 17 and 18. From Table 17, one can observe that the order of L_1 scheme is little scattered, it seems like $1 + \beta$, when β is near to 0 and $2 - \beta$ when β go away from the 0. On the other hand, from Table 18, one*

TABLE 14. Order of convergence and errors of $u(x,t)$ ($u_t = 0, \Delta t = h^2, \delta = 0.5$).

β	h	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/4	2.17886e-03	*	1.6682e-02	*	1.6682e-02	*	1.40198e-02	*	6.54202e-03	*
	1/8	1.35664e-04	4.00546	1.08376e-03	3.94417	1.08376e-03	3.94417	8.73879e-04	4.00388	4.06547e-04	4.00824
0.5	1/16	8.4724e-06	4.00112	6.76878e-05	4.001	6.76878e-05	4.001	5.45958e-05	4.00057	2.53898e-05	4.0011
	1/32	5.29777e-07	3.99931	4.22974e-06	4.00026	4.22974e-06	4.00026	3.41183e-06	4.00017	1.58784e-06	3.99912
	1/4	2.04135e-03	*	1.62954e-02	*	1.62954e-02	*	1.3617e-02	*	6.11043e-03	*
	1/8	1.26872e-04	4.00546	1.06276e-03	3.94417	1.06276e-03	3.94417	8.52478e-04	4.00388	3.80196e-04	4.00824
0.8	1/16	7.92247e-06	4.00128	6.63629e-05	4.0013	6.63629e-05	4.0013	5.32519e-05	4.00076	2.37413e-05	4.00127
	1/32	4.95483e-07	3.999041	4.14688e-06	4.00028	4.14688e-06	4.00028	3.32777e-06	4.0002	1.48509e-06	3.99878

TABLE 15. L_1 scheme temporal order of convergence and errors of $u(x,t)$ ($h = 1/500$, $\delta = 0.2$).

β	Δt	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/10	1.4416e-02	*	4.30319e-02	*	4.30319e-02	*	4.27561e-02	*	5.5239e-02	*
	1/20	1.14214e-02	0.335938	3.87916e-02	0.14966	3.87916e-02	0.14966	3.82938e-02	0.159019	4.76558e-02	0.213035
0.2	1/40	8.78781e-03	0.378158	3.49728e-02	0.14951	3.49728e-02	0.14951	3.4299e-02	0.158944	3.20711e-02	0.571376
	1/80	5.88397e-03	0.578713	3.16592e-02	0.143611	3.16592e-02	0.143611	3.08896e-02	0.151047	2.13674e-02	0.585861
	1/10	7.49017e-03	*	1.64018e-02	*	1.64018e-02	*	1.62592e-02	*	2.90877e-02	*
	1/20	4.49354e-03	0.737145	1.0196e-02	0.685856	1.0196e-02	0.685856	1.021e-02	0.671273	1.64975e-02	0.81816
0.8	1/40	2.55749e-03	0.813128	6.31061e-03	0.692147	6.31061e-03	0.692147	6.25563e-03	0.706761	9,50808e-03	0.795022
	1/80	1.46959e-03	0.799316	3.80945e-03	0.728198	3.80945e-03	0.728198	3.74429e-03	0.740461	5.37863e-03	0.821916

TABLE 16. WSGD scheme temporal order of convergence and errors of $u(x,t)$ ($h = 1/500$, $\delta = 0.2$).

β	Δt	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/10	1.10908e-01	*	3.23293e-01	*	3.23293e-01	*	3.16156e-01	*	3.24417e-01	*
	1/20	7.85924e-02	0.496896	2.72327e-01	0.2475	2.72327e-01	0.2475	2.64662e-01	0.256485	1.8708e-01	0.794197
0.2	1/40	5.36009e-02	0.552131	2.30537e-01	0.240343	2.30537e-01	0.240343	2.2265e-01	0.249374	1.13372e-01	0.722591
	1/80	3.68593e-02	0.540228	1.96899e-01	0.227541	1.96899e-01	0.227541	1.89239e-01	0.234563	6.42057e-02	0.820288
	1/10	2.28375e-03	*	6.4179e-03	*	6.4179e-03	*	6.19607e-03	*	6.68023e-03	*
	1/20	1.04468e-03	1.12835	3.31821e-03	0.951694	3.31821e-03	0.951694	3.20275e-03	0.952041	3.43163e-03	0.961004
0.8	1/40	4.91559e-04	1.08762	1.77435e-03	0.903118	1.77435e-03	0.903118	1.71691e-03	0.8995	1.54229e-03	1.15382
	1/80	2.43959e-04	1.01073	9.72774e-04	0.867112	9.72774e-04	0.867112	9.45491e-04	0.860677	5.54406e-04	1.47606

can easily observed that WSGD scheme has order $1+\beta$ when β is near to 0 while WSGD scheme attain exactly second order when β away from 0.

TABLE 17. L_1 scheme temporal order of convergence and errors of $u(x,t)$ ($h = 1/500$, $\delta = 0.2$).

β	Δt	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/10	1.14573e-03	*	2.47934e-03	*	2.47934e-03	*	2.25886e-03	*	4.42912e-03	*
	1/20	4.57194e-04	1.32538	1.02325e-03	1.2768	1.02325e-03	1.2768	1.027e-03	1.25955	1.65256e-03	1.42232
0.2	1/40	1.74523e-04	1.38939	4.23052e-04	1.27425	4.23052e-04	1.27425	4.20899e-04	1.28689	6.4067e-04	1.36705
	1/80	6.68011e-05	1.38548	1.70352e-04	1.31232	1.70352e-04	1.31232	1.67821e-04	1.32655	2.44288e-04	1.39099
	1/10	1.70938e-04	*	4.73691e-04	*	4.73691e-04	*	4.70139e-04	*	7.03472e-04	*
	1/20	7.87213e-05	1.11865	2.2238e-04	1.09092	2.2238e-04	1.09092	2.19337e-04	1.09968	3.07623e-04	1.19333
0.8	1/40	3.1755e-05	1.30977	1.04107e-04	1.09497	1.04107e-04	1.09497	1.02068e-04	1.10389	1.14692e-04	1.4234
	1/80	1.14849e-05	1.46725	4.87318e-05	1.09513	4.87318e-05	1.09513	4.75427e-05	1.10223	4.1765e-05	1.4574

Remark 3. *If $p \geq 2$ in Example 4, then WSGD and L_1 schemes can attain their original order of convergence that is 2 and $\Delta t^{2-\beta}$, respectively. This condition is sufficient but unnecessary.*

6. Conclusion

This paper established two augmented finite volume schemes on uniform grids for solving the nonlinear TFDPE. The advantage of the proposed second and fourth-order schemes is that we got the best order of convergence and error for this singular equation, which can not be obtained with the help of traditional numerical methods

TABLE 18. WSGD scheme temporal order of convergence and errors of $u(x,t)$ ($h = 1/500$, $\delta = 0.2$).

β	Δt	$E_s^{L_\infty}u$	Order	$E_r^{L_\infty}u$	Order	$E^{L_\infty}u$	Order	$E_r^{L_2}u$	Order	$E_{Aug}^{L_\infty}$	Order
	1/10	2.07e-03	*	5.98347e-03	*	5.98347e-03	*	5.86001e-03	*	6.05498e-03	*
	1/20	7.42881e-04	1.47843	2.53386e-03	1.23964	2.53386e-03	1.23964	2.4669e-03	1.24821	1.76834e-03	1.77573
0.2	1/40	2.55631e-04	1.53907	1.0743e-03	1.23794	1.0743e-03	1.23794	1.0396e-03	1.24667	6.28224e-04	1.49304
	1/80	8.81176e-05	1.53656	4.57371e-04	1.23196	4.57371e-04	1.23196	4.4048e-04	1.23888	1.71955e-04	1.86924
	1/10	8.06233e-04	*	2.05115e-03	*	2.05115e-03	*	1.98865e-03	*	3.28113e-03	*
	1/20	1.88912e-04	2.09349	4.74058e-04	2.1133	4.74058e-04	2.1133	4.5814e-04	2.11793	7.08797e-04	2.21075
0.8	1/40	4.69295e-05	2.00914	1.1368e-04	2.06008	1.1368e-04	2.06008	1.09906e-04	2.05951	1.71375e-04	2.04821
	1/80	1.1768e-05	1.99562	2.85014e-05	1.99588	2.85014e-05	1.99588	2.76556e-05	1.99063	4.31e-05	1.9914

because of the singularity of the proposed equation. It is essential that we do not use the mesh points in the singular subdomain near the intermediate point because we converted multiple augmented variables involved in the Puiseux series to a single augmented variable, which makes the method simpler, and we can choose an intermediate point independently because the choice of a suitable intermediate point is very difficult, or we cannot choose in the case of multiple augmented variables. We showed the effectiveness and efficiency of the proposed schemes by giving numerical examples of weakly and strongly degenerate cases. We also gave an interesting example with coefficient blow-up at the degenerate point and showed that the second and fourth-order schemes are also valid for this interesting case. In the numerical examples, we used different values of β and observed that the order of convergence is stable and does not depend on the fractional parameter.

Data availability

Data will be made available on request.

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