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ENERGY-CONSERVATIVE FINITE DIFFERENCE METHOD FOR THE COUPLED NONLINEAR KLEIN-GORDON EQUATION IN THE NONRELATIVISTIC LIMIT REGIME

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Abstract. In this paper, we propose an energy-conservative finite difference time domain (FDTD) method for solving the coupled nonlinear Klein-Gordon equations (CNKGEs) in the nonrelativistic limit regime, involving a small parameter $0 < \varepsilon \ll 1$ which is inversely proportional to the speed of light. Employing cut-off technique, we analyze rigorously error estimates for the numerical method. Numerical results are reported to confirm the energy-conservative property and the error results in l^2 norm and H^1 norm under different values of ε.

Key words. Coupled nonlinear Klein-Gordon equations, finite difference time domain method, energy-conservative, cut-off technique, nonrelativistic regime.

1. Introduction

The Klein-Gordon equations (KGEs), proposed in 1927 by physicists Oskar Klein and Walter Gordon, can be used to describe the motion of a spin-0 particle with the mass m. It is a fundamental equation in relativistic quantum mechanics and quantum field theory and can be regarded as the relativistic version of the Schrödinger equation.

The following coupled nonlinear Klein-Gordon equations in d-dimensions ($d = 1, 2, 3$) can be used to describe the interaction of two fields (ϕ and ψ) with the mass m_1 and the mass m_2, respectively,

$$\frac{1}{c^2}\partial_{tt}\phi - \triangle\phi + \frac{m_1^2c^2}{\hbar^2}\phi + \eta_1|\phi|^2\phi + \eta_2|\psi|^2\phi = 0, \tag{1a}$$

$$\frac{1}{c^2}\partial_{tt}\psi - \triangle\psi + \frac{m_2^2c^2}{\hbar^2}\psi + \eta_2|\phi|^2\psi + \eta_3|\psi|^2\psi = 0, \tag{1b}$$

where t is time, $\mathbf{x}$ is the spatial coordinate, $\phi := \phi(\mathbf{x}, t)$ and $\psi := \psi(\mathbf{x}, t)$ are functions representing electron-positron fields, $m_2 = \alpha m_1, 0 < \alpha \leq 1$, $\hbar$ is Planck's constant, c is speed of light, and η_1, η_2, η_3 are the interaction constants of electron-positron fields. Introduce the notions

$$\tilde{t} := \frac{t}{t_s},\ \tilde{\mathbf{x}} := \frac{\mathbf{x}}{\mathbf{x}_s},\ u(\tilde{\mathbf{x}}, \tilde{t}) := \frac{\phi(\mathbf{x}, t)}{\phi_s}, v(\tilde{\mathbf{x}}, \tilde{t}) := \frac{\psi(\mathbf{x}, t)}{\psi_s}, \tag{2}$$

plugging (2) into (1a) - (1b), we have

$$\frac{\mathbf{x}_s^2}{c^2t_s^2}\partial_{tt}u - \triangle u + \frac{m_1^2c^2\mathbf{x}_s^2}{\hbar^2}u + \eta_1\phi_s^2\mathbf{x}_s^2|u|^2u + \eta_2^2\psi_s^2\mathbf{x}_s^2|v|^2u = 0, \tag{3a}$$

$$\frac{\mathbf{x}_s^2}{c^2t_s^2}\partial_{tt}v - \triangle v + \frac{\alpha m_1^2c^2\mathbf{x}_s^2}{\hbar^2}v + \eta_2\phi_s^2\mathbf{x}_s^2|u|^2v + \eta_3\psi_s^2\mathbf{x}_s^2|v|^2v = 0, \tag{3b}$$

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where $t_s, \mathbf{x}_s, \phi_s, \psi_s$ are the dimensionless time unit, length unit, and filed unit. Setting $\varepsilon = \frac{\hbar}{cm_1\mathbf{x}_s} = \frac{\mathbf{x}_s}{ct_s} = \frac{v_s}{c}$, $\beta_1 = \eta_1\phi_s^2\mathbf{x}_s^2$, $\beta_2 = \eta_2^2\psi_s^2\mathbf{x}_s^2 = \eta_2^2\phi_s^2\mathbf{x}_s^2$, $\beta_3 = \eta_3\psi_s^2\mathbf{x}_s^2$, $\phi_s = \psi_s = \frac{1}{\mathbf{x}_s\sqrt{\eta}}$ with $\eta = \min\{\eta_1, \eta_2, \eta_3\}$, the dimensionless CNKGEs are obtained

$$\varepsilon^2\partial_{tt}u - \triangle u + \frac{1}{\varepsilon^2}u + \beta_1|u|^2u + \beta_2|v|^2u = 0, \tag{4a}$$

$$\varepsilon^2\partial_{tt}v - \triangle v + \frac{\alpha}{\varepsilon^2}v + \beta_2|u|^2v + \beta_3|v|^2v = 0, \tag{4b}$$

with initial conditions

$$u(\mathbf{x}, 0) = \xi(\mathbf{x}), \qquad \partial_t u(\mathbf{x}, 0) = \frac{1}{\varepsilon^2}\zeta(\mathbf{x}), \tag{4c}$$

$$v(\mathbf{x}, 0) = \rho(\mathbf{x}), \qquad \partial_t v(\mathbf{x}, 0) = \frac{\alpha}{\varepsilon^2}\eta(\mathbf{x}). \tag{4d}$$

Here $u(\mathbf{x}, t)$ and $v(\mathbf{x}, t)$ are unknown wave functions with temporal wavelength of $\mathcal{O}(\varepsilon^2)$, where $\varepsilon \in (0, 1)$ is a constant inversely proportional to the light-speed constant c. In addition, $\xi(\mathbf{x})$, $\rho(\mathbf{x})$, $\zeta(\mathbf{x})$ and $\eta(\mathbf{x})$ are given functions that are independent of ε. Fundamentally, nonlinear wave phenomena occur in various areas of physical science. The KGEs, widely applied in quantum and particle physics, have garnered significant attention in researching solitons and condensed matter physics [12], the interaction of solitons in plasma collisions [13], the recurrence of initial states [18], and lattice nonlinear dynamics [14]. Schiff [32] made efforts to consider nuclear saturation and shell structure in terms of many-body forces which were derived from mesons obeying a nonlinear wave equation.

The system (4) is time symmetric or time reversible. Additionally, under periodic or homogeneous Dirichlet boundary conditions, the system (4) is energy-conservative in the sense that

$$E(t) \equiv E(0), \qquad t > 0, \tag{5}$$

where $E(t)$ is the total energy defined by

$$\begin{aligned} E(t) = \int_\Omega \bigg[& \varepsilon^2(\partial_t u)^2 + (\nabla u)^2 + \frac{1}{\varepsilon^2}u^2 + \varepsilon^2(\partial_t v)^2 + (\nabla v)^2 \\ & + \frac{\alpha}{\varepsilon^2}v^2 + \frac{\beta_1}{2}u^4 + \beta_2 u^2v^2 + \frac{\beta_3}{2}v^4 \bigg] d\mathbf{x}. \end{aligned} \tag{6}$$

It is well-known that conservative numerical schemes consistently outperform nonconservative ones. The crux of their superiority lies in their ability to preserve important invariant properties, allowing for a more detailed representation of physical processes. From this perspective, the numerical simulation can be measured by the extent to maintain the invariant properties of the original continuous model. To achieve an appropriate numerical method that ensures energy conservation, the classical approach involves constructing a fully implicit scheme, which always bring significant challenges for convergence analysis. In this paper, we analyze rigourously the unconditional convergence results for the energy-conservative implicit scheme.

In the regime of $O(1)$-speed of light, where the parameter $\varepsilon > 0$ is fixed, the KGEs have garnered substantial interest, experiencing a notable surge in both analytical and numerical research. Along the analytical front, Scott [30] outlined several physical implementations and described the construction of analog models. The global classical solutions of the KGEs were investigated in [20, 33]. Moreover, the Cauchy problem for the KGEs were studied, we refer the readers to [19, 31] and therein references. Researchers proposed and analyzed standard finite difference time domain (FDTD) methods for the KGEs in [2, 15, 21, 27, 28]. Bao et al. [9] found the time-splitting Fourier pseudospectral numerical scheme and established uniform error bounds of this numerical method. Wang et al. [35] used variational method and finite element methods (FEMs) for the KGEs with Dirichlet boundary condition. We also can learn about some research results involved in KGEs coupled with other equations. For example, Benci et al. [5] analyzed the solitary waves of the KGEs coupled with Maxwell equations. Li et al. [22, 23] studied the conservative FEMs for the KGEs coupled with Schrödinger system.

To elucidate the behavior of charged mesons in the presence of an electromagnetic field, the CNKGEs were introduced by I. Segal for the first time in [29]. For the Cauchy problem for the CNKGEs, Liu [24] analyzed the sharp criteria for global existence and blow-up. With the help of variational argument, potential well argument, and concavity method, Zhang [37] studied the existence and the sharp criterion for blow-up of CNKGEs. T. Alagesa [3] et al. delved into the examination of the bilinear form and one-soliton solutions of CNKGEs. In the numerical aspect, multiple numerical schemes were put forward and analyzed in many literature. For example, Tai [34] et al. and Xu [36] et al. introduced the numerical method under Gautschi-type integrator sine spectral discretization. Doha [16] et al. proposed the Jacobi-Gauss-Lobatto collocation method for the CNKGEs.

However, in the nonrelativistic limit regime, characterized by $0 < \varepsilon \ll 1$ or an infinitely large speed of light, the analysis and efficient computation of the KGEs or the CNKGEs are complicated issues. These challenges stem primarily from the unbounded nature of the energy $E(t)$ as ε approaches zero. Recently, Machihara et al. [26] investigated the nonrelativistic limit of the Cauchy problem for the KGEs and proved the convergence of finite energy solutions to the corresponding solutions of the nonlinear Schrödinger equation in the energy space. Bao et al. [6, 7, 8] studied the numerical methods and provided rigorous error estimates for the KGEs in the nonrelativistic limit regime. In Bao's research(such as [6]), the nonlinear term f of the KGEs is assumed to satisfy certain conditions, such that $\|f^{'}\|_{L^\infty} + \|f^{''}\|_{L^\infty} + \|f^{'''}\|_{L^\infty} \lesssim 1$. There exists no work focusing on the numerical method for the CNKGEs in the nonrelativistic limit regime. The highly oscillatory nature in the solution makes the computation in the nonrelativistic limit regime extremely challenging.

The main contributions of this work include: (i) we accomplish the scaling of the CNKGEs in the nonrelativistic limit regime by the dimensionless unit. (ii) An energy-conservative FDTD method for the above proposed dimensionless CNKGEs is established. Further, under some certain conditions and with the help of energy conservation of the discrete system, the boundedness of the numerical solutions in different norms are derived. (iii) The constructed scheme is fully implicit, which increases the difficulty of the error analysis. In this work, with the help of the cut-off technique, the error estimates are derived, without any time-space step ratio restriction. Moreover, compared with the reference [6], the boundedness of the numerical solutions in L^∞-norm is no longer required in this work. For the higher-dimensional cases, our results are also suitable.

The outline of the paper is as follows. In Section 2, we present a fully discrete FDTD numerical scheme and the energy-conservative is proved in the nonrelativistic limit regime. In Section 3, the rigorous error analysis are obtained by the cut-off technology. In Section 4, several numerical simulation results are reported to verify the theoretical analysis. Conclusions are given in the final section.

2. FDTD method

The purpose of this section is to propose the FDTD method for the CNKGEs (4) and to conduct a comprehensive analysis of the stability and convergence properties of the numerical scheme in the nonrelativistic limit regime. For simplicity of notations, we consider the numerical analysis in the case of one dimension (1D). The results can be generalized for the case of higher dimensions. In practical computation, we truncate the problem onto an interval $\Omega = (a, b)$ with periodic boundary conditions. Then the CNKGEs (4) with periodic boundary conditions collapses to

$$\varepsilon^2\partial_{tt}u - \partial_{xx}u + \frac{1}{\varepsilon^2}u + \beta_1|u|^2u + \beta_2|v|^2u = 0, \qquad x \in \Omega, \quad t > 0, \tag{7a}$$

$$\varepsilon^2\partial_{tt}v - \partial_{xx}v + \frac{\alpha}{\varepsilon^2}v + \beta_2|u|^2v + \beta_3|v|^2v = 0, \qquad x \in \Omega, \quad t > 0, \tag{7b}$$

with the boundary and initial conditions

$$u(a,t) = u(b,t), \quad \partial_t u(a,t) = \partial_t u(b,t), \qquad t \geq 0, \tag{7c}$$

$$v(a,t) = v(b,t), \quad \partial_t v(a,t) = \partial_t v(b,t), \qquad t \geq 0, \tag{7d}$$

$$u(x,0) = \xi(x), \quad \partial_t u(x,0) = \frac{1}{\varepsilon^2}\zeta(x), \qquad x \in \overline{\Omega} = [a,b], \tag{7e}$$

$$v(x,0) = \rho(x), \quad \partial_t v(x,0) = \frac{\alpha}{\varepsilon^2}\eta(x), \qquad x \in \overline{\Omega} = [a,b], \tag{7f}$$

where the initial-value functions satisfy

$$\xi(a) = \xi(b), \xi^{'}(a) = \xi^{'}(b), \zeta(a) = \zeta(b), \zeta^{'}(a) = \zeta^{'}(b), \tag{8}$$

$$\rho(a) = \rho(b), \rho^{'}(a) = \rho^{'}(b), \eta(a) = \eta(b), \eta^{'}(a) = \eta^{'}(b). \tag{9}$$

2.1. Energy-conservative property. We have the following theorem concerning the property of energy-conservative of the system (7) under the conditions $u(\cdot,t), v(\cdot,t) \in \mathcal{H}^1(\Omega)$ and $\partial_t u(\cdot,t), \partial_t v(\cdot,t) \in L^2(\Omega)$.

Theorem 2.1. *The energy, which is defined by*

$$\begin{aligned} E = \int_\Omega \Big[&\varepsilon^2(\partial_t u)^2 + (\partial_x u)^2 + \frac{1}{\varepsilon^2}u^2 + \varepsilon^2(\partial_t v)^2 + (\partial_x v)^2 \\ &+ \frac{\alpha}{\varepsilon^2}v^2 + \frac{\beta_1}{2}u^4 + \beta_2 u^2v^2 + \frac{\beta_3}{2}v^4\Big]dx, \end{aligned} \tag{10}$$

is conservative.

Proof. Multiplying (7a) and (7b) by $\partial_t u$ and $\partial_t v$ respectively, and using the initial-boundary conditions (7c)-(7f), we have

$$\begin{aligned} &\partial_t \int_\Omega \left[\frac{\varepsilon^2}{2}(\partial_t u)^2 + \frac{1}{2}(\partial_x u)^2 + \frac{1}{2\varepsilon^2}u^2\right]dx \\ &+ \beta_1 \int_\Omega |u|^2u\partial_t u dx + \beta_2 \int_\Omega |v|^2u\partial_t u dx = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} &\partial_t \int_\Omega \left[\frac{\varepsilon^2}{2}(\partial_t v)^2 + \frac{1}{2}(\partial_x v)^2 + \frac{\alpha}{2\varepsilon^2}v^2\right]dx \\ &+ \beta_2 \int_\Omega |u|^2v\partial_t v dx + \beta_3 \int_\Omega |v|^2v\partial_t v dx = 0. \end{aligned} \tag{12}$$

Noticing that

$$\partial_t \left(\frac{\beta_1}{4}u^4 + \frac{\beta_2}{2}u^2v^2 + \frac{\beta_3}{4}v^4\right) = \beta_1|u|^2u\partial_t u + \beta_2|v|^2u\partial_t u + \beta_2|u|^2v\partial_t v + \beta_3|v|^2v\partial_t v,$$

from (11) and (12), we have

$$\frac{1}{2}\partial_t\int_\Omega\Big[\varepsilon^2\partial_t u\partial_t u+\partial_x u\partial_x u+\frac{1}{\varepsilon^2}uu+\varepsilon^2\partial_t v\partial_t v+\partial_x v\partial_x v$$
$$+\frac{\alpha}{\varepsilon^2}vv+\frac{\beta_1}{2}u^4+\beta_2u^2v^2+\frac{\beta_3}{2}v^4\Big]=0. \tag{13}$$

Namely,

$$\partial_t E=0.$$

This proof is completed. □

2.2. Energy-conservative FDTD method. From Theorem 2.1, we obtain that the continuous model is energy-conservative. In this subsection, we will construct an energy-conservative numerical scheme for the model (7) and the energy conservation property will be studied in detail.

The mesh size h and time size τ can be chosen as $h:=\triangle x=\frac{(b-a)}{M},\tau:=\Delta t>0$, where M is a positive integer. The grid points and times step can be denoted as

$$x_j=a+jh,\quad j=0,1,\cdots,M,$$
$$t_n:=n\tau,\quad n=0,1,2,\cdots.$$

Denote $X_M=\{\iota=(\iota_0,\iota_1,\cdots,\iota_M)\,|\,\iota_0=\iota_M\}\in\mathcal{R}^{M+1}$, and $\iota_0=\iota_M,\iota_{M-1}=\iota_{-1}$. We equip the spaces X_M with the standard discrete l^2 norm semi-H^1 norm semi-H^2 norm and l^∞ norm, these norms are defined by

$$\|\iota^n\|_{l^2}^2=h\sum_{j=0}^{M-1}|\iota_j^n|^2,\quad \|\delta_x^+\iota^n\|_{l^2}^2=h\sum_{j=0}^{M-1}|\delta_x^+\iota_j^n|^2,$$
$$\|\delta_x^2\iota^n\|_{l^2}^2=h\sum_{j=0}^{M-1}|\delta_x^2\iota_j^n|^2,\quad \|\iota^n\|_{l^\infty}=\max_{0\le j\le M-1}|\iota_j^n|.$$

Let ι_j^n be the numerical solution $\iota(x_j,t_n)\,(j=0,1,\cdots,M,n=0,1,2,\cdots)$ and introduce the finite difference discretization operators as

$$\delta_t^+\iota_j^n=\frac{\iota_j^{n+1}-\iota_j^n}{\tau},\qquad \delta_t^-\iota_j^n=\frac{\iota_j^n-\iota_j^{n-1}}{\tau},\qquad \delta_t^2\iota_j^n=\frac{\iota_j^{n+1}-2\iota_j^n+-\iota_j^{n-1}}{\tau^2},$$
$$\delta_x^+\iota_j^n=\frac{\iota_{j+1}^n-\iota_j^n}{h},\qquad \delta_x^-\iota_j^n=\frac{\iota_j^n-\iota_{j-1}^n}{h},\qquad \delta_x^2\iota_j^n=\frac{\iota_{j+1}^n-2\iota_j^n+\iota_{j-1}^n}{h^2}.$$

Here, we consider the implicit energy-conservative FDTD method for the coupled equation (7), which is to find $(u^{n+1},v^{n+1})\in X_M\times X_M$ for $n=1,2,\cdots$, such that

$$\varepsilon^2\delta_t^2u_j^n-\frac{1}{2}\delta_x^2(u_j^{n+1}+u_j^{n-1})$$
$$+\frac{1}{2\varepsilon^2}(u_j^{n+1}+u_j^{n-1})+G_1(u_j^{n+1},u_j^{n-1},v_j^n)=0, \tag{14a}$$
$$\varepsilon^2\delta_t^2v_j^n-\frac{1}{2}\delta_x^2(v_j^{n+1}+v_j^{n-1})$$
$$+\frac{\alpha}{2\varepsilon^2}(v_j^{n+1}+v_j^{n-1})+G_2(u_j^n,v_j^{n+1},v_j^{n-1})=0, \tag{14b}$$

with the initial and boundary conditions

$$u_0^n=u_M^n,u_{-1}^n=u_{M+1}^n,n\ge0,\quad u_j^0=\xi(x_j), \tag{14c}$$
$$v_0^n=v_M^n,v_{-1}^n=v_{M+1}^n,n\ge0,\quad v_j^0=\rho(x_j), \tag{14d}$$

$$u_j^1 = \xi(x_j) + \frac{\tau}{\varepsilon^2}\zeta(x_j) + \frac{\tau^2}{2\varepsilon^2}\left[\delta_x^2\xi(x_j) - \frac{1}{\varepsilon^2}\xi(x_j) + f_1(\xi(x_j),\rho(x_j))\right], \tag{14e}$$

$$v_j^1 = \rho(x_j) + \frac{\tau\alpha}{\varepsilon^2}\eta(x_j) + \frac{\tau^2}{2\varepsilon^2}\left[\delta_x^2\rho(x_j) - \frac{\alpha}{\varepsilon^2}\rho(x_j) + f_2(\xi(x_j),\rho(x_j))\right], \tag{14f}$$

where

$$G_1(u_1,u_2,v) = \int_0^1 f_1(\theta u_1 + (1-\theta)u_2, v)d\theta = \frac{F(u_1,v)-F(u_2,v)}{u_1-u_2}, \tag{15}$$

$$G_2(u,v_1,v_2) = \int_0^1 f_2(u, \theta v_1 + (1-\theta)v_2)d\theta = \frac{F(u,v_1)-F(u,v_2)}{v_1-v_2}, \tag{16}$$

$$f_1(u,v) = \beta_1|u|^2u + \beta_2|v|^2u, \qquad f_2(u,v) = \beta_2|u|^2v + \beta_3|v|^2v, \tag{17}$$

$$F(u,v) = \frac{\beta_1}{4}|u|^4 + \frac{\beta_2}{2}|u|^2|v|^2 + \frac{\beta_3}{4}|v|^4. \tag{18}$$

We have the following theorem about energy conservation of the numerical scheme (14).

Theorem 2.2. *(Energy conservation) Define numerical energy*

$$\begin{aligned} E^n =& \varepsilon^2\|\delta_t^+u^n\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+u^{n+1}\|_{l^2}^2 + \|\delta_x^+u^n\|_{l^2}^2\right) + \frac{1}{2\varepsilon^2}\left(\|u^{n+1}\|_{l^2}^2 + \|u^n\|_{l^2}^2\right) \\ &+ \varepsilon^2\|\delta_t^+v^n\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+v^{n+1}\|_{l^2}^2 + \|\delta_x^+v^n\|_{l^2}^2\right) + \frac{\alpha}{2\varepsilon^2}\left(\|v^{n+1}\|_{l^2}^2 + \|v^n\|_{l^2}^2\right) \\ &+ h\sum_{j=0}^{M-1}\left[F(u_j^{n+1},v_j^n) + F(u_j^n,v_j^{n+1})\right], \end{aligned} \tag{19}$$

we have $E^n = E^{n-1}$.

Proof. Multiplying both sides of equations (14a) - (14b) by $h(u_j^{n+1}-u_j^{n-1})$ and $h(v_j^{n+1}-v_j^{n-1})$, respectively, then summing up for $j = 1, \cdots, M-1$, we have

$$\begin{aligned} & h\sum_{j=0}^{M-1}(\delta_t^+u_j^n - \delta_t^+u_j^{n-1})(\delta_t^+u_j^n + \delta_t^+u_j^{n-1}) \\ &+ \frac{h}{2}\sum_{j=0}^{M-1}(\delta_x^+u_j^{n+1} - \delta_x^+u_j^{n-1})(\delta_x^+u_j^{n+1} + \delta_x^+u_j^{n-1}) \\ &+ \frac{h}{2\varepsilon^2}\sum_{j=0}^{M-1}(u_j^{n+1} + u_j^{n-1})(u_j^{n+1} - u_j^{n-1}) \\ &+ h\sum_{j=0}^{M-1}\left[F(u_j^{n+1},v_j^n) - F(u_j^{n-1},v_j^n)\right] = 0, \end{aligned} \tag{20}$$

$$\begin{aligned} & h\sum_{j=0}^{M-1}(\delta_t^+v_j^n - \delta_t^+v_j^{n-1})(\delta_t^+v_j^n + \delta_t^+v_j^{n-1}) \\ &+ \frac{h}{2}\sum_{j=0}^{M-1}(\delta_x^+v_j^{n+1} - \delta_x^+v_j^{n-1})(\delta_x^+v_j^{n+1} + \delta_x^+v_j^{n-1}) \\ &+ \frac{h}{2\varepsilon^2}\sum_{j=0}^{M-1}(v_j^{n+1} + v_j^{n-1})(v_j^{n+1} - v_j^{n-1}) \end{aligned}$$

$$+ h \sum_{j=0}^{M-1} \left[F(u_j^n, v_j^{n+1}) - F(u_j^n, v_j^{n-1})\right] = 0. \tag{21}$$

From (20) - (21), we get

$$\begin{aligned}
&\varepsilon^2\|\delta_t^+ u^n\|_{l^2}^2 + \varepsilon^2\|\delta_t^+ v^n\|_{l^2}^2\\
&+ \frac{1}{2}\left(\|\delta_x^+ u^{n+1}\|_{l^2}^2 + \|\delta_x^+ u^n\|_{l^2}^2\right) + \frac{1}{2\varepsilon^2}\left(\|u^{n+1}\|_{l^2}^2 + \|u^n\|_{l^2}^2\right)\\
&+ \frac{1}{2}\left(\|\delta_x^+ v^{n+1}\|_{l^2}^2 + \|\delta_x^+ v^n\|_{l^2}^2\right) + \frac{\alpha}{2\varepsilon^2}\left(\|v^{n+1}\|_{l^2}^2 + \|v^n\|_{l^2}^2\right)\\
&+ h \sum_{j=0}^{M-1} \left[F(u_j^{n+1}, v_j^n) + F(u_j^n, v_j^{n+1})\right]\\
=&\varepsilon^2\|\delta_t^+ u^{n-1}\|_{l^2}^2 + \varepsilon^2\|\delta_t^+ v^{n-1}\|_{l^2}^2\\
&+ \frac{1}{2}\left(\|\delta_x^+ u^n\|_{l^2}^2 + \|\delta_x^+ u^{n-1}\|_{l^2}^2\right) + \frac{1}{2\varepsilon^2}\left(\|u^n\|_{l^2}^2 + \|u^{n-1}\|_{l^2}^2\right)\\
&+ \frac{1}{2}\left(\|\delta_x^+ v^n\|_{l^2}^2 + \|\delta_x^+ v^{n-1}\|_{l^2}^2\right) + \frac{\alpha}{2\varepsilon^2}\left(\|v^n\|_{l^2}^2 + \|v^{n-1}\|_{l^2}^2\right)\\
&+ h \sum_{j=0}^{M-1} \left[F(u_j^n, v_j^{n-1}) + F(u_j^{n-1}, v_j^n)\right].
\end{aligned}$$

Therefore, we have $E^n = E^{n-1}$. Hence, this proof is completed. □

Theorem 2.3. *(Boundedness) Assume one of the following conditions holds*

(a) $\min\{\beta_1, \beta_2, \beta_3\} > 0$;

(b) $\beta_1 > 0,\ \beta_3 > 0,\ \beta_1\beta_3 \le \beta_2^2$.

Then the solutions of system (14) are bounded in the sense that

(22)

$$\begin{aligned}
&\varepsilon^2\|\delta_t^+ u^n\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+ u^{n+1}\|_{l^2}^2 + \|\delta_x^+ u^n\|_{l^2}^2\right) + \frac{1}{2\varepsilon^2}\left(\|u^{n+1}\|_{l^2}^2 + \|u^n\|_{l^2}^2\right)\\
&\quad + \varepsilon^2\|\delta_t^+ v^n\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+ v^{n+1}\|_{l^2}^2 + \|\delta_x^+ v^n\|_{l^2}^2\right) + \frac{\alpha}{2\varepsilon^2}\left(\|v^{n+1}\|_{l^2}^2 + \|v^n\|_{l^2}^2\right) \lesssim 1.
\end{aligned}$$

Proof. From Theorem 2.2, we have $E^n = E^0$. Under the condition (a), and noticing that

$$h \sum_{j=0}^{M-1} \left[F(u_j^{n+1}, v_j^n) + F(u_j^n, v_j^{n+1})\right] > 0,$$

we obtain

(23)

$$\begin{aligned}
&\varepsilon^2\|\delta_t^+ u^n\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+ u^{n+1}\|_{l^2}^2 + \|\delta_x^+ u^n\|_{l^2}^2\right) + \frac{1}{2\varepsilon^2}\left(\|u^{n+1}\|_{l^2}^2 + \|u^n\|_{l^2}^2\right)\\
&+ \varepsilon^2\|\delta_t^+ v^n\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+ v^{n+1}\|_{l^2}^2 + \|\delta_x^+ v^n\|_{l^2}^2\right) + \frac{\alpha}{2\varepsilon^2}\left(\|v^{n+1}\|_{l^2}^2 + \|v^n\|_{l^2}^2\right)\\
\le&\varepsilon^2\|\delta_t^+ u^0\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+ u^1\|_{l^2}^2 + \|\delta_x^+ u^0\|_{l^2}^2\right) + \frac{1}{2\varepsilon^2}\left(\|u^1\|_{l^2}^2 + \|u^0\|_{l^2}^2\right)\\
&+ \varepsilon^2\|\delta_t^+ v^0\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+ v^1\|_{l^2}^2 + \|\delta_x^+ v^0\|_{l^2}^2\right) + \frac{\alpha}{2\varepsilon^2}\left(\|v^1\|_{l^2}^2 + \|v^0\|_{l^2}^2\right)\\
&+ h \sum_{j=0}^{M-1} \left[\frac{\beta_1}{4}|u_j^1|^4 + \frac{\beta_2}{2}|u_j^1|^2|v_j^0|^2 + \frac{\beta_3}{4}|v_j^0|^4 + \frac{\beta_1}{4}|u_j^0|^4 + \frac{\beta_2}{2}|u_j^0|^2|v_j^1|^2 + \frac{\beta_3}{4}|v_j^1|^4\right]
\end{aligned}$$

$\lesssim 1.$

Under $|u_j^n|^2 \neq 0, |v_j^n|^2 \neq 0$, we get

$$\begin{aligned}
&\varepsilon^2\|\delta_t^+ u^n\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+ u^{n+1}\|_{l^2}^2 + \|\delta_x^+ u^n\|_{l^2}^2\right) + \frac{1}{2\varepsilon^2}\left(\|u^{n+1}\|_{l^2}^2 + \|u^n\|_{l^2}^2\right) \\
&+ \varepsilon^2\|\delta_t^+ v^n\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+ v^{n+1}\|_{l^2}^2 + \|\delta_x^+ v^n\|_{l^2}^2\right) + \frac{\alpha}{2\varepsilon^2}\left(\|v^{n+1}\|_{l^2}^2 + \|v^n\|_{l^2}^2\right) \\
=&E^0 - h\sum_{j=0}^{M-1}\left[\frac{\beta_1}{4}\left|u_j^{n+1}\right|^4 + \frac{\beta_2}{2}\left|u_j^{n+1}\right|^2\left|v_j^n\right|^2 + \frac{\beta_3}{4}\left|v_j^n\right|^4\right. \\
&\left.+\frac{\beta_1}{4}\left|u_j^n\right|^4 + \frac{\beta_2}{2}\left|u_j^n\right|^2\left|v_j^{n+1}\right|^2 + \frac{\beta_3}{4}\left|v_j^{n+1}\right|^4\right] \\
\lesssim&1 - h\sum_{j=0}^{M-1}\left[\frac{\beta_1}{4}\left|\frac{u_j^{n+1}}{v_j^n}\right|^4 + \frac{\beta_2}{2}\left|\frac{u_j^{n+1}}{v_j^n}\right|^2 + \frac{\beta_3}{4}\right] \\
&- h\sum_{j=0}^{M-1}\left[\frac{\beta_1}{4} + \frac{\beta_2}{2}\left|\frac{v_j^{n+1}}{u_j^n}\right|^2 + \frac{\beta_3}{4}\left|\frac{v_j^{n+1}}{u_j^n}\right|^4\right]. \qquad (24)
\end{aligned}$$

Under the condition (b), it follows from (24) that

$$\begin{aligned}
&\varepsilon^2\|\delta_t^+ u^n\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+ u^{n+1}\|_{l^2}^2 + \|\delta_x^+ u^n\|_{l^2}^2\right) + \frac{1}{2\varepsilon^2}\left(\|u^{n+1}\|_{l^2}^2 + \|u^n\|_{l^2}^2\right) \\
&+ \varepsilon^2\|\delta_t^+ v^n\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+ v^{n+1}\|_{l^2}^2 + \|\delta_x^+ v^n\|_{l^2}^2\right) + \frac{\alpha}{2\varepsilon^2}\left(\|v^{n+1}\|_{l^2}^2 + \|v^n\|_{l^2}^2\right) \\
&\lesssim 1 + \frac{\beta_1\beta_3 - \beta_2^2}{-4\beta_1} + \frac{\beta_1\beta_3 - \beta_2^2}{-4\beta_3}.
\end{aligned}$$

Note that the above process also holds when $\beta_1\beta_3 = \beta_2^2$.

The proof is completed. □

Remark 2.1. *If the coefficients $\beta_1, \beta_2, \beta_3$ satisfy that $\beta_1 < 0, \beta_3 < 0, \beta_1\beta_3 > \beta_2^2$, the boundedness of the numerical solutions cannot be obtained. Indeed, from (24), we get*

$$\begin{aligned}
&\varepsilon^2\|\delta_t^+ u^n\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+ u^{n+1}\|_{l^2}^2 + \|\delta_x^+ u^n\|_{l^2}^2\right) + \frac{1}{2\varepsilon^2}\left(\|u^{n+1}\|_{l^2}^2 + \|u^n\|_{l^2}^2\right) \\
&+ \varepsilon^2\|\delta_t^+ v^n\|_{l^2}^2 + \frac{1}{2}\left(\|\delta_x^+ v^{n+1}\|_{l^2}^2 + \|\delta_x^+ v^n\|_{l^2}^2\right) + \frac{\alpha}{2\varepsilon^2}\left(\|v^{n+1}\|_{l^2}^2 + \|v^n\|_{l^2}^2\right) \\
>&\frac{\beta_1\beta_3 - \beta_2^2}{-4\beta_1} + \frac{\beta_1\beta_3 - \beta_2^2}{-4\beta_3}.
\end{aligned}$$

In this case, the boundedness of the numerical solution in the L^∞-norm directly could not be obtained. Hence, the error estimates are difficult to derive. In next section, the cut-off technique [10, 11, 25] *will be adopted to get the error estimates of the numerical scheme, and the analytical method can be directly applied to the higher dimensional cases.*

3. Error estimates

In this section, we will give the error analysis for the numerical method. We first make the following assumptions on the exact solutions u and v of the system (7):

(25a)

$$u \in C^4([0,T]; W^{1,\infty}) \cap C^3([0,T]; W^{2,\infty}) \cap C^2([0,T]; W^{3,\infty}) \cap C([0,T]; W_p^{5,\infty}),$$

(25b)
$$v \in C^4([0,T];W^{1,\infty}) \cap C^3([0,T];W^{2,\infty}) \cap C^2([0,T];W^{3,\infty}) \cap C([0,T];W_p^{5,\infty}),$$
(25c)
$$\left\| \frac{\partial^{r+s}}{\partial t^r \partial x^s} u(x,t) \right\|_{L^\infty(\Omega_T)} \lesssim \frac{1}{\varepsilon^{2r}},\ 0 \le r \le 4,\ 0 \le r+s \le 5,$$

where $W_p^{m,\infty}$ is the $W^{m,\infty}$ with periodic boundary condition, $\Omega_T = \Omega \times [0,T]$.

Denote the errors ${}_u e_j^n, {}_v e_j^n$ and local truncation errors for the numerical scheme (14) as

$$\begin{aligned}
{}_u e_j^n =& u(x_j,t_n) - u_j^n, \quad {}_v e_j^n = v(x_j,t_n) - v_j^n,\\
A_j^0 =& \delta_t^+ u(x_j,0) - \frac{1}{\varepsilon^2}\zeta(x_j) - \frac{\tau}{2\varepsilon^2}\left[\delta_x^2 \xi(x_j) - \frac{1}{\varepsilon^2}\xi(x_j) - f_1(\xi(x_1),\rho(x_j))\right],\\
B_j^0 =& \delta_t^+ v(x_j,0) - \frac{\alpha}{\varepsilon^2}\eta(x_j) - \frac{\tau}{2\varepsilon^2}\left[\delta_x^2 \rho(x_j) - \frac{\alpha}{\varepsilon^2}\rho(x_j) - f_2(\xi(x_1),\rho(x_j))\right],\\
A_j^n =& \varepsilon^2 \delta_t^2 u(x_j,t_n) - \frac{1}{2}\left[\delta_x^2 u(x_j,t_{n+1}) + \delta_x^2 u(x_j,t_{n-1})\right]\\
&+ \frac{1}{2\varepsilon^2}\left[u(x_j,t_{n+1}) + u(x_j,t_{n-1})\right] + G_1\left(u(x_j,t_{n+1}), u(x_j,t_{n-1}), v(x_j,t_n)\right),\\
B_j^n =& \varepsilon^2 \delta_t^2 v(x_j,t_n) - \frac{1}{2}\left[\delta_x^2 v(x_j,t_{n+1}) + \delta_x^2 v(x_j,t_{n-1})\right]\\
&+ \frac{\alpha}{2\varepsilon^2}\left[v(x_j,t_{n+1}) + v(x_j,t_{n-1})\right] + G_2\left(u(x_j,t_n), v(x_j,t_{n+1}), v(x_j,t_{n-1})\right).
\end{aligned}$$

The error estimate of the numerical scheme (14) can be obtained as follows.

Theorem 3.1. *Under assumptions (25), there exist sufficiently small, positive constants τ_0, h_0 independent of ε such that for any $0 < \varepsilon \le 1$, we have the following error estimate, for $\tau \in (0,\tau_0], h \in (0,h_0]$,*

(26) $$\|{}_u e_j^n\|_{l^2} + \alpha\|{}_v e_j^n\|_{l^2} + \|\delta_x^+ {}_u e_j^n\|_{l^2} + \alpha\|\delta_x^+ {}_v e_j^n\|_{l^2} \lesssim \frac{\tau^2}{\varepsilon^6} + h^2.$$

To prove the convergence of this discrete problem, we need the following lemmas.

Lemma 3.1. *(Local truncation errors) Under the assumption* (25), *we have*

(27) $$|A_j^0| \lesssim \frac{\tau^2}{\varepsilon^6} + h^2, \quad |\delta_x^+ A_j^0| \lesssim \frac{\tau^2}{\varepsilon^6} + h^2, \quad |\delta_x^2 A_j^0| \lesssim \frac{\tau^2}{\varepsilon^6} + h^2,$$

(28) $$|A_j^n| \lesssim h^2 + \frac{\tau^2}{\varepsilon^6}, \quad |\delta_x^+ A_j^n| \lesssim h^2 + \frac{\tau^2}{\varepsilon^6},$$

(29) $$|B_j^0| \lesssim \frac{\tau^2}{\varepsilon^6} + h^2, \quad |\delta_x^+ B_j^0| \lesssim \frac{\tau^2}{\varepsilon^6} + h^2, \quad |\delta_x^2 B_j^0| \lesssim \frac{\tau^2}{\varepsilon^6} + h^2,$$

(30) $$|B_j^n| \lesssim h^2 + \frac{\tau^2}{\varepsilon^6}, \quad |\delta_x^+ B_j^n| \lesssim h^2 + \frac{\tau^2}{\varepsilon^6}.$$

Proof. Noticing that

$$\begin{aligned}
\varepsilon^2 \partial_{tt} u(x_j,0) - \partial_{xx} u(x_j,0) &= -\frac{1}{\varepsilon^2} u(x_j,0) - f_1(u(x_j,0), v(x_j,0)),\\
\varepsilon^2 \partial_{tt} v(x_j,0) - \partial_{xx} v(x_j,0) &= -\frac{\alpha}{\varepsilon^2} v(x_j,0) - f_2(u(x_j,0), v(x_j,0)),
\end{aligned}$$

and using Taylor's expansion, we obtain

$$A_j^0 = \frac{u(x_j,t_1) - u(x_j,0)}{\tau} - \partial_t u(x_j,0) - \frac{\tau}{2}\partial_{tt} u(x_j,0)$$

$$
\begin{aligned}
&- \frac{\tau}{2\varepsilon^2}\left[\delta_x^2 u(x_j,0) - \partial_{xx}u(x_j,0)\right] \\
=&\frac{1}{2\tau}\int_0^{t_1}(t_1-s)^2 ds - \frac{\tau}{2\varepsilon^2}\left[\frac{1}{2h^2}\int_{x_{j+1}}^{x_j}(x_{j+1}-s)^2\partial_{xxx}u(s,0)ds\right. \\
&\left.+\frac{1}{2h^2}\int_{x_{j-1}}^{x_j}(x_{j-1}-s)^2\partial_{xxx}u(s,0)ds\right]. \qquad (31)
\end{aligned}
$$

By Young's inequality, we get

$$
|A_j^0| \le \frac{\tau^2}{6}\|\partial_{ttt}u\|_{L^\infty} + \frac{\tau h}{6\varepsilon^2}\|\partial_{xxx}u\|_{L^\infty} \le \frac{\tau^2}{6\varepsilon^6} + \frac{\tau h}{6\varepsilon^2} \lesssim \frac{\tau^2}{\varepsilon^6} + h^2, \qquad (32)
$$

$$
|\delta_x^+ A_j^0| \le \frac{\tau^2}{6}\|\partial_{tttx}u\|_{L^\infty} + \frac{\tau h}{6\varepsilon^2}\|\partial_{xxxx}u\|_{L^\infty} \lesssim \frac{\tau^2}{\varepsilon^6} + h^2, \qquad (33)
$$

$$
|\delta_x^2 A_j^0| \le \frac{\tau^2}{6}\|\partial_{tttxx}u\|_{L^\infty} + \frac{\tau h}{6\varepsilon^2}\|\partial_{xxxxx}u\|_{L^\infty} \lesssim \frac{\tau^2}{\varepsilon^6} + h^2. \qquad (34)
$$

For the term A_j^n, we have

$$
\begin{aligned}
A_j^n =&\varepsilon^2\delta_t^2 u(x_j,t_n) - \varepsilon^2\partial_{tt}u(x_j,t_n) + \partial_{xx}u(x_j,t_n) - \frac{1}{\varepsilon^2}u(x_j,t_n) \\
&- \frac{1}{2}\left[\delta_x^2 u(x_j,t_{n+1}) + \delta_x^2 u(x_j,t_{n-1})\right] + \frac{1}{2\varepsilon^2}\left[u(x_j,t_{n+1}) + u(x_j,t_{n-1})\right] \\
&+ G_1\left(u(x_j,t_{n+1}),u(x_j,t_{n-1}),v(x_j,t_n)\right) - f_1\left(u(x_j,t_n),v(x_j,t_n)\right). \qquad (35)
\end{aligned}
$$

By Taylor's expansion, it yields that

$$
\begin{aligned}
&G_1\left(u(x_j,t_{n+1}),u(x_j,t_{n-1}),v(x_j,t_n)\right) - f_1\left(u(x_j,t_n),v(x_j,t_n)\right) \\
=&\int_0^1 f_1\left(\theta u(x_j,t_{n+1}) + (1-\theta)u(x_j,t_{n-1}),v(x_j,t_n)\right)d\theta \\
&- f_1\left(u(x_j,t_n),v(x_j,t_n)\right) \\
=&\left[\frac{1}{2}u(x_j,t_{n+1}) + \frac{1}{2}u(x_j,t_{n-1}) - u(x_j,t_n)\right]\partial_u f_1\left(u(x_j,t_n),v(x_j,t_n)\right) \\
&+\left[\frac{1}{3}u^2(x_j,t_{n+1})\frac{1}{3}u^2(x_j,t_{n-1}) + \frac{1}{3}u(x_j,t_{n+1})u(x_j,t_{n-1}) + u^2(x_j,t_n)\right. \\
&\left.- u(x_j,t_{n+1})u(x_j,t_n) - u(x_j,t_n)u(x_j,t_{n-1})\right]\partial_{uu}f_1\left(u(x_j,t_n),v(x_j,t_n)\right) \\
&+ R_n. \qquad (36)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&|G_1(u(x_j,t_{n+1}),u(x_j,t_{n-1}),v(x_j,t_n)) - f_1(u(x_j,t_n),v(x_j,t_n))| \\
\le&\frac{\tau^2}{2}\|\partial_{tt}u\|_{L^\infty} + \tau^2\|\partial_t u\|_{L^\infty}. \qquad (37)
\end{aligned}
$$

For the other term of A_j^n, we obtain

$$
\varepsilon^2|\delta_t^2 u(x_j,t_n) - \partial_{tt}u(x_j,t_n)| \le \frac{\varepsilon^2\tau^2}{12}\|\partial_{tttt}u\|_{L^\infty}, \qquad (38)
$$

$$
\begin{aligned}
&\left|-\frac{1}{2}\left[\delta_x^2 u(x_j,t_{n+1}) + \delta_x^2 u(x_j,t_{n-1})\right] + \partial_{xx}u(x_j,t_n)\right| \\
\le&\frac{\tau^2}{2}\|\partial_{xxtt}u\|_{L^\infty} + \frac{h^2}{12}\|\partial_{xxxx}u\|_{L^\infty}, \qquad (39)
\end{aligned}
$$

$$\frac{1}{\varepsilon^2}\left[\frac{1}{2}\left(u(x_j,t_{n+1})+u(x_j,t_{n-1})\right)-u(x_j,t_n)\right]\le\frac{\tau^2}{2\varepsilon^2}\|\partial_{tt}u\|_{L^\infty}. \tag{40}$$

We can get

$$\begin{aligned}|A_j^n|\le&\frac{\varepsilon^2\tau^2}{12}\|\partial_{tttt}u\|_{L^\infty}+\frac{\tau^2}{2}\|\partial_{xxtt}u\|_{L^\infty}+\frac{h^2}{12}\|\partial_{xxxx}u\|_{L^\infty}\\&+\frac{\tau^2}{2\varepsilon^2}\|\partial_{tt}u\|_{L^\infty}+\frac{\tau^2}{2}\|\partial_{tt}u\|_{L^\infty}\|\partial_u f_1\|_{L^\infty}+\tau^2\|\partial_t u\|^2_{L^\infty}\|\partial_{uu}f_1\|_{L^\infty}\\\le&\frac{\varepsilon^2\tau^2}{12}\frac{1}{\varepsilon^8}+\frac{\tau^2}{2}\frac{1}{\varepsilon^4}+\frac{h^2}{12}+\frac{\tau^2}{2\varepsilon^2}\frac{1}{\varepsilon^4}+\frac{\tau^2}{2\varepsilon^4}+\frac{\tau^2}{\varepsilon^4}\\\lesssim&h^2+\frac{\tau^2}{\varepsilon^6}.\end{aligned} \tag{41}$$

Similarly, we have

$$|\delta_x^+A_j^n|\lesssim h^2+\frac{\tau^2}{\varepsilon^6}. \tag{42}$$

Also, we can obtain

$$\begin{aligned}&|B_j^0|\lesssim\frac{\tau^2}{\varepsilon^6}+h^2,\qquad|\delta_x^+B_j^0|\lesssim\frac{\tau^2}{\varepsilon^6}+h^2,\qquad|\delta_x^2B_j^0|\lesssim\frac{\tau^2}{\varepsilon^6}+h^2,\\&|B_j^n|\lesssim\frac{\tau^2}{\varepsilon^6}+h^2,\qquad|\delta_x^+B_j^n|\lesssim\frac{\tau^2}{\varepsilon^6}+h^2.\end{aligned}$$

This proof is completed. □

Next, we introduce the cut-off technique to truncate the nonlinearity into a global Lipschitz function with compact support.

Define $M_0=\max\{\|u(x,t)\|_{L^\infty},\|v(x,t)\|_{L^\infty}\},\ B=M_0+1$ and

$$\rho(s)\ =\ s\begin{cases}1, & 0\le|s|\le1,\\ \in[0,1], & 1\le|s|\le2,\\ 0, & |s|\ge2.\end{cases} \tag{43}$$

Define

$$f_{1B}(s,v)=f_1(s,v)\rho\left(\frac{s}{B}\right),\qquad f_{2B}(u,s)=f_2(u,s)\rho(\frac{s}{B}), \tag{44}$$

and

$$F_{1B}(s,v)=\int_0^s f_{1B}(\sigma,v)d\sigma,\qquad F_{2B}(u,s)=\int_0^s f_{2B}(u,\sigma)d\sigma. \tag{45}$$

Setting $\hat{u}^0=u^0,\hat{u}^1=u^1,\hat{v}^0=v^0,\hat{v}^1=v^1$, then we construct the following numerical scheme, which is to find $\hat{u}^{n+1}\in X_M$ and $\hat{v}^{n+1}\in X_M$, such that

$$\varepsilon^2\delta_t^2\hat{u}_j^n-\frac{1}{2}\delta_x^2(\hat{u}_j^{n+1}+\hat{u}_j^{n-1})+\frac{1}{2\varepsilon^2}(\hat{u}_j^{n+1}+\hat{u}_j^{n-1})+\hat{G}_1(\hat{u}_j^{n+1},\hat{u}_j^{n-1},\hat{v}_j^n)=0, \tag{46}$$

$$\varepsilon^2\delta_t^2\hat{v}_j^n-\frac{1}{2}\delta_x^2(\hat{v}_j^{n+1}+\hat{v}_j^{n-1})+\frac{\alpha}{2\varepsilon^2}(\hat{v}_j^{n+1}+\hat{v}_j^{n-1})+\hat{G}_2(\hat{u}_j^n,\hat{v}_j^{n+1},\hat{v}_j^{n-1})=0, \tag{47}$$

where

$$\begin{aligned}\hat{G}_1(u_1,u_2,v)&=\int_0^1 f_{1B}(\theta u_1+(1-\theta)u_2,v)d\theta=\frac{F_{1B}(u_1,v)-F_{1B}(u_2,v)}{u_1-u_2},\\\hat{G}_2(u,v_1,v_2)&=\int_0^1 f_{2B}(u,\theta v_1+(1-\theta)v_2)d\theta=\frac{F_{2B}(u,v_1)-F_{2B}(u,v_2)}{v_1-v_2}.\end{aligned}$$

By the definitions of $\hat{G}_1$ and $\hat{G}_2$, we have

$$G_1\left(u(x_j,t_{n+1}),u(x_j,t_{n-1}),v(x_j,t_n)\right)=\hat{G}_1\left(u(x_j,t_{n+1}),u(x_j,t_{n-1}),v(x_j,t_n)\right),$$
$$G_2\left(u(x_j,t_n),v(x_j,t_{n+1}),v(x_j,t_{n-1})\right)=\hat{G}_2\left(u(x_j,t_n),v(x_j,t_{n+1}),v(x_j,t_{n-1})\right).$$

Define

$$\begin{aligned}
{}_u\hat{e}_j^n &= u(x_j,t_n)-\hat{u}_j^n,\\
{}_v\hat{e}_j^n &= v(x_j,t_n)-\hat{v}_j^n,\\
\eta_{1j}^n &= \hat{G}_1\left(u(x_j,t_{n+1}),u(x_j,t_{n-1}),v(x_j,t_n)\right)-\hat{G}_1(\hat{u}_j^{n+1},\hat{u}_j^{n-1},\hat{v}_j^n),\\
\eta_{2j}^n &= \hat{G}_2\left(u(x_j,t_n),v(x_j,t_{n+1}),v(x_j,t_{n-1})\right)-\hat{G}_2(\hat{u}_j^n,\hat{v}_j^{n+1},\hat{u}_j^{n-1}).
\end{aligned}$$

We can get the following error equations

$$\varepsilon^2\delta_t^2\,{}_u\hat{e}_j^n-\frac{1}{2}\delta_x^2({}_u\hat{e}_j^{n+1}+{}_u\hat{e}_j^{n-1})+\frac{1}{2\varepsilon^2}({}_u\hat{e}_j^{n+1}+{}_u\hat{e}_j^{n-1})=A_j^n-\eta_{1j}^n, \tag{48a}$$

$$\varepsilon^2\delta_t^2\,{}_v\hat{e}_j^n-\frac{1}{2}\delta_x^2({}_v\hat{e}_j^{n+1}+{}_v\hat{e}_j^{n-1})+\frac{\alpha}{2\varepsilon^2}({}_v\hat{e}_j^{n+1}+{}_v\hat{e}_j^{n-1})=B_j^n-\eta_{2j}^n, \tag{48b}$$

$${}_u\hat{e}_j^0=0,\qquad {}_u\hat{e}_j^1=\tau A_j^0, \tag{48c}$$

$${}_v\hat{e}_j^0=0,\qquad {}_v\hat{e}_j^1=\tau B_j^0. \tag{48d}$$

Lemma 3.2. *For η_{1j}^n and η_{2j}^n, there holds*

$$\|\eta_1^n\|_{l^2}^2\leq\|{}_u\hat{e}^{n+1}\|_{l^2}^2+\|{}_u\hat{e}^{n-1}\|_{l^2}^2+\|{}_v\hat{e}^n\|_{l^2}^2, \tag{49a}$$

$$\begin{aligned}\|\delta_x^+\eta_1^n\|_{l^2}^2\leq&\|{}_u\hat{e}^{n+1}\|_{l^2}^2+\|\delta_x^+\,{}_u\hat{e}^{n+1}\|_{l^2}^2+\|{}_u\hat{e}^{n-1}\|_{l^2}^2\\
&+\|\delta_x^+\,{}_u\hat{e}^{n-1}\|_{l^2}^2+\|{}_v\hat{e}^n\|_{l^2}^2+\|\delta_x^+\,{}_v\hat{e}^n\|_{l^2}^2,\end{aligned} \tag{49b}$$

$$\|\eta_2^n\|_{l^2}^2\leq\|{}_v\hat{e}^{n+1}\|_{l^2}^2+\|{}_v\hat{e}^{n-1}\|_{l^2}^2+\|{}_u\hat{e}^n\|_{l^2}^2, \tag{49c}$$

$$\begin{aligned}\|\delta_x^+\eta_2^n\|_{l^2}^2\leq&\|{}_v\hat{e}^{n+1}\|_{l^2}^2+\|\delta_x^+\,{}_v\hat{e}^{n+1}\|_{l^2}^2+\|{}_v\hat{e}^{n-1}\|_{l^2}^2\\
&+\|\delta_x^+\,{}_v\hat{e}^{n-1}\|_{l^2}^2+\|{}_u\hat{e}^n\|_{l^2}^2+\|\delta_x^+\,{}_u\hat{e}^n\|_{l^2}^2.\end{aligned} \tag{49d}$$

Proof. A direct calculation gives

$$\begin{aligned}|\eta_{1j}^n|=&\left|\hat{G}_1(u(x_j,t_{n+1}),u(x_j,t_{n-1}),v(x_j,t_n))-\hat{G}_1(\hat{u}_j^{n+1},\hat{u}_j^{n-1},\hat{v}_j^n)\right|\\
=&\left|\int_0^1 f_{1B}(\theta u(x_j,t_{n+1})+(1-\theta)u(x_j,t_{n-1}),v(x_j,t_n))\right.\\
&\left.-f_{1B}(\theta\hat{u}_j^{n+1}+(1-\theta)\hat{u}_j^{n-1},\hat{v}_j^n)d\theta\right|\\
\lesssim&|{}_u\hat{e}^{n+1}|+|{}_u\hat{e}^{n-1}|+|{}_v\hat{e}^n|.\end{aligned}$$

Then, (49a)-(49d) hold obviously. Thus, this proof is completed. □

Lemma 3.3. *For the error equations* (48)*, there holds*

$$\|{}_u\hat{e}^n\|_{l^2}^2+\alpha\|{}_v\hat{e}^n\|_{l^2}^2+\|\delta_x^+\,{}_u\hat{e}^n\|_{l^2}^2+\alpha\|\delta_x^+\,{}_v\hat{e}^n\|_{l^2}^2\lesssim\left(\frac{\tau^2}{\varepsilon^6}+h^2\right)^2. \tag{50}$$

Proof. Define the 'energy' for the error

$$\begin{aligned}\mathbb{E}^n=&\varepsilon^2\|\delta_t^+\,{}_u\hat{e}^n\|_{l^2}^2+\frac{1}{2}\left(\|\delta_x^+\,{}_u\hat{e}^{n+1}\|_{l^2}^2+\|\delta_x^+\,{}_u\hat{e}^n\|_{l^2}^2\right)+\frac{1}{2\varepsilon^2}\left(\|{}_u\hat{e}^{n+1}\|_{l^2}^2+\|{}_u\hat{e}^n\|_{l^2}^2\right)\\
&+\varepsilon^2\|\delta_t^+\,{}_v\hat{e}^n\|_{l^2}^2+\frac{1}{2}\left(\|\delta_x^+\,{}_v\hat{e}^{n+1}\|_{l^2}^2+\|\delta_x^+\,{}_v\hat{e}^n\|_{l^2}^2\right)+\frac{\alpha}{2\varepsilon^2}\left(\|{}_v\hat{e}^{n+1}\|_{l^2}^2+\|{}_v\hat{e}^n\|_{l^2}^2\right).\end{aligned} \tag{51}$$

Multiplying (48a) and (48b) by $h({}_u\hat{e}^{n+1} - {}_u\hat{e}^{n-1})$ and $h({}_v\hat{e}^{n+1} - {}_v\hat{e}^{n-1})$, respectively, and summing up for $j = 0, \cdots, M-1$, we have

$$\begin{aligned}
&\varepsilon^2 h \sum_{j=0}^{M-1} \delta_t^2 {}_u\hat{e}_j^n ({}_u\hat{e}^{n+1} - {}_u\hat{e}^{n-1}) \\
&- \frac{h}{2} \sum_{j=0}^{M-1} \delta_x^2 ({}_u\hat{e}_j^{n+1} + {}_u\hat{e}_j^{n-1})({}_u\hat{e}^{n+1} - {}_u\hat{e}^{n-1}) \\
&+ \frac{h}{2\varepsilon^2} \sum_{j=0}^{M-1} ({}_u\hat{e}_j^{n+1} + {}_u\hat{e}_j^{n-1})({}_u\hat{e}^{n+1} - {}_u\hat{e}^{n-1}) \\
=& h \sum_{j=0}^{M-1} (A_j^n - \eta_{1j}^n)({}_u\hat{e}^{n+1} - {}_u\hat{e}^{n-1}),
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
&\varepsilon^2 h \sum_{j=0}^{M-1} \delta_t^2 {}_v\hat{e}_j^n ({}_v\hat{e}^{n+1} - {}_v\hat{e}^{n-1}) \\
&- \frac{h}{2} \sum_{j=0}^{M-1} \delta_x^2 ({}_v\hat{e}_j^{n+1} + {}_v\hat{e}_j^{n-1})({}_v\hat{e}^{n+1} - {}_v\hat{e}^{n-1}) \\
&+ \frac{\alpha h}{2\varepsilon^2} \sum_{j=0}^{M-1} ({}_v\hat{e}_j^{n+1} + {}_v\hat{e}_j^{n-1})({}_v\hat{e}^{n+1} - {}_v\hat{e}^{n-1}) \\
=& h \sum_{j=0}^{M-1} (B_j^n - \eta_{2j}^n)({}_v\hat{e}^{n+1} - {}_v\hat{e}^{n-1}).
\end{aligned} \tag{53}$$

From (52) and (53), we have

$$\begin{aligned}
\mathbb{E}^n - \mathbb{E}^{n-1} &= h \sum_{j=0}^{M-1} (A_j^n - \eta_{1j}^n)({}_u\hat{e}^{n+1} - {}_u\hat{e}^{n-1}) + h \sum_{j=0}^{M-1} (B_j^n - \eta_{2j}^n)({}_v\hat{e}^{n+1} - {}_v\hat{e}^{n-1}) \\
&:= \mathscr{A}_1 + \mathscr{A}_2.
\end{aligned} \tag{54}$$

For the term $\mathscr{A}_1$,

$$\begin{aligned}
\mathscr{A}_1 &= h \sum_{j=0}^{M-1} (A_j^n - \eta_{1j}^n)({}_u\hat{e}^{n+1} - {}_u\hat{e}^{n-1}) \\
&\leq h \sum_{j=0}^{M-1} \left(|A_j^n| + |\eta_{1j}^n|\right) \left|{}_u\hat{e}^{n+1} - {}_u\hat{e}^{n-1}\right| \\
&= \tau h \sum_{j=0}^{M-1} \left(|A_j^n| + |\eta_{1j}^n|\right) \left|\delta_t^+ {}_u\hat{e}^n + \delta_t^+ {}_u\hat{e}^{n-1}\right| \\
&\leq \tau \left[\frac{1}{\varepsilon^2} (\|A^n\|_{l^2}^2 + \|\eta_1^n\|_{l^2}^2) + \varepsilon^2 (\|\delta_t^+ {}_u\hat{e}^n\|_{l^2}^2 + \|\delta_t^+ {}_u\hat{e}^{n-1}\|_{l^2}^2) \right].
\end{aligned} \tag{55}$$

Similarly, we have

$$\mathscr{A}_2 \leq \tau \left[\frac{1}{\varepsilon^2} (\|B^n\|_{l^2}^2 + \|\eta_2^n\|_{l^2}^2) + \varepsilon^2 (\|\delta_t^+ {}_v\hat{e}^n\|_{l^2}^2 + \|\delta_t^+ {}_v\hat{e}^{n-1}\|_{l^2}^2) \right]. \tag{56}$$

Plugging (55) and (56) into (54), and using (49a) and (49c), we obtain

$$\begin{aligned}
&\mathbb{E}^n - \mathbb{E}^{n-1} \\
\le&\tau\Big[\frac{1}{\varepsilon^2}(\|A^n\|_{l^2}^2 + \|B^n\|_{l^2}^2) \\
&+ \varepsilon^2(\|\delta_t^+ {}_u\hat{e}^n\|_{l^2}^2 + \|\delta_t^+ {}_u\hat{e}^{n-1}\|_{l^2}^2 + \|\delta_t^+ {}_v\hat{e}^n\|_{l^2}^2 + \|\delta_t^+ {}_v\hat{e}^{n-1}\|_{l^2}^2)\Big] \\
&+ \frac{\tau}{\varepsilon^2}\big(\|{}_u\hat{e}^{n+1}\|_{l^2}^2 + \|{}_u\hat{e}^{n-1}\|_{l^2}^2 + \|{}_v\hat{e}^n\|_{l^2}^2 \\
&+\|{}_v\hat{e}^{n+1}\|_{l^2}^2 + \|{}_v\hat{e}^{n-1}\|_{l^2}^2 + \|{}_u\hat{e}^n\|_{l^2}^2\big) \\
\le&\tau(\mathbb{E}^n + \mathbb{E}^{n-1}) + \frac{\tau}{\varepsilon^2}\left(h^2 + \frac{\tau^2}{\varepsilon^2}\right)^2. \qquad (57)
\end{aligned}$$

There exists a positive constant τ_0 independent on ε and h, such that for $\tau \in (0, \tau_0)$,

$$\mathbb{E}^n - \mathbb{E}^{n-1} \le \tau\mathbb{E}^{n-1} + \frac{\tau}{\varepsilon^2}\left(h^2 + \frac{\tau^2}{\varepsilon^2}\right)^2. \qquad (58)$$

Form (58), we have

$$\mathbb{E}^n - \mathbb{E}^0 \le \tau\sum_{m=0}^{n-1}\mathbb{E}^m + \frac{T}{\varepsilon^2}\left(h^2 + \frac{\tau^2}{\varepsilon^2}\right)^2. \qquad (59)$$

By Gronwall's inequality in (59), we get

$$\mathbb{E}^n \le \mathbb{E}^0 + \frac{T}{\varepsilon^2}\left(h^2 + \frac{\tau^2}{\varepsilon^2}\right)^2. \qquad (60)$$

For $n = 0$, we have

$$\begin{aligned}
\mathbb{E}^0 =&\varepsilon^2\|\delta_t^+ {}_u\hat{e}^0\|_{l^2}^2 + \frac{1}{2}(\|\delta_x^+ {}_u\hat{e}^1\|_{l^2}^2 + \|\delta_x^+ {}_u\hat{e}^0\|_{l^2}^2) + \frac{1}{2\varepsilon^2}(\|{}_u\hat{e}^1\|_{l^2}^2 + \|{}_u\hat{e}^0\|_{l^2}^2) \\
&+ \varepsilon^2\|\delta_t^+ {}_v\hat{e}^0\|_{l^2}^2 + \frac{1}{2}(\|\delta_x^+ {}_v\hat{e}^1\|_{l^2}^2 + \|\delta_x^+ {}_v\hat{e}^0\|_{l^2}^2) + \frac{\alpha}{2\varepsilon^2}(\|{}_v\hat{e}^1\|_{l^2}^2 + \|{}_v\hat{e}^0\|_{l^2}^2) \\
=&\varepsilon^2\|A^0\|_{l^2}^2 + \frac{\tau^2}{2}\|\delta_x^+ A^0\|_{l^2}^2 + \frac{\tau^2}{2\varepsilon^2}\|A^0\|_{l^2}^2 \\
&+ \varepsilon^2\|B^0\|_{l^2}^2 + \frac{\tau^2}{2}\|\delta_x^+ B^0\|_{l^2}^2 + \frac{\alpha\tau^2}{2\varepsilon^2}\|B^0\|_{l^2}^2 \\
\lesssim&\left[1 + \frac{(1+\alpha)\tau^2}{\varepsilon^2}\right]\left(\frac{\tau^2}{\varepsilon^6} + h^2\right)^2. \qquad (61)
\end{aligned}$$

From (60) and (61), we get

$$\mathbb{E}^n \lesssim \frac{1}{\varepsilon^2}\left(\frac{\tau^2}{\varepsilon^6} + h^2\right)^2. \qquad (62)$$

In addition, define another 'energy' for the error as

$$\begin{aligned}
\widetilde{\mathbb{E}}^n =&\varepsilon^2\|\delta_x^+\delta_t^+ {}_u\hat{e}^n\|_{l^2}^2 + \frac{1}{2}(\|\delta_{x}^2{}_u\hat{e}^{n+1}\|_{l^2}^2 + \|\delta_{x}^2{}_u\hat{e}^n\|_{l^2}^2) \\
&+ \frac{1}{2\varepsilon^2}(\|\delta_x^+ {}_u\hat{e}^{n+1}\|_{l^2}^2 + \|\delta_x^+ {}_u\hat{e}^n\|_{l^2}^2) + \varepsilon^2\|\delta_x^+\delta_t^+ {}_v\hat{e}^n\|_{l^2}^2 x \\
&+ \frac{1}{2}(\|\delta_{x}^2{}_v\hat{e}^{n+1}\|_{l^2}^2 + \|\delta_{x}^2{}_v\hat{e}^n\|_{l^2}^2) + \frac{\alpha}{2\varepsilon^2}(\|\delta_x^+ {}_v\hat{e}^{n+1}\|_{l^2}^2 + \|\delta_x^+ {}_v\hat{e}^n\|_{l^2}^2). \qquad (63)
\end{aligned}$$

Multiplying (48a) and (48b) by $h(\delta_{xu}^2\hat{e}^{n+1}-\delta_{xu}^2\hat{e}^{n-1})$ and $h(\delta_{xv}^2\hat{e}^{n+1}-\delta_{xv}^2\hat{e}^{n-1})$, respectively, and similar to the above procedure, we have

$$\widetilde{\mathbb{E}}^n \lesssim \frac{1}{\varepsilon^2}\left(\frac{\tau^2}{\varepsilon^6}+h^2\right)^2. \tag{64}$$

Noticing that

$$(\|_u\hat{e}^{n+1}\|_{l^2}^2+\|_u\hat{e}^n\|_{l^2}^2)+\alpha(\|_v\hat{e}^{n+1}\|_{l^2}^2+\|_v\hat{e}^n\|_{l^2}^2)\le 2\varepsilon^2\mathbb{E}^n$$

and

$$(\|\delta_x^+{}_u\hat{e}^{n+1}\|_{l^2}^2+\|\delta_x^+{}_u\hat{e}^n\|_{l^2}^2)+\alpha(\|\delta_x^+{}_v\hat{e}^{n+1}\|_{l^2}^2+\|\delta_x^+{}_v\hat{e}^n\|_{l^2}^2)\le 2\varepsilon^2\widetilde{\mathbb{E}}^n,$$

we obtain the error estimate

$$\|_u\hat{e}^n\|_{l^2}^2+\alpha\|_v\hat{e}^n\|_{l^2}^2+\|\delta_x^+{}_u\hat{e}^n\|_{l^2}^2+\alpha\|\delta_x^+{}_v\hat{e}^n\|_{l^2}^2\lesssim\left(\frac{\tau^2}{\varepsilon^6}+h^2\right)^2. \tag{65}$$

From (65), we have

$$\|_u\hat{e}^n\|_{l^\infty}+\|_v\hat{e}^n\|_{l^\infty}\le\|\delta_x^+{}_u\hat{e}^n\|_{l^2}+\|\delta_x^+{}_v\hat{e}^n\|_{l^2}\le 1, \tag{66}$$

provided that $h\le h_0$ with a positive constant h_0, and $\tau=o(\varepsilon^3)$. Hence, it follows that

$$\begin{aligned}\|\hat{u}^n\|_{l^\infty}+\|\hat{v}^n\|_{l^\infty}&\le\|u(\cdot,t_n)\|_{l^\infty}+\|v(\cdot,t_n)\|_{l^\infty}+\|_u\hat{e}^n\|_{l^\infty}+\|_v\hat{e}^n\|_{l^\infty}\\&\le M_0+1,\end{aligned} \tag{67}$$

This proof is completed. □

Based on the above analysis, we can get the result given in Theorem 3.1.

Table 1 The discrete energy and relative error at different times with $\varepsilon=0.05, h=1/64, \tau=1/100$.

t	$E_h(t)$	$\Delta E_h(t)$
0	3.234473587834926E+04	0
5	3.234473587833564E+04	4.213318872851796E-13
8	3.234473587832729E+04	6.792373911679658E-13
11	3.234473587831894E+04	9.375927956244681E-13
14	3.234473587831049E+04	1.198647603523267E-12
18	3.234473587829935E+04	1.543046492702984E-12

Table 2 The discrete energy and relative error at different times with $\varepsilon=0.1, h=1/64, \tau=1/100$.

t	$E_h(t)$	$\Delta E_h(t)$
0	2.366525815162189E+02	0
5	2.366525815161711E+02	1.985955040633904E-12
8	2.366525815154697E+02	3.165926513434345E-12
11	2.366525815152006E+02	4.302902588040124E-12
14	2.366525815149426E+02	5.393040100366803E-12
18	2.366525815146064E+02	6.813809822832881E-12

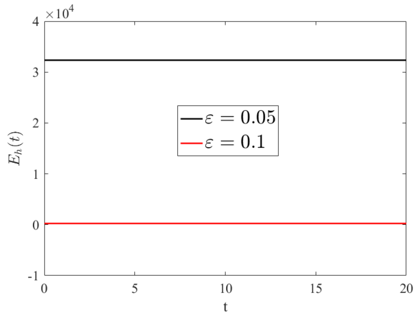

Figure 1 The discrete energy with different ε. Here $h = 1/64, \tau = 1/100$.

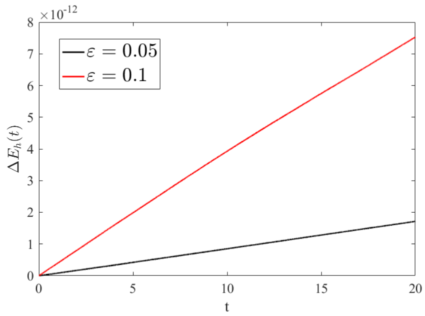

Figure 2 The relative error of energy with different ε. Here $h = 1/64, \tau = 1/100$.

4. Numerical experiments

In this section, we present the numerical results to confirm the energy-conservative property in Section 2 and the error estimation in Section 3. Choose a computational interval $[a, b]$ large enough that periodic boundary conditions do not introduce significant aliasing errors relative to the problem in the entire space. In the numerical experiment, we take $a = -8, b = 8, \alpha = 1, \beta_1 = \beta_2 = \beta_3 = 1$ and choose the initial conditions as

(68) $$u(x, 0) = sech(x^2 - 1), \quad \partial_t u(x, 0) = 0,$$

(69) $$v(x, 0) = sech(x^2 + 1), \quad \partial_t v(x, 0) = 0.$$

Table 3 Spatial discretization errors of FDTD method at time $T = 0.5$ in the case with $\varepsilon = 0.1, \tau = 1e - 3$.

h	1/32	1/64	1/128
$\|\|e_u\|\|_{l^2}$	9.768E-04	2.445E-04	6.114E-05
$\|\|\delta_x^+ e_u\|\|_{l^2}$	6.731E-03	1.689E-03	4.227E-04
$\|\|e_v\|\|_{l^2}$	1.031E-04	2.579E-05	6.448E-06
$\|\|\delta_x^+ e_v\|\|_{l^2}$	4.259E-04	1.065E-04	2.662E-05

Table 4 Temporal discretization errors of FDTD method at time $T = 0.5$ in the case with $\varepsilon = 0.1, h = \frac{1}{128}$.

τ	1/1000	1/2000	1/4000
$\|e_u\|_{l^2}$	3.476E-02	8.419E-03	2.096E-03
$\|\delta_x^+ e_u\|_{l^2}$	9.175E-02	2.289E-02	5.723E-03
$\|e_v\|_{l^2}$	1.790E-02	4.647E-03	1.176E-03
$\|\delta_x^+ e_v\|_{l^2}$	3.547E-02	9.265E-03	2.342E-03

Table 5 Spatial discretization errors of FDTD method at time $T = 0.5$ in the case with $\varepsilon = 0.05, \tau = \frac{1}{8000}$.

h	1/32	1/64	1/128
$\|e_u\|_{l^2}$	1.159E-03	2.902E-04	7.259E-05
$\|\delta_x^+ e_u\|_{l^2}$	8.146E-03	2.047E-03	5.123E-04
$\|e_v\|_{l^2}$	9.650E-05	2.412E-05	6.031E-06
$\|\delta_x^+ e_v\|_{l^2}$	4.740E-04	1.185E-04	2.963E-05

Table 6 Temporal discretization errors of FDTD method at time $T = 0.5$ in the case with $\varepsilon = 0.05, h = \frac{1}{128}$.

τ	1/8000	1/16000	1/32000
$\|e_u\|_{l^2}$	8.742E-02	2.110E-02	5.227E-03
$\|\delta_x^+ e_u\|_{l^2}$	9.078E-02	2.256E-02	5.633E-03
$\|e_v\|_{l^2}$	4.132E-02	9.897E-03	2.446E-03
$\|\delta_x^+ e_v\|_{l^2}$	3.122E-02	7.558E-03	1.875E-03

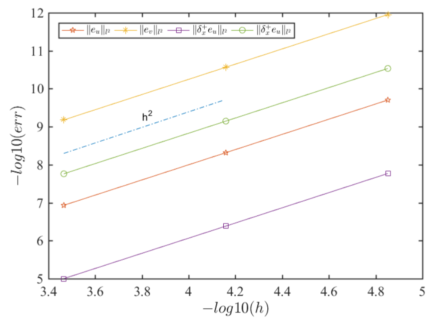

Figure 3 Convergence rates of spatial discretization errors under $\varepsilon = 0.1, T = 0.5$.

For different ε ($0 < \varepsilon \ll 1$), we present the discrete energy $E_h(t)$ and its relative error $\Delta E_h(t)$ at different times in Tables 1-2, where $\Delta E_h(t)$ is defined by

$$\Delta E_h(t) = \frac{|E_h(t) - E_h(0)|}{E_h(0)}.$$

$E_h(t)$ and $\Delta E_h(t)$ are plotted in Figs.1-2. The numerical results are coinsistent with Theorem 2.2.

In order to quantify the convergence, we choose different ε and τ under $\varepsilon-$scalability $\tau = \mathcal{O}(\varepsilon^3)$. Denote $\{u_j^N(\tau,h)|0 \le j \le M-1\}, \{v_j^N(\tau,h)|0 \le j \le M-1\}$ as the numerical solutions of time grid τ and space grid h at time t_N. Then, define

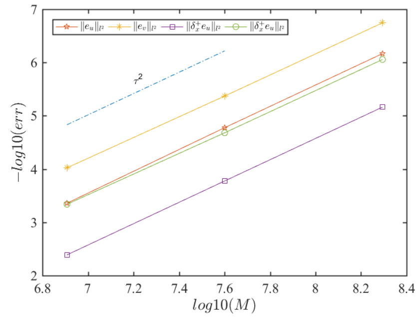

Figure 4 Convergence rates of temporal discretization errors under $\varepsilon = 0.1, T = 0.5$.

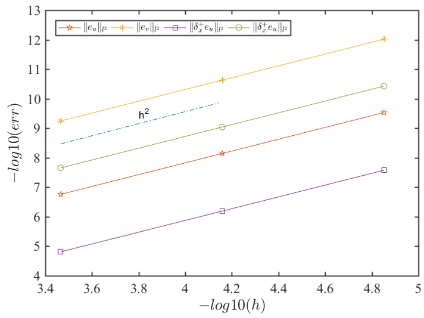

Figure 5 Convergence rates of spatial discretization errors under $\varepsilon = 0.05, T = 0.5$.

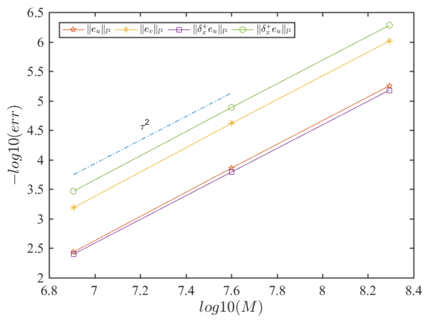

Figure 6 Convergence rates of temporal discretization errors under $\varepsilon = 0.05, T = 0.5$.

the errors in the spatial direction with sufficiently small τ by

$$\|e_u(h)\|_{l^2}^2 = h \sum_{j=0}^{M-1} |u_j^N(\tau, h) - u_j^N(\tau, \frac{h}{2})|^2,$$

$$\|e_v(h)\|_{l^2}^2 = h \sum_{j=0}^{M-1} |v_j^N(\tau, h) - v_j^N(\tau, \frac{h}{2})|^2,$$

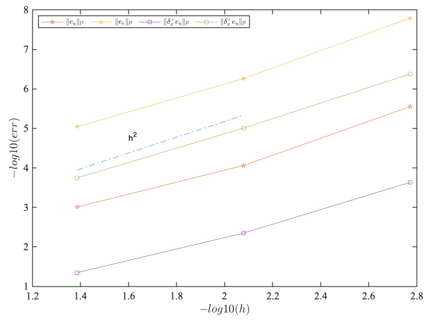

Figure 7 Convergence rates of spatial discretization errors under $\varepsilon = 0.1, T = 0.5$.

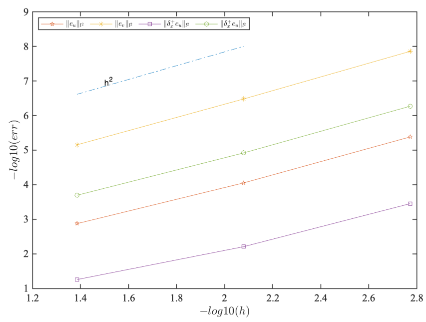

Figure 8 Convergence rates of spatial discretization errors under $\varepsilon = 0.05, T = 0.5$.

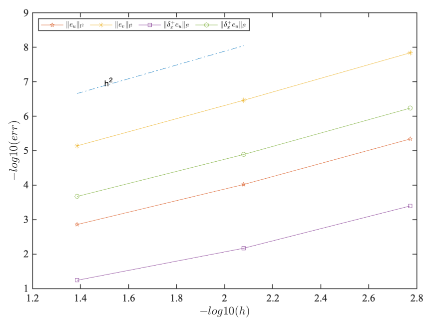

Figure 9 Convergence rates of spatial discretization errors under $\varepsilon = 0.025, T = 0.5$.

$$\|\delta_x^+ e_u(h)\|_{l^2}^2 = h \sum_{j=0}^{M-1} |\delta_x^+ u_j^N(\tau, h) - \delta_x^+ u_j^N(\tau, \frac{h}{2})|^2,$$
$$\|\delta_x^+ e_v(h)\|_{l^2}^2 = h \sum_{j=0}^{M-1} |\delta_x^+ v_j^N(\tau, h) - \delta_x^+ v_j^N(\tau, \frac{h}{2})|^2,$$

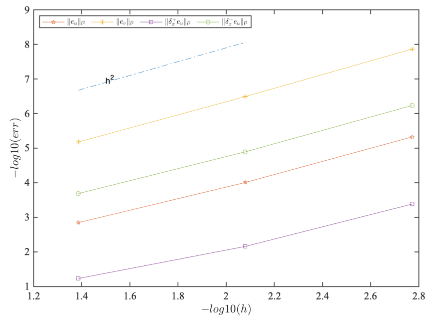

Figure 10 Convergence rates of spatial discretization errors under $\varepsilon = 0.0125, T = 0.5$.

and the errors in the temporal direction with sufficiently small h by

$$\|e_u(\tau)\|_{l^2}^2 = h \sum_{j=0}^{M-1} |u_j^N(\tau, h) - u_j^N(\frac{\tau}{2}, h)|^2,$$

$$\|e_v(\tau)\|_{l^2}^2 = h \sum_{j=0}^{M-1} |v_j^N(\tau, h) - v_j^N(\frac{\tau}{2}, h)|^2,$$

$$\|\delta_x^+ e_u(\tau)\|_{l^2}^2 = h \sum_{j=0}^{M-1} |\delta_x^+ u_j^N(\tau, h) - \delta_x^+ u_j^N(\frac{\tau}{2}, h)|^2,$$

$$\|\delta_x^+ e_v(h)\|_{l^2}^2 = h \sum_{j=0}^{M-1} |\delta_x^+ v_j^N(\tau, h) - \delta_x^+ v_j^N(\frac{\tau}{2}, h)|^2.$$

Table 3 and Table 4 show the errors of FDTD method at $t = 0.5$ in the case with $\varepsilon = 0.1$ under $\tau = \mathcal{O}(\varepsilon^3)$. Fig.3 and Fig.4 show the errors behaves with $\mathcal{O}(h^2 + \frac{\tau^2}{\varepsilon^6})$.

For the case with $\varepsilon = 0.05$, we can draw the following observations: (1) in Table 5 and Fig.5, the spatial error is second-order accurate. (2) The convergence order of temporal errors, under $\tau = \mathcal{O}(\varepsilon^3)$, can be found from Table 6 and Fig.6.

We choose the different ε ($\varepsilon = 0.1, 0.05, 0.025, 0.0125$) and $\tau = 2e - 5$ at time T is fixed. Figs.7-10 show the convergence rates of spatial discretization errors of FDTD method at $t = 0.5$ in the case with $h = 1/4, 1/8, 1/16, 1/32$.

5. Conclusions

In this paper, we have developed and analyzed the energy-conservative FDTD method for CNKGEs in the nonrelativistic limit regime. We analyzed the energy-conservative property of the numerical schemes. With the help of the cut-off technique, we exhibit a rigorous analysis of error estimates and obtain the convergence result $\mathcal{O}(h^2 + \frac{\tau^2}{\varepsilon^6})$. Numerical experiments are carried out to support the theoretical claims.

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