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A VOIGT-REGULARIZATION OF THE THERMALLY COUPLED INVISCID, RESISTIVE MAGNETOHYDRODYNAMIC

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Abstract. In this paper, we prove the existence of weak solution and the uniqueness of strong solution to a Voigt-regularization of the three-dimensional thermally coupled inviscid, resistive MHD equations. We also propose a fully discrete scheme for the considered problem, which is proven to be stable and convergent. All computational results support the theoretical analysis and demonstrate the effectiveness of the presented scheme.

Key words. Thermally coupled magnetohydrodynamic, inviscid, resistive, Voigt-regularization, finite element method, three-dimensional MHD equations.

1. Introduction

The incompressible magnetohydrodynamic (MHD) describes the dynamic behavior of an electrically conducting fluid under the influence of a magnetic field, and has a wide range of applications in scientific and engineering, such as electromagnetic pumping, liquid metal, electrolyte, and so on (see [1, 2, 3, 4, 5]). It consists of a viscous, incompressible fluid which owns the property of electric current conduction and interacting with electromagnetic induction. The MHD flow has a multi-physics phenomenon: the magnetic field changes the momentum of the fluid through the Lorenz force, and conversely, the conducting fluid influences the magnetic field through electric currents. Additionally, if the buoyancy effect cannot be neglected in the momentum equation due to temperature differences in the conductive flow, then the incompressible MHD equations are usually coupled to the heat equation. In this way, multiple physical fields (velocity, pressure, magnetic and temperature) will be coupled in the MHD system.

Usually, the thermally coupled incompressible MHD system is given as follows [6]:

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla(p + \frac{1}{2}|\boldsymbol{B}|^2) - (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} = \boldsymbol{f} + \boldsymbol{\beta}\theta, \tag{1a}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{1b}$$

$$\boldsymbol{B}_t - \mu\Delta\boldsymbol{B} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} + \nabla q = \nabla\times\boldsymbol{g}, \tag{1c}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{1d}$$

$$\theta_t - \kappa\Delta\theta + \boldsymbol{u}\cdot\nabla\theta = \Psi, \tag{1e}$$

with appropriate boundary and initial conditions. Here, $\nu \geq 0$ is the fluid viscosity, $\mu \geq 0$ is the magnetic resistivity, κ is the thermal conductivity, $\boldsymbol{\beta}$ is the thermal expansion coefficient, and the unknowns are the fluid velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, the fluid pressure $p(\boldsymbol{x},t)$, the magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$, the magnetic pressure $q(\boldsymbol{x},t)$, and the temperature field $\theta(\boldsymbol{x},t)$. In fact, by *a posteriori*, one can drive that $\nabla q \equiv \mathbf{0}$. Besides, the given function $\boldsymbol{f}$ is the external force, $\boldsymbol{g}$ is the known applied current, and Ψ is the heat source. Note that these equations contain the three dimensional

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Navier-Stokes equations as a special case (namely $\boldsymbol{B} \equiv \boldsymbol{0}$ and $\theta \equiv 0$), and the mathematical theory is far from complete.

Denote by $P := p + \frac{1}{2}|\boldsymbol{B}|^2$ a modified pressure, in this paper we study the following Voigt-regularization of (1) in inviscid and resistive case (i.e., $\nu = 0$ and $\mu \neq 0$).

$$\boldsymbol{u}_t - \alpha^2 \Delta \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla P - (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} = \boldsymbol{\beta}\theta, \tag{2a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2b}$$

$$\boldsymbol{B}_t - \mu \Delta \boldsymbol{B} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} + \nabla q = 0, \tag{2c}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2d}$$

$$\theta_t - \kappa \Delta \theta + \boldsymbol{u} \cdot \nabla \theta = 0, \tag{2e}$$

$$(\boldsymbol{u}, \boldsymbol{B}, \theta)|_{t=0} = (\boldsymbol{u}_0, \boldsymbol{B}_0, \theta_0), \tag{2f}$$

where $\alpha^2 \Delta \boldsymbol{u}_t$ is the Voigt term and $\alpha > 0$ is a regularization parameter. Moreover, when $\alpha = 0$, we formally retrieve (1) by adding forcing terms to (2a), (2c) and (2e), and reintroducing a viscous term $\nu \Delta \boldsymbol{u}$ to the right-hand side of (2a).

The Voigt (also written Voight) term was originally proposed by Voigt in [7] for viscoelastic fluids. Viscoelasticity is the property of a material that, under stress and deformation, exhibits both viscous and elastic characteristics. Unlike the Kelvin stress-strain relation [8], Voigt has derived a system of equations that governs the behavior of elastic solids with viscous properties, which is known today as the Kelvin-Voigt equations [7]. In [9], the Navier-Stokes-Voigt (NSV) equations were firstly introduced by Oskolkov as a model of Kelvin-Voigt fluids in which α denotes a material parameter connected to a characteristic length of viscoelastic effects. The authors also pointed out that the NSV equations have a real physical sense, and describe the flow of a viscous incompressible Newtonian fluid which requires $\frac{\alpha^2}{\nu}$ units of time in order to be set in motion under the action of a suddenly applied force. The NSV equations were proposed by Cao et al. [10] as a regularizaion of the Euler equations and Navier-Stokes equations which suggested a smaller resolution requirement in large scale computations.

In the last several decades, many analyses and applications regarding the Voigt regularization have been studied (see, e.g., [11, 12, 13, 14, 15, 16, 17]). The Voigt regularization enjoys a feature that it is inviscid and does not require any artificial boundary conditions to prove the global existence and uniqueness of strong solutions [10]. It is also simpler than the nonlinear viscosity model of Ladyzhenskaya [18] and Smagorinsky [19]. Due to these benefits, the ability to adapt an existing CFD code to the Voigt regularization without intrusion has great interest. In [20], Kuberry et al. proposed a Voigt regularization algorithm for the Navier-Stokes equations. Numerical tests show that the Voigt regularization algorithm on a coarse mesh produces good approximations to higher Reynolds number. Later on, Layton and Rebholz [21] found that the regularization parameter α has effect of slowing the temporal evolution; that is, compared to the usual solutions of the Navier-Stokes equations, the NSV approximations have a longer relaxation time and damped effects decay more slowly. The statistical properties of the NSV model have also been investigated computationally, using a phenomenological model of turbulence known as the Sabra shell model [22]. The results indicate that the NSV model may capture important statistical features of the Navier-Stokes equations, and therefore give motivation for it to be investigated for use in numerical simulations. Furthermore, in the context of numerical computations, the NSV system appears to have less stiffness than the Navier-Stokes system (see, e.g., [23, 22]). For more discussion of the NSV equations, the authors refer readers to [24, 25, 26, 27, 28, 29].

The MHD-Voigt model was introduced and studied [30], where the global well-posedness was established for the three dimensional case, even with zero fluid viscosity and zero magnetic resistivity. A similar model with Voigt regularization only on the momentum equation, but with non-zero magnetic resistivity, was studied in [31, 32]. Similar to the case of the NSV system, it has also been noted in [32] that the three dimensional MHD-Voigt system (with non-zero viscosity and magnetic resistivity) is globally well-posed under physical boundary conditions. Kuberry et al. [20] proposed a second-order time discretization method and obtained unconditional stability and optimal convergence results for the MHD-Voigt model. Additionally, Lu et al. [33] studied the MHD-Voigt equations with the finite element scheme for spatial discretization and the Crank-Nicolson-type scheme for temporal discretization.

In this work, we consider a Voigt-regularization of the thermally coupled inviscid, resistive MHD problem, and propose some theoretical analysis and numerical results. This paper is organized as follows. Section 2 introduces some basic notations and mathematical preliminaries. In Section 3, we present the existence of weak solution, and uniqueness of strong solution along with strong solution continuous dependence on initial value. In the next section, we present a fully discrete scheme of the considered thermally coupled MHD-Voigt problem and prove both stability and convergence. Numerical results are exhibited in Section 5, which demonstrate the effectiveness of the presented scheme.

2. Preliminaries

In this section, we introduce some preliminary material and notations which are commonly used in the mathematical study of the considered fluid. We consider the problem with periodic boundary conditions as [32]. Next, denote the function space

$$\mathcal{V} := \left\{ \boldsymbol{\phi} \in \mathcal{F} | \nabla \cdot \boldsymbol{\phi} = 0 \quad \text{and} \quad \int_{\Omega} \boldsymbol{\phi}(\boldsymbol{x}) d\boldsymbol{x} = 0 \right\},$$

where $\mathcal{F}$ is the set of all three dimensional vector-valued trigonometric polynomials with periodic bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3). Here, we note that the restriction to the region is only for the sake of simplifying the exposition, but essentially the same computations in Section 3 can be carried out for an open bounded subset of $\mathbb{R}^d$ with smooth boundary, as [34] for the Navier-Stokes equations.

We denote by L^p and W_p^k the usual Lebesgue and Sobolev spaces over Ω, $H^k := W_2^k$ and define H and V to be the closures of $\mathcal{V}$ in L^2 and H^1, respectively. Additionally, we denote by V' the dual space of V, and denote the action of V' on V by $\langle \cdot, \cdot \rangle_{V'}$. Next, we introduce the usual L^2 norm and its inner product by $\| \cdot \|$ and $(\cdot, \cdot)$. The H^k norm and L^p ($p \neq 2$) norm are denoted by $\| \cdot \|_k$ and $\| \cdot \|_{L^p}$, respectively.

Now, we denote by $P_\sigma : L^2 \to H$ the Leray-Helmholtz projection operator and define the Stokes operator $A := -P_\sigma \Delta$. Notice that in the case of periodic boundary condition, we have $A = -\Delta$ (see, e.g., [35]). It is known that $A^{-1} : H \to \mathcal{D}(A) := H^2 \cap V$ is a positive-definite, self-adjoint, compact operator, and there is an orthonormal basis $\{w_i\}_{i=1}^{\infty}$ of H consisting of eigenvectors of A corresponding to eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ such that $Aw_j = \lambda_i w_j$ (see, e.g. [35, 36]). Next, let $H_m :=$ span$\{w_1, \dots, w_m\}$ and $P_m : H \to H_m$ be the L^2 orthogonal projection onto H_m with respect to $\{w_i\}_{i=1}^{\infty}$.

Moreover, it will use the following notation for the nonlinear term

$$B(\boldsymbol{u},\boldsymbol{v}) := P_\sigma((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}), \tag{3}$$

for $\boldsymbol{u},\boldsymbol{v}\in\mathcal{V}$. The operator B defined in (3) is a bilinear form which can be extended as a continuous map $B: V\times V\to V'$. Furthermore, for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V$,

$$\langle B(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}\rangle_{V'} = -\langle B(\boldsymbol{u},\boldsymbol{w}),\boldsymbol{v}\rangle_{V'} \quad \text{and} \quad \langle B(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{v}\rangle_{V'} = 0. \tag{4}$$

Now, we give the following trilinear form $b(\cdot,\cdot,\cdot): V\times V\times V\to\mathbb{R}$ by

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) := \langle B(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}\rangle_{V'}.$$

Throughout the paper, C is used to denote the generic constant different in different occurrences and independent of mesh size, time step and the regularization parameter α. The following lemmas, for instance see [37, 38] are largely used in numerical analysis.

Lemma 2.1 (Differential Grönwall's inequality)**.** *Let $\eta(t)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality*

$$\eta'(t)\le\phi(t)\eta(t)+\psi(t)+C,$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0,T]$. Then

$$\eta(t)\le\exp(\int_0^t\phi(s)ds)\left(\eta(0)+\int_0^t\psi(s)ds+C\right), \tag{5}$$

for all $0\le t\le T$.

Lemma 2.2 (Discrete Grönwall's inequality)**.** *Let Δt, K and a_n,b_n,c_n,d_n be nonnegative numbers such that for $M\ge 0$*

$$a_M+\Delta t\sum_{n=0}^M b_n\le\Delta t\sum_0^M d_na_n+\Delta t\sum_{n=0}^M c_n+K.$$

Furthermore, suppose that the time step satisfies $\Delta t d_n<1$ for each n. Then

$$a_M+\Delta t\sum_{n=0}^M b_n\le\exp(\Delta t\sum_{n=0}^M d_n)\left(\Delta t\sum_{n=0}^M c_n+K\right). \tag{6}$$

3. Well-posedness of solutions

This section is devoted to stating and proving the well-posedness of (2). We mainly show the the existence of weak solution and uniqueness of strong solution. Noticing the Helmholtz-Weyl decomposition theorem [34], we can derive the unique decomposition of the temperature $\boldsymbol{\beta}\theta=\boldsymbol{\beta}\tilde\theta+\boldsymbol{\beta}\bar\theta$, where $\tilde\theta := P_\sigma\theta\in H$ and $\boldsymbol{\beta}\bar\theta\in G := \{\boldsymbol{s}\in L^2 : \boldsymbol{s}=\nabla g,\forall g\in H^1\}$. Then, based on the de Rham theorem [36, 39, 30], we have

$$\boldsymbol{s}=\nabla g \;\text{ if and only if }\; \langle\boldsymbol{s},\boldsymbol{v}\rangle_{\mathcal{V}'}=0 \quad \forall\boldsymbol{v}\in\mathcal{V}. \tag{7}$$

In the following part, assume $\boldsymbol{u}_0\in V$ and $\boldsymbol{B}_0,\tilde\theta_0\in H$ to show existence without uniqueness of weak solution. Additionally, for strong solution, assume $\boldsymbol{u}_0,\boldsymbol{B}_0,\tilde\theta_0\in V$ to obtain existence, uniqueness, and continuous dependence on initial data.

3.1. Existence of weak solution. Consider (2) in a functional form. Making P_σ act on (2), we obtain the following system (8), which is equivalent to (2) (see, e.g., [36] for the Navier-Stokes equations). Recalling the Helmholtz-Weyl decomposition of the temperature above, we consider $\tilde{\theta}$.

$$\frac{d}{dt}(\boldsymbol{u}+\alpha^2 A\boldsymbol{u}) = B(\boldsymbol{B},\boldsymbol{B}) - B(\boldsymbol{u},\boldsymbol{u}) + \boldsymbol{\beta}\tilde{\theta}, \tag{8a}$$

$$\frac{d}{dt}\boldsymbol{B} + \mu A\boldsymbol{B} = B(\boldsymbol{B},\boldsymbol{u}) - B(\boldsymbol{u},\boldsymbol{B}), \tag{8b}$$

$$\frac{d}{dt}\tilde{\theta} + \kappa A\tilde{\theta} = -B(\boldsymbol{u},\tilde{\theta}), \tag{8c}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \boldsymbol{B}(0) = \boldsymbol{B}_0, \tilde{\theta}(0) = \tilde{\theta}_0. \tag{8d}$$

Note that one can recover p, q and $\bar{\theta}$ by using de Rham's theorem (7) (see, e.g., [36, 39] for the Navier-Stokes equations).

Definition 3.1. Let $\boldsymbol{u}_0 \in V, \boldsymbol{B}_0, \tilde{\theta}_0 \in H$. Then $(\boldsymbol{u},\boldsymbol{B},\tilde{\theta})$ is called a weak solution to (8), on the time interval $[0,T]$, if

$$\boldsymbol{u} \in C([0,T],V), \quad \boldsymbol{B},\tilde{\theta} \in L^2((0,T),V) \cap C_w([0,T],H),$$
$$\frac{d\boldsymbol{u}}{dt} \in L^4((0,T),H), \quad \frac{d\boldsymbol{B}}{dt}, \frac{d\tilde{\theta}}{dt} \in L^2((0,T),V'),$$

and furthermore $(\boldsymbol{u},\boldsymbol{B},\tilde{\theta})$ satisfies (8a) in the sense of $L^{\frac{4}{3}}([0,T],V')$ and (8b), (8c) in the sense of $L^2([0,T],V')$.

Under definition of the weak solution, we are now ready to state and prove the existence of weak solution.

Theorem 3.1. *Let $\boldsymbol{u}_0 \in V$ and $\boldsymbol{B}_0, \tilde{\theta}_0 \in H$. Then* (8) *has a weak solution $(\boldsymbol{u},\boldsymbol{B},\tilde{\theta})$ for arbitrary $T > 0$.*

Proof. We consider the following finite dimensional Galerkin approximation of (8). Find $\boldsymbol{u}_m, \boldsymbol{B}_m, \tilde{\theta}_m \in C^1(0,T;H_m)$ by solving

$$\frac{d}{dt}(\boldsymbol{u}_m+\alpha^2 A\boldsymbol{u}_m) = P_m B(\boldsymbol{B}_m,\boldsymbol{B}_m) - P_m B(\boldsymbol{u}_m,\boldsymbol{u}_m) + \boldsymbol{\beta}\tilde{\theta}_m, \tag{9a}$$

$$\frac{d}{dt}\boldsymbol{B}_m + \mu A\boldsymbol{B}_m = P_m B(\boldsymbol{B}_m,\boldsymbol{u}_m) - P_m B(\boldsymbol{u}_m,\boldsymbol{B}_m), \tag{9b}$$

$$\frac{d}{dt}\tilde{\theta}_m + \kappa A\tilde{\theta}_m = -P_m B(\boldsymbol{u}_m,\tilde{\theta}_m), \tag{9c}$$

$$\boldsymbol{B}_m(0) = P_m\boldsymbol{B}_0, \boldsymbol{u}_m(0) = P_m\boldsymbol{u}_0, \tilde{\theta}_m(0) = P_m\tilde{\theta}_0. \tag{9d}$$

Applying the operator $(I+\alpha^2 A)^{-1}$ to (9a), we see that (9) is equivalent to an ordinary differential equations $\dot{\boldsymbol{y}} = F(\boldsymbol{y})$. By the Picard-Lindelöf Theorem [40], this system has a unique solution on $[0,T^{\max})$, which is the maximal interval for the existence and uniqueness of solution.

Now, we prove $T^{\max} = \infty$ by contradiction. Supposing $T^{\max} < \infty$, we have

$$\lim_{t\to T^{\max}} \|\boldsymbol{u}_m(t)\| = \infty \quad \text{or} \quad \lim_{t\to T^{\max}} \|\boldsymbol{B}_m(t)\| = \infty \quad \text{or} \quad \lim_{t\to T^{\max}} \|\tilde{\theta}_m(t)\| = \infty, \tag{10}$$

otherwise we could use the Picard-Lindelöf Theorem to extend the solution further in time, contradicting the definition of $T^{\max}$. In fact, take the inner product of (9a) with $\boldsymbol{u}_m(t)$, (9b) with $\boldsymbol{B}_m(t)$ and (9c) with $\tilde{\theta}_m(t)$ for $t \in [0, T^{\max})$, respectively. Using (4) on the ensuing equations, we have

$$\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{u}_m\|^2 + \alpha^2\|\nabla \boldsymbol{u}_m\|^2) = b(\boldsymbol{B}_m, \boldsymbol{B}_m, \boldsymbol{u}_m) + (\boldsymbol{\beta}\tilde{\theta}_m, \boldsymbol{u}_m), \tag{11a}$$

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{B}_m\|^2 + \mu\|\nabla \boldsymbol{B}_m\|^2 = -b(\boldsymbol{B}_m, \boldsymbol{B}_m, \boldsymbol{u}_m), \tag{11b}$$

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\theta}_m\|^2 + \kappa\|\nabla\tilde{\theta}_m\|^2 = 0. \tag{11c}$$

Integrating (11c) with respect to t from 0 to t yields

$$\|\tilde{\theta}_m(t)\|^2 + 2\kappa\int_0^t \|\nabla\tilde{\theta}_m\|^2 dt = \|\tilde{\theta}_m(0)\|^2 \le \|\tilde{\theta}_0\|^2 < \infty. \tag{12}$$

Next, adding (11a) and (11b), and utilizing (12) give

$$\frac{d}{dt}(\|\boldsymbol{u}_m\|^2 + \alpha^2\|\nabla \boldsymbol{u}_m\|^2 + \|\boldsymbol{B}_m\|^2) + 2\mu\|\nabla \boldsymbol{B}_m\|^2 \le 2\|\boldsymbol{u}_m\|^2 + 2\|\boldsymbol{\beta}\tilde{\theta}_0\|^2. \tag{13}$$

Apply the differential Grönwall's inequality (5) on (13)

$$\|\boldsymbol{u}_m\|^2 \le \exp(2T^{\max})\left(\|\boldsymbol{u}_0\|^2 + 2\|\boldsymbol{\beta}\tilde{\theta}_0\|^2\right) =: K_1, \tag{14}$$

and then, combining (13) and (4.3), we arrive at

$$\frac{d}{dt}(\|\boldsymbol{u}_m\|^2 + \alpha^2\|\nabla \boldsymbol{u}_m\|^2 + \|\boldsymbol{B}_m\|^2) + 2\mu\|\nabla \boldsymbol{B}_m\|^2 \le 2K_1 + 2\|\boldsymbol{\beta}\tilde{\theta}_0\|^2 =: K_2. \tag{15}$$

Integrating (15) with respect to t from 0 to t leads to

$$\begin{aligned} &\|\boldsymbol{u}_m(t)\|^2 + \alpha^2\|\nabla \boldsymbol{u}_m(t)\|^2 + \|\boldsymbol{B}_m(t)\|^2 + 2\mu\int_0^t \|\nabla \boldsymbol{B}_m\|^2 ds \\ &= K_2 t + \|\boldsymbol{u}_m(0)\|^2 + \alpha^2\|\nabla \boldsymbol{u}_m(0)\|^2 + \|\boldsymbol{B}_m(0)\|^2 \\ &\le (K_3^\alpha)^2 := K_2 T^{\max} + \|\boldsymbol{u}_0\|^2 + \alpha^2\|\nabla \boldsymbol{u}_0\|^2 + \|\boldsymbol{B}_0\|^2 < \infty, \end{aligned} \tag{16}$$

which and (12) contradict (10). Hence, $T^{\max} = \infty$ and (9) has a unique solution $(\boldsymbol{u}_m, \boldsymbol{B}_m, \tilde{\theta}_m)$ for arbitrary $T > 0$.

Additionally, according to (12) and (16), for fixed but arbitrary $T > 0$, we gain

$$\boldsymbol{u}_m \text{ is bounded in } L^\infty([0,T], V), \tag{17a}$$

$$\boldsymbol{B}_m \text{ is bounded in } L^\infty([0,T], H) \cap L^2([0,T], V), \tag{17b}$$

$$\tilde{\theta}_m \text{ is bounded in } L^\infty([0,T], H) \cap L^2([0,T], V), \tag{17c}$$

uniformly with respect to m.

Moreover, we will extract subsequences of $\{\boldsymbol{u}_m\}$, $\{\boldsymbol{B}_m\}$ and $\{\tilde{\theta}_m\}$ which converge in $L^2((0,T), H)$ by using the Aubin compactness theorem (see, e.g., [41, 36]). To satisfy the hypotheses of this theorem, we will show that $\dfrac{d\boldsymbol{u}_m}{dt}$ is uniformly bounded in $L^4((0,T), H) \hookrightarrow L^2((0,T), V')$, and $\dfrac{d\boldsymbol{B}_m}{dt}, \dfrac{d\tilde{\theta}_m}{dt}$ are uniformly bounded in $L^2((0,T), V')$, with respect to m.

In fact, from (9a) and (16), (12), (4), we arrive at

$$\begin{aligned} &\|(I + \alpha^2 A)\frac{d\boldsymbol{u}_m}{dt}\|_{\mathcal{D}(A)'} \\ &\le \|P_m B(\boldsymbol{B}_m, \boldsymbol{B}_m)\|_{\mathcal{D}(A)'} + \|P_m B(\boldsymbol{u}_m, \boldsymbol{u}_m)\|_{\mathcal{D}(A)'} + \|\boldsymbol{\beta}\tilde{\theta}_m\|_{\mathcal{D}(A)'} \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\boldsymbol{v}\in\mathcal{D}(A)} \frac{|b(\boldsymbol{B}_m,\boldsymbol{B}_m,P_m\boldsymbol{v})|}{\|\boldsymbol{v}\|_{\mathcal{D}(A)}} + \sup_{\boldsymbol{v}\in\mathcal{D}(A)} \frac{|b(\boldsymbol{u}_m,\boldsymbol{u}_m,P_m\boldsymbol{v})|}{\|\boldsymbol{v}\|_{\mathcal{D}(A)}} + \sup_{\boldsymbol{v}\in\mathcal{D}(A)} \frac{\left|(\boldsymbol{\beta}\tilde{\theta}_m,\boldsymbol{v})\right|}{\|\boldsymbol{v}\|_{\mathcal{D}(A)}}\\
&= \sup_{\boldsymbol{v}\in\mathcal{D}(A)} \frac{|b(\boldsymbol{B}_m,P_m\boldsymbol{v},\boldsymbol{B}_m)|}{\|\boldsymbol{v}\|_{\mathcal{D}(A)}} + \sup_{\boldsymbol{v}\in\mathcal{D}(A)} \frac{|b(\boldsymbol{u}_m,P_m\boldsymbol{v},\boldsymbol{u}_m)|}{\|\boldsymbol{v}\|_{\mathcal{D}(A)}} + \sup_{\boldsymbol{v}\in\mathcal{D}(A)} \frac{\left|(\boldsymbol{\beta}\tilde{\theta}_m,\boldsymbol{v})\right|}{\|\boldsymbol{v}\|_{\mathcal{D}(A)}}\\
&\le C(\|\boldsymbol{B}_m\|^{\frac{3}{2}}\|\nabla\boldsymbol{B}_m\|^{\frac{1}{2}} + \|\boldsymbol{u}_m\|^{\frac{3}{2}}\|\nabla\boldsymbol{u}_m\|^{\frac{1}{2}} + \|\boldsymbol{\beta}\tilde{\theta}_m\|^2)\\
&\le C((K_3^\alpha)^{\frac{3}{2}}\|\nabla\boldsymbol{B}_m\|^{\frac{1}{2}} + (K_3^\alpha)^2\alpha^{-\frac{1}{2}} + \|\boldsymbol{\beta}\tilde{\theta}_0\|^2),
\end{aligned}
\tag{18}
$$

as well as

$$
\begin{aligned}
\|(I+\alpha^2A)\frac{d\boldsymbol{u}_m}{dt}\|_{V'} \le& \|P_mB(\boldsymbol{B}_m,\boldsymbol{B}_m)\|_{V'} + \|P_mB(\boldsymbol{u}_m,\boldsymbol{u}_m)\|_{V'} + \|\boldsymbol{\beta}\tilde{\theta}_m\|_{V'}\\
=& \sup_{\boldsymbol{v}\in V} \frac{|b(\boldsymbol{B}_m,\boldsymbol{B}_m,P_m\boldsymbol{v})|}{\|\boldsymbol{v}\|_V}\\
&+ \sup_{\boldsymbol{v}\in V} \frac{|b(\boldsymbol{u}_m,\boldsymbol{u}_m,P_m\boldsymbol{v})|}{\|\boldsymbol{v}\|_V} + \sup_{\boldsymbol{v}\in V} \frac{\left|(\boldsymbol{\beta}\tilde{\theta}_m,\boldsymbol{v})\right|}{\|\boldsymbol{v}\|_V}\\
\le& C(\|\boldsymbol{B}_m\|^{\frac{1}{2}}\|\nabla\boldsymbol{B}_m\|^{\frac{3}{2}} + \|\boldsymbol{u}_m\|^{\frac{1}{2}}\|\nabla\boldsymbol{u}_m\|^{\frac{3}{2}} + \|\boldsymbol{\beta}\tilde{\theta}_m\|^2)\\
\le& C((K_3^\alpha)^{\frac{1}{2}}\|\nabla\boldsymbol{B}_m\|^{\frac{3}{2}} + (K_3^\alpha)^2\alpha^{-\frac{3}{2}} + \|\boldsymbol{\beta}\tilde{\theta}_0\|^2),
\end{aligned}
\tag{19}
$$

where we have applied the Poincaré inequality, and the Sobolev embeddings and interpolation inequality. Hence, the right-hand side (RHS) of (18) and (19) is uniformly bounded in $L^4(0,T)$ and $L^{\frac{4}{3}}(0,T)$, respectively. Then $(I+\alpha^2A)\frac{d\boldsymbol{u}_m}{dt}$ is uniformly bounded in $L^4([0,T],\mathcal{D}(A)') \cap L^{\frac{4}{3}}([0,T],V')$ with respect to m. By applying the operator $(I+\alpha^2A)^{-1}$, we have

$$
\frac{d\boldsymbol{u}_m}{dt} \text{ is bounded in } L^4([0,T],H) \cap L^{\frac{4}{3}}([0,T],V),
\tag{20}
$$

uniformly with respect to m.

Now, we consider $\frac{d\boldsymbol{B}_m}{dt}$ and $\frac{d\tilde{\theta}_m}{dt}$. From (9b) and (9c) we discover

$$
\begin{aligned}
\|\frac{d\boldsymbol{B}_m}{dt}\|_{V'} &\le \|P_mB(\boldsymbol{B}_m,\boldsymbol{u}_m)\|_{V'} + \|P_mB(\boldsymbol{u}_m,\boldsymbol{B}_m)\|_{V'} + \mu\|A\boldsymbol{B}_m\|_{V'}\\
&\le C\|\boldsymbol{B}_m\|^{\frac{1}{2}}\|\nabla\boldsymbol{B}_m\|^{\frac{1}{2}}\|\nabla\boldsymbol{u}_m\| + \mu\|\nabla\boldsymbol{B}_m\|\\
&\le C(K_3^\alpha)^{\frac{3}{2}}\alpha^{-1}\|\nabla\boldsymbol{B}_m\|^{\frac{1}{2}} + \mu\|\nabla\boldsymbol{B}_m\|,
\end{aligned}
$$

and

$$
\begin{aligned}
\|\frac{d\tilde{\theta}_m}{dt}\|_{V'} &\le \|P_mB(\boldsymbol{u}_m,\tilde{\theta}_m)\|_{V'} + \kappa\|A\tilde{\theta}_m\|_{V'}\\
&\le \sup_{\boldsymbol{v}\in V} \frac{\left|b(\boldsymbol{u}_m,\tilde{\theta}_m,P_m\boldsymbol{v})\right|}{\|\boldsymbol{v}\|_V} + \kappa\|\nabla\tilde{\theta}_m\|\\
&\le \|\nabla\boldsymbol{u}_m\|\|\nabla\tilde{\theta}_m\| + \kappa\|\nabla\tilde{\theta}_m\| \le ((K_3^\alpha)\alpha^{-1}+\kappa)\|\nabla\tilde{\theta}_m\|.
\end{aligned}
$$

Combine the above results, to deduce

$$
\frac{d\boldsymbol{B}_m}{dt} \text{ is bounded in } L^2([0,T],V'),
\tag{21}
$$

and

$$\frac{d\tilde{\theta}_m}{dt} \text{ is bounded in } L^2([0,T],V'), \tag{22}$$

uniformly with respect to m.

Therefore, by the Aubin compactness theorem, there exists a subsequence of $(\boldsymbol{u}_m, \boldsymbol{B}_m, \tilde{\theta}_m)$ (relabel as $(\boldsymbol{u}_m, \boldsymbol{B}_m, \tilde{\theta}_m)$) and elements $\boldsymbol{u}, \boldsymbol{B}, \tilde{\theta} \in L^2([0,T],H)$ such that

$$\boldsymbol{u}_m \to \boldsymbol{u} \text{ strongly in } L^2([0,T],H), \tag{23a}$$

$$\boldsymbol{B}_m \to \boldsymbol{B} \text{ strongly in } L^2([0,T],H), \tag{23b}$$

$$\tilde{\theta}_m \to \tilde{\theta} \text{ strongly in } L^2([0,T],H). \tag{23c}$$

Furthermore, in view of (17), (20), (21) and (22), then recalling the Banach-Alaoglu theorem [42], we can pass to additional subsequence (we again relabel as $(\boldsymbol{u}_m, \boldsymbol{B}_m, \tilde{\theta}_m)$), to show that $\boldsymbol{u} \in L^\infty([0,T],V)$, $\boldsymbol{B} \in L^\infty([0,T],H) \cap L^2([0,T],V)$, $\tilde{\theta} \in L^\infty([0,T],H) \cap L^2([0,T],V)$, $\boldsymbol{u}_t \in L^4([0,T],H) \cap L^{\frac{4}{3}}([0,T],V)$, $\boldsymbol{B}_t \in L^2([0,T],V')$, $\tilde{\theta}_t \in L^2([0,T],V')$, and

$$\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}, \boldsymbol{B}_m \rightharpoonup \boldsymbol{B}, \tilde{\theta}_m \rightharpoonup \tilde{\theta} \text{ weakly in } L^2([0,T],V), \quad \text{as } m \to \infty. \tag{24}$$

Now fix k and take $m \geq k$. Let $\boldsymbol{w} \in C^1([0,T],H_k)$ with $\boldsymbol{w}(T) = 0$ be arbitrarily given. Take the inner product of (9) with $\boldsymbol{w}$. Integrating with respect to t from 0 to T, we deduce

$$\begin{aligned} &- (\boldsymbol{u}_m(0), \boldsymbol{w}(0)) - \alpha^2(\nabla \boldsymbol{u}_m(0), \nabla \boldsymbol{w}(0)) \\ &- \int_0^T (\boldsymbol{u}_m, \boldsymbol{w}_t)dt + \alpha^2 \int_0^T (\nabla \boldsymbol{u}_m, \nabla \boldsymbol{w}_t)dt \end{aligned}$$

$$= \int_0^T b(\boldsymbol{B}_m, \boldsymbol{B}_m, P_m\boldsymbol{w})dt - \int_0^T b(\boldsymbol{u}_m, \boldsymbol{u}_m, P_m\boldsymbol{w})dt + \int_0^T (\boldsymbol{\beta}\tilde{\theta}_m, \boldsymbol{w})dt, \tag{25a}$$

$$- (\boldsymbol{B}_m(0), \boldsymbol{w}(0)) - \int_0^T (\boldsymbol{B}_m, \boldsymbol{w}_t)dt + \mu \int_0^T (\nabla \boldsymbol{B}_m, \nabla \boldsymbol{w}_t)dt$$

$$= \int_0^T b(\boldsymbol{B}_m, \boldsymbol{u}_m, P_m\boldsymbol{w})dt - \int_0^T b(\boldsymbol{u}_m, \boldsymbol{B}_m, P_m\boldsymbol{w})dt, \tag{25b}$$

$$\int_0^T b(\boldsymbol{u}_m, \tilde{\theta}_m, P_m w)dt + \kappa \int_0^T (\nabla \tilde{\theta}_m, \nabla w_t)dt$$

$$=(\tilde{\theta}_m(0), w(0)) + \int_0^T (\tilde{\theta}_m, w_t)dt. \tag{25c}$$

Note that (25) holds with $(\boldsymbol{u}_m, \boldsymbol{B}_m, \tilde{\theta}_m, P_m)$ replaced by $(\boldsymbol{u}, \boldsymbol{B}, \tilde{\theta}, I)$, where I is the identity operator. In fact, employ (24) to discover

$$\int_0^T (\boldsymbol{u}_m, \boldsymbol{w}_t)dt \to \int_0^T (\boldsymbol{u}, \boldsymbol{w}_t)dt, \quad \alpha^2 \int_0^T (\nabla \boldsymbol{u}_m, \nabla \boldsymbol{w}_t)dt \to \alpha^2 \int_0^T (\nabla \boldsymbol{u}(t), \nabla \boldsymbol{w}_t)dt,$$

$$\int_0^T (\boldsymbol{\beta}\tilde{\theta}_m, \boldsymbol{w})dt \to \int_0^T (\boldsymbol{\beta}\tilde{\theta}, \boldsymbol{w})dt, \quad \int_0^T (\boldsymbol{B}_m, \boldsymbol{w}_t)dt \to \int_0^T (\boldsymbol{B}, \boldsymbol{w}_t)dt,$$

$$\mu \int_0^T (\nabla \boldsymbol{B}_m, \nabla \boldsymbol{w})dt \to \mu \int_0^T (\nabla \boldsymbol{B}, \nabla \boldsymbol{w})dt, \quad \int_0^T (\tilde{\theta}_m, \boldsymbol{w}_t)dt \to \int_0^T (\tilde{\theta}, \boldsymbol{w}_t)dt,$$

$$\kappa \int_0^T (\nabla \tilde{\theta}_m(t), \nabla \boldsymbol{w})dt \to \kappa \int_0^T (\nabla \tilde{\theta}, \nabla \boldsymbol{w})dt.$$

In addition, we will prove the convergence of the trilinear terms. Here, we only consider the convergence in one case, i.e.,

$$I(m) := \int_0^T b(\boldsymbol{u}_m, \boldsymbol{u}_m, P_m\boldsymbol{w})dt - \int_0^T b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w})dt \to 0, \quad \text{as } m \to \infty.$$

The convergence of the rest trilinear terms can be obtained by similar arguments as the Navier-Stokes equations (see, e.g., [36]). Note that $\boldsymbol{w} \in C^1([0,T], H_k)$ and $k \leq m$ implies $P_m\boldsymbol{w} = \boldsymbol{w}$. Rewrite $I(m)$ as

$$I(m) = \int_0^T b(\boldsymbol{u}_m - \boldsymbol{u}, \boldsymbol{u}_m, \boldsymbol{w})dt + \int_0^T b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{u} - \boldsymbol{u}_m)dt =: I_1(m) + I_2(m),$$

where we have used (4). In fact, $I(m) \to 0$ for $\boldsymbol{w} \in C^1([0,T], H_k)$. Since by standard estimate on the trilinear term, (17a), the Hölder inequality and (23a) shows that $I_1(m) \to 0$ and $I_2(m) \to 0$.

Note that $\boldsymbol{u}_m(0) = P_m\boldsymbol{u}_0 \to \boldsymbol{u}_0$ in V, $\boldsymbol{B}_m(0) = P_m\boldsymbol{B}_0 \to \boldsymbol{B}_0$ and $\tilde{\theta}_m(0) = P_m\tilde{\theta}_0 \to \tilde{\theta}_0$ in H. Thus, passing to limit as $m \to \infty$ in (25), we gain for all $\boldsymbol{w} \in C^1([0,T], H_k)$ with $\boldsymbol{w}(T) = 0$

$$\begin{aligned} &- (\boldsymbol{u}(0), \boldsymbol{w}(0)) - \alpha^2(\nabla\boldsymbol{u}(0), \nabla\boldsymbol{w}(0)) \\ &- \int_0^T (\boldsymbol{u}, \boldsymbol{w}_t)dt + \alpha^2 \int_0^T (\nabla\boldsymbol{u}, \nabla\boldsymbol{w}_t)dt \\ &= \int_0^T b(\boldsymbol{B}, \boldsymbol{B}, \boldsymbol{w})dt - \int_0^T b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w})dt + \int_0^T (\boldsymbol{\beta}\tilde{\theta}, \boldsymbol{w})dt, \end{aligned} \tag{26a}$$

$$\begin{aligned} &- (\boldsymbol{B}_m(0), \boldsymbol{w}(0)) - \int_0^T (\boldsymbol{B}, \boldsymbol{w}_t)dt + \mu \int_0^T (\nabla\boldsymbol{B}, \nabla\boldsymbol{w}_t)dt \\ &= \int_0^T b(\boldsymbol{B}, \boldsymbol{u}, \boldsymbol{w})dt - \int_0^T b(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{w})dt, \end{aligned} \tag{26b}$$

$$\begin{aligned} &\int_0^T b(\boldsymbol{u}, \tilde{\theta}, w)dt + \kappa \int_0^T (\nabla\tilde{\theta}, \nabla w_t)dt \\ &=(\tilde{\theta}(0), w(0)) + \int_0^T (\tilde{\theta}, w_t)dt. \end{aligned} \tag{26c}$$

Moreover, utilize $\boldsymbol{u} \in L^\infty([0,T], V)$, $\boldsymbol{B}, \tilde{\theta} \in L^\infty((0,T), H) \cap L^2((0,T), V)$ and a standard estimate on the trilinear term, to get the fact that (26) holds for all $\boldsymbol{w} \in C^1([0,T], V)$ with $\boldsymbol{w}(T) = 0$ due that $C^1([0,T], H_k)$ is dense in $C^1([0,T], V)$. Acting (8a), (8b) and (8c) on $\boldsymbol{w}$ and comparing with (26), we discover $\boldsymbol{u}(0) + \alpha^2 A\boldsymbol{u}(0) = \boldsymbol{u}_0 + \alpha^2 A\boldsymbol{u}_0$, $\boldsymbol{B}(0) = \boldsymbol{B}_0$ and $\tilde{\theta}(0) = \tilde{\theta}_0$. According to $(I + \alpha^2 A)^{-1}$, one has $\boldsymbol{u}(0) = \boldsymbol{u}_0$.

Finally, we prove that $\boldsymbol{u}$, $\boldsymbol{B}$ and $\tilde{\theta}$ satisfy the requirements for continuity in time in Definition 3.1. Taking the action of (26b) and (26c) with $\boldsymbol{v} \in \mathcal{V}$ and integrating in time, we obtain, for $a.e.$ $t_0, t_1 \in [0,T]$, we get

$$\begin{aligned} &(\boldsymbol{B}(t_1) - \boldsymbol{B}(t_0), \boldsymbol{v}) + \mu \int_{t_0}^{t_1} (\nabla\boldsymbol{B}, \nabla\boldsymbol{v})dt \\ &= \int_{t_0}^{t_1} b(\boldsymbol{B}, \boldsymbol{u}, \boldsymbol{v})dt - \int_{t_0}^{t_1} b(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})dt, \end{aligned} \tag{27}$$

$$(\tilde{\theta}(t_1) - \tilde{\theta}(t_0), \boldsymbol{v}) + \kappa \int_{t_0}^{t_1} (\nabla\tilde{\theta}, \nabla\boldsymbol{v})dt = - \int_{t_0}^{t_1} b(\boldsymbol{u}, \tilde{\theta}, \boldsymbol{v})dt. \tag{28}$$

Observe that the integrands are in $L^1(0,T)$. Then (27) and (28) implies $\boldsymbol{B}, \tilde{\theta} \in C_w([0,T], \mathcal{V})$ by sending $t_1 \to t_0$. Further, due to the density of $\mathcal{V}$ in H and the fact that $\boldsymbol{B}, \tilde{\theta} \in L^\infty([0,T], H)$, we deduce that $\boldsymbol{B}, \tilde{\theta} \in C_w([0,T], H)$ with help of the triangle inequality. On the other hand, recalling (20) implies $\dfrac{d\boldsymbol{u}}{dt} \in L^4([0,T], H) \hookrightarrow L^2([0,T], V')$. Observe that $\boldsymbol{u} \in C([0,T], H)$ due to $\boldsymbol{u} \in L^2([0,T], V)$. □

Remark 3.1. *We only proved the existence of* $\tilde{\theta}$ *in Theorem 3.1. But the existence of* θ *also holds. Indeed, as the approach to recovering the pressure term in* [36]*, it follows* (7) *that* $(\boldsymbol{\beta}\bar{\theta}, \boldsymbol{w}) = 0$. *Adding this term to the RHS of* (26a)*, we get* $(\boldsymbol{\beta}\theta, \boldsymbol{w})$. *For* (26c)*, we can argue almost exactly as* (26a).

3.2. Existence and uniqueness of strong solution. In this subsection, we consider existence and uniqueness of strong solution. For our purpose herein, we firstly give definition of a strong solution in the following sense.

Definition 3.2. *Let* $\boldsymbol{u}_0, \boldsymbol{B}_0, \tilde{\theta}_0 \in V$. *Then* $(\boldsymbol{u}, \boldsymbol{B}, \tilde{\theta})$ *is called a strong solution to* (8) *if it is a weak solution and*

$$\boldsymbol{B}, \tilde{\theta} \in L^2((0,T), \mathcal{D}(A)) \cap C([0,T], V), \ \frac{d\boldsymbol{u}}{dt} \in C([0,T], V), \ \frac{d\boldsymbol{B}}{dt}, \frac{d\tilde{\theta}}{dt} \in L^2((0,T), H).$$

Next, we show in Theorem 3.2 that the strong solution exists globally.

Theorem 3.2. *Let* $\boldsymbol{u}_0$ *and* $\boldsymbol{B}_0, \tilde{\theta}_0 \in V$. *Then* (8) *has a strong solution* $(\boldsymbol{u}, \boldsymbol{B}, \tilde{\theta})$ *for arbitrary* $T > 0$.

Proof. Firstly, take the inner product of (9b) and (9c) with $A\boldsymbol{B}_m$ and $A\tilde{\theta}_m$, respectively. Then employing (16) and the Young's inequality, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{B}_m\|^2 + \mu \|A\boldsymbol{B}_m\|^2 = b(\boldsymbol{B}_m, \boldsymbol{u}_m, A\boldsymbol{B}_m) - b(\boldsymbol{u}_m, \boldsymbol{B}_m, A\boldsymbol{B}_m)\\
&\le C(\|\nabla \boldsymbol{u}_m\|\|\nabla \boldsymbol{B}_m\|^{\frac{1}{2}}\|A\boldsymbol{B}_m\|^{\frac{1}{2}}\|A\boldsymbol{B}_m\| + \|\nabla \boldsymbol{B}_m\|^{\frac{1}{2}}\|A\boldsymbol{B}_m\|^{\frac{1}{2}}\|\nabla \boldsymbol{u}_m\|\|A\boldsymbol{B}_m\|)\\
&\le CK_3^\alpha \alpha^{-1}\|\nabla \boldsymbol{B}_m\|^{\frac{1}{2}}\|A\boldsymbol{B}_m\|^{\frac{3}{2}} \le CK_4^\alpha \|\nabla \boldsymbol{B}_m\|^2 + \frac{\mu}{2}\|A\boldsymbol{B}_m\|^2,\\
&\frac{1}{2}\frac{d}{dt}\|\nabla \tilde{\theta}_m\|^2 + \kappa\|A\tilde{\theta}_m\|^2 = -b(\boldsymbol{u}_m, \tilde{\theta}_m, A\tilde{\theta}_m) \le C\|\nabla \boldsymbol{u}_m\|\|\nabla \tilde{\theta}_m\|^{\frac{1}{2}}\|A\tilde{\theta}_m\|^{\frac{3}{2}}\\
&\le CK_3^\alpha \alpha^{-1}\|\nabla \tilde{\theta}_m\|^{\frac{1}{2}}\|A\tilde{\theta}_m\|^{\frac{3}{2}} \le CK_{4'}^\alpha \|\nabla \tilde{\theta}_m\|^2 + \frac{\kappa}{2}\|A\tilde{\theta}_m\|^2,
\end{aligned}$$

where $K_4^\alpha := C(\alpha^{-1}K_3^\alpha)^4\mu^{-3}$ and $K_{4'}^\alpha := C(\alpha^{-1}K_3^\alpha)^4\kappa^{-3}$, thereby obtaining the estimates

$$\frac{d}{dt}\|\nabla \boldsymbol{B}_m\|^2 + \mu\|A\boldsymbol{B}_m\|^2 \le CK_4^\alpha \|\nabla \boldsymbol{B}_m\|^2, \tag{29}$$

$$\frac{d}{dt}\|\nabla \tilde{\theta}_m\|^2 + \kappa\|A\tilde{\theta}_m\|^2 \le CK_{4'}^\alpha \|\nabla \tilde{\theta}_m\|^2. \tag{30}$$

Then, integrate (29) and (30) over $[0,t]$ to give

$$\begin{aligned}
\|\nabla \boldsymbol{B}_m(t)\|^2 + \mu \int_0^t \|A\boldsymbol{B}_m(s)\|^2 ds &\le \|\nabla \boldsymbol{B}_m(0)\|^2 + CK_4^\alpha \int_0^t \|\nabla \boldsymbol{B}_m(s)\|^2 ds\\
&\le \|\nabla \boldsymbol{B}_0\|^2 + C\mu^{-1}K_4^\alpha (K_3^\alpha)^2 =: K_5^\alpha,
\end{aligned} \tag{31}$$

$$\begin{aligned}
\|\nabla \tilde{\theta}_m(t)\|^2 + \kappa \int_0^t \|A\tilde{\theta}_m\|^2 ds &\le \|\nabla \tilde{\theta}_0\|^2 + CK_{4'}^\alpha \int_0^t \|\nabla \tilde{\theta}_m\|^2 ds\\
&\le \|\nabla \tilde{\theta}_0\|^2 + CK_{4'}^\alpha \frac{\|\tilde{\theta}_0\|^2}{\kappa} =: K_{5'}^\alpha.
\end{aligned} \tag{32}$$

Consequently (31) and (32) imply

$$\boldsymbol{B}_m, \tilde{\theta}_m \in L^\infty([0,T],V) \cap L^2([0,T],\mathcal{D}(A)), \tag{33}$$

uniformly with respect to m. Moreover, remembering (19), we deduce

$$\frac{d\boldsymbol{u}_m}{dt} \in L^\infty([0,T],V),$$

uniformly with respect to m, with help of the improved bound (33).

Secondly, we consider $\frac{d\boldsymbol{B}_m}{dt}$ and $\frac{d\tilde{\theta}_m}{dt}$. Combining (9b), (9c) and (16), we have

$$\begin{aligned}
\|\frac{d\boldsymbol{B}_m}{dt}\| &\le \|P_m B(\boldsymbol{B}_m, \boldsymbol{u}_m)\| + \|P_m B(\boldsymbol{u}_m, \boldsymbol{B}_m)\| + \mu\|A\boldsymbol{B}_m\| \\
&= \sup_{\boldsymbol{w}\in H} \frac{|b(\boldsymbol{B}_m, \boldsymbol{u}_m, P_m\boldsymbol{w})|}{\|\boldsymbol{w}\|} + \sup_{\boldsymbol{w}\in H} \frac{|b(\boldsymbol{u}_m, \boldsymbol{B}_m, P_m\boldsymbol{w})|}{\|\boldsymbol{w}\|} + \mu\|A\boldsymbol{B}_m\| \\
&\le C\|\nabla \boldsymbol{B}_m\|^{\frac{1}{2}}\|A\boldsymbol{B}_m\|^{\frac{1}{2}}\|\nabla\boldsymbol{u}_m\| + C\|\nabla\boldsymbol{u}_m\|\|A\boldsymbol{B}_m\| + \mu\|A\boldsymbol{B}_m\| \\
&\le C(K_3^\alpha \alpha^{-1} + \mu)\|A\boldsymbol{B}_m\|, \\
\|\frac{d\tilde{\theta}_m}{dt}\| &\le \|B(\boldsymbol{u}_m, \tilde{\theta}_m)\| + \kappa\|A\tilde{\theta}_m\| = \sup_{\boldsymbol{w}\in H} \frac{\left|b(\boldsymbol{u}_m, \tilde{\theta}_m, \boldsymbol{w})\right|}{\|\boldsymbol{w}\|} + \kappa\|A\tilde{\theta}_m\| \\
&\le C\|\boldsymbol{u}_m\|^{\frac{1}{2}}\|\nabla\boldsymbol{u}_m\|^{\frac{1}{2}}\|A\tilde{\theta}_m\| + \kappa\|A\tilde{\theta}_m\| \\
&\le C(K_3^\alpha \alpha^{-1} + \kappa)\|A\tilde{\theta}_m\|,
\end{aligned}$$

which and (33) yields

$$\frac{d\boldsymbol{B}_m}{dt}, \frac{d\tilde{\theta}_m}{dt} \in L^2([0,T],H), \tag{34}$$

uniformly with respect to m.

Finally, according to the proof of Theorem 3.1, we find that there exists a weak solution $(\boldsymbol{u}, \boldsymbol{B}, \tilde{\theta})$ to (8) such that $\boldsymbol{B}_m \to \boldsymbol{B}$, $\tilde{\theta}_m \to \tilde{\theta}$ in $L^\infty([0,T],H) \cap L^2([0,T],V)$ for subsequence $\{\boldsymbol{B}_m, \tilde{\theta}_m\}$. Next, using (33) and (34) and applying the Aubin compactness theorem to extract a subsequence (relabeled as $(\boldsymbol{B}_m, \tilde{\theta}_m)$) such that

$$\boldsymbol{B}_m \to \boldsymbol{B},\ \tilde{\theta}_m \to \tilde{\theta} \text{ strongly in } L^2([0,T],V).$$

By the Banach-Alaoglu Theorem, (33) and the uniqueness of limit, it follows that $\boldsymbol{B}, \tilde{\theta} \in L^\infty([0,T],V) \cap L^2([0,T],\mathcal{D}(A))$. It is easy to find from (34) that $\frac{d\boldsymbol{B}}{dt}, \frac{d\tilde{\theta}}{dt} \in L^2([0,T],H)$, which implies that $\boldsymbol{B}, \tilde{\theta} \in C([0,T],V)$. Because of $\boldsymbol{u}$, $\boldsymbol{B}$, $\tilde{\theta} \in C([0,T],V)$, we deduce that the RHS of (8a) belongs to $C([0,T],V')$. Inverting $I + \alpha^2 A$ gives $\frac{d\boldsymbol{u}}{dt} \in C([0,T],V)$. □

In the last part of this subsection, we show the uniqueness of strong solution. We first recall a lemma that will be used later.

Lemma 3.1. [32] *Let $\boldsymbol{v} \in C((0,T),H)$ and $\frac{d\boldsymbol{v}}{dt} \in L^p((0,T),H)$ for some $p \in [1,\infty]$. Then the following equality holds on $(0,T)$,*

$$\frac{d}{dt}\|\boldsymbol{v}\|^2 = 2\left(\frac{d\boldsymbol{v}}{dt}, \boldsymbol{v}\right).$$

Moreover, $\|\boldsymbol{v}\|^2$ is absolutely continuous.

We now turn to show the uniqueness of strong solution and its continuous dependence on initial data.

Theorem 3.3. *Let* $(\boldsymbol{u}_i, \boldsymbol{B}_i, \tilde{\theta}_i)$ $(i = 1, 2)$ *be two strong solutions to* (8) *with initial data* $\boldsymbol{u}_0^i, \boldsymbol{B}_0^i, \tilde{\theta}_0^i \in V$. *Then one gets*

$$\begin{aligned} &\|\delta \boldsymbol{u}(t)\|^2 + \alpha^2 \|\nabla \delta \boldsymbol{u}(t)\|^2 + \|\delta \boldsymbol{B}(t)\|^2 + \|\delta \tilde{\theta}(t)\|^2 \\ &\leq (\|\delta \boldsymbol{u}_0\|^2 + \alpha^2 \|\nabla \delta \boldsymbol{u}_0\|^2 + \|\delta \boldsymbol{B}_0\|^2 + \|\delta \tilde{\theta}_0\|) \exp(K_6^\alpha t), \end{aligned} \tag{35}$$

where $\delta v := v_1 - v_2$ *and* $K_6^\alpha(K_3^\alpha, K_5^\alpha, K_{5'}^\alpha, \alpha, \nu, \mu, \kappa) > 0$ *is a constant. Especially, if* $\boldsymbol{u}_0^1 = \boldsymbol{u}_0^2$, $\boldsymbol{B}_0^1 = \boldsymbol{B}_0^2$ *and* $\tilde{\theta}_0^1 = \tilde{\theta}_0^2$, *then* $\delta \boldsymbol{u} = \boldsymbol{0}$, $\delta \boldsymbol{B} = \boldsymbol{0}$ *and* $\delta \tilde{\theta} = 0$.

Proof. We show that from (8a) we have

$$(I + \alpha^2 A) \frac{d\delta \boldsymbol{u}}{dt} = B(\boldsymbol{B}_1, \boldsymbol{B}_1) - B(\boldsymbol{u}_1, \boldsymbol{u}_1) - B(\boldsymbol{B}_2, \boldsymbol{B}_2) + B(\boldsymbol{u}_2, \boldsymbol{u}_2) + \boldsymbol{\beta} \delta \tilde{\theta}. \tag{36}$$

Multiply both sides of (36) with $(I + \alpha^2 A)^{-\frac{1}{2}}$.

$$\begin{aligned} (I + \alpha^2 A)^{\frac{1}{2}} \frac{d\delta \boldsymbol{u}}{dt} =& (I + \alpha^2 A)^{-\frac{1}{2}} \Big(B(\boldsymbol{B}_1, \boldsymbol{B}_1) \\ & - B(\boldsymbol{u}_1, \boldsymbol{u}_1) - B(\boldsymbol{B}_2, \boldsymbol{B}_2) + B(\boldsymbol{u}_2, \boldsymbol{u}_2) + \boldsymbol{\beta} \delta \tilde{\theta} \Big). \end{aligned} \tag{37}$$

Since $\delta \boldsymbol{u} \in C([0, T], V)$ and $\frac{d\delta \boldsymbol{u}}{dt} \in L^{\frac{4}{3}}([0, T], V)$, we have $(I + \alpha^2 A)^{\frac{1}{2}} \frac{d}{dt} \delta \boldsymbol{u} \in L^{\frac{4}{3}}([0, T], H)$. Then, taking the inner product with $(I + \alpha^2 A)^{\frac{1}{2}} \delta \boldsymbol{u} \in C([0, T], H)$, we get from (37)

$$\begin{aligned} &\left((I + \alpha^2 A)^{\frac{1}{2}} \frac{d\delta \boldsymbol{u}}{dt}, (I + \alpha^2 A)^{\frac{1}{2}} \delta \boldsymbol{u} \right) = \frac{1}{2} \frac{d}{dt} \|(I + \alpha^2 A)^{\frac{1}{2}} \delta \boldsymbol{u}\|^2 \\ &= b(\delta \boldsymbol{B}, \boldsymbol{B}_1, \delta \boldsymbol{u}) + b(\boldsymbol{B}_2, \delta \boldsymbol{B}, \delta \boldsymbol{u}) - b(\delta \boldsymbol{u}, \boldsymbol{u}_1, \delta \boldsymbol{u}) + (\boldsymbol{\beta} \delta \tilde{\theta}, \delta \boldsymbol{u}), \end{aligned} \tag{38}$$

where we have used Lemma 3.1 and (4). Arguing in a similar way to (38), from (8b) and (8b), we gain

$$\frac{1}{2} \frac{d}{dt} \|\delta \boldsymbol{B}\|^2 + \mu \|\nabla \delta \boldsymbol{B}\|^2 = b(\delta \boldsymbol{B}, \boldsymbol{u}_1, \delta \boldsymbol{B}) + b(\boldsymbol{B}_2, \delta \boldsymbol{u}, \delta \boldsymbol{B}) - b(\delta \boldsymbol{u}, \boldsymbol{B}_1, \delta \boldsymbol{B}),$$

$$\frac{1}{2} \frac{d}{dt} \|\delta \tilde{\theta}\|^2 + \kappa \|\nabla \delta \tilde{\theta}\|^2 = -b(\delta \boldsymbol{u}, \tilde{\theta}_2, \delta \tilde{\theta}).$$

Moreover, together these equations lead to

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|(I + \alpha^2 A)^{\frac{1}{2}} \delta \boldsymbol{u}\|^2 + \|\delta \boldsymbol{B}\|^2 + \|\delta \tilde{\theta}\|^2) + \mu \|\nabla \delta \boldsymbol{B}\|^2 + \kappa \|\nabla \delta \tilde{\theta}\|^2 \\ &= b(\delta \boldsymbol{B}, \boldsymbol{B}_1, \delta \boldsymbol{u}) - b(\delta \boldsymbol{u}, \boldsymbol{u}_1, \delta \boldsymbol{u}) + b(\delta \boldsymbol{B}, \boldsymbol{u}_1, \delta \boldsymbol{B}) \\ &\quad - b(\delta \boldsymbol{u}, \boldsymbol{B}_1, \delta \boldsymbol{B}) - b(\delta \boldsymbol{u}, \tilde{\theta}_2, \delta \tilde{\theta}) + (\boldsymbol{\beta} \delta \tilde{\theta}, \delta \boldsymbol{u}) \\ &\leq C \|\nabla \delta \boldsymbol{B}\| \|\nabla \boldsymbol{B}_1\| \|\nabla \delta \boldsymbol{u}\| + C \|\delta \boldsymbol{u}\|^{\frac{1}{2}} \|\nabla \delta \boldsymbol{u}\|^{\frac{3}{2}} \|\nabla \boldsymbol{u}_1\| \\ &\quad + C \|\delta \boldsymbol{B}\|^{\frac{1}{2}} \|\nabla \delta \boldsymbol{B}\|^{\frac{3}{2}} \|\nabla \boldsymbol{u}_1\| + C \|\nabla \delta \boldsymbol{u}\| \|\nabla \boldsymbol{B}_1\| \|\delta \boldsymbol{B}\|^{\frac{1}{2}} \|\nabla \delta \boldsymbol{B}\|^{\frac{1}{2}} \\ &\quad + C \|\nabla \delta \boldsymbol{u}\| \|\nabla \tilde{\theta}_2\| \|\delta \tilde{\theta}\|^{\frac{1}{2}} \|\nabla \delta \tilde{\theta}\|^{\frac{1}{2}} + \|\boldsymbol{\beta} \delta \tilde{\theta}\| \|\delta \boldsymbol{u}\| \\ &\leq C K_5^\alpha \|\nabla \delta \boldsymbol{B}\| \|\nabla \delta \boldsymbol{u}\| + C K_3^\alpha \|\delta \boldsymbol{u}\|^{\frac{1}{2}} \|\nabla \delta \boldsymbol{u}\|^{\frac{3}{2}} + C K_3^\alpha \|\delta \boldsymbol{B}\|^{\frac{1}{2}} \|\nabla \delta \boldsymbol{B}\|^{\frac{3}{2}} \\ &\quad + C K_5^\alpha \|\nabla \delta \boldsymbol{u}\| \|\delta \boldsymbol{B}\|^{\frac{1}{2}} \|\nabla \delta \boldsymbol{B}\|^{\frac{1}{2}} + C K_{5'}^\alpha \|\nabla \delta \boldsymbol{u}\| \|\delta \tilde{\theta}\|^{\frac{1}{2}} \|\nabla \delta \tilde{\theta}\|^{\frac{1}{2}} + \|\boldsymbol{\beta} \delta \tilde{\theta}\| \|\delta \boldsymbol{u}\| \\ &\leq C K_6^\alpha (\|\delta \boldsymbol{u}\|^2 + \alpha^2 \|\nabla \delta \boldsymbol{u}\|^2 + \|\delta \boldsymbol{B}\|^2 + \|\delta \tilde{\theta}\|^2) + \frac{\mu}{2} \|\nabla \delta \boldsymbol{B}\|^2 + \frac{\kappa}{2} \|\nabla \delta \tilde{\theta}\|^2, \end{aligned}$$

where $K_6^\alpha = K_6^\alpha(K_3^\alpha, K_5^\alpha, K_{5'}^\alpha, \alpha, \nu, \mu, \kappa)$ is a positive constant. Here, we used standard estimates on the trilinear term, the Young inequality, (16), (31) and (32) (hold in the limit as $m \to \infty$ for $\boldsymbol{u}_i, \boldsymbol{B}_i, \tilde{\theta}_i$).

Therefore, one deduces that

$$\frac{d}{dt}(\|(I+\alpha^2 A)^{\frac{1}{2}}\delta\boldsymbol{u}\|^2 + \|\delta\boldsymbol{B}\|^2 + \|\delta\tilde{\theta}\|^2) + \mu\|\nabla\delta\boldsymbol{B}\|^2 + \kappa\|\nabla\delta\tilde{\theta}\|^2 \le CK_6^\alpha(\|\delta\boldsymbol{u}\|^2 + \alpha^2\|\nabla\delta\boldsymbol{u}\|^2 + \|\delta\boldsymbol{B}\|^2 + \|\delta\tilde{\theta}\|^2).$$

Using the differential Grönwall's inequality (5) together with the identity

$$\|(I+\alpha^2 A)^{\frac{1}{2}}\delta\boldsymbol{u}\|^2 = \|\delta\boldsymbol{u}\|^2 + \alpha^2\|\nabla\delta\boldsymbol{u}\|^2,$$

we finishes the proof. □

4. A fully discrete approximation

In this section, we will design a fully discrete scheme for the Voigt-regularization of the thermally coupled inviscid, resistive MHD equations (2).

Denote by τ_h a regular, conforming mesh of a polygon Ω with maximum element diameter h. Let P_k be the set of continuous piecewise polynomials those are of degree k on each element, and P_{k-1}^{dc} be the set of discontinuous piecewise polynomials those are of degree $k-1$ on each element, respectively. The finite element spaces used throughout will be the Scott-Vogelius (SV) pair [43], $(\boldsymbol{X}_h, Q_h) = ((P_k)^d, P_{k-1}^{dc})$, which enforces strong (pointwise) divergence free, and will approximate velocity and pressure, as well as the magnetic field and corresponding Lagrange multiplier. The temperature will be approximated by $X_h = P_k$. The mesh will be built from a barycenter refinement of a regular mesh if $k \ge d$, and if $k = d-1$, the mesh will be the Powell-Sabin mesh for the inf-sup stability [44, 45].

The numerical scheme is now derived with a Galerkin finite element method for spatial discretization and the Crank-Nicolson scheme for temporal discretization. For simplicity, we require the discrete initial conditions $\boldsymbol{u}_h^0 = \boldsymbol{u}_0$, $\boldsymbol{B}_h^0 = \boldsymbol{B}_0$ and $\theta_h^0 = \theta_0$. Define $\boldsymbol{u}_h^{-1} := \boldsymbol{u}_h^0$, $\boldsymbol{B}_h^{-1} := \boldsymbol{B}_h^0$, $\theta_h^{-1} := \theta_h^0$, and $s_h^{n+\frac{1}{2}} := \frac{1}{2}(s_h^n + s_h^{n+1})$ with $s = \boldsymbol{u}$, $\boldsymbol{B}$, P, q and θ.

The resulting scheme reads: for all $(\boldsymbol{v}_h, \boldsymbol{\chi}_h, \lambda_h, r_h, \phi_h) \in (\boldsymbol{X}_h, \boldsymbol{X}_h, Q_h, Q_h, X_h)$, find $(\boldsymbol{u}_h^{n+1}, \boldsymbol{B}_h^{n+1}, P_h^{n+\frac{1}{2}}, q_h^{n+\frac{1}{2}}, \theta_h^{n+1}) \in (\boldsymbol{X}_h, \boldsymbol{X}_h, Q_h, Q_h, X_h)$, for $n = 0, 1, 2, \cdots, M-1$, with $M = \frac{T}{\Delta t}$,

$$\frac{1}{\Delta t}(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n, \boldsymbol{v}_h) + \frac{\alpha^2}{\Delta t}(\nabla(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n), \nabla\boldsymbol{v}_h) + ((\boldsymbol{u}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{u}_h^{n+\frac{1}{2}}, \boldsymbol{v}_h) - ((\boldsymbol{B}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{B}_h^{n+\frac{1}{2}}, \boldsymbol{v}_h) - \left(P_h^{n+\frac{1}{2}}, \nabla\cdot\boldsymbol{v}_h\right) = (\boldsymbol{\beta}\theta_h^{n+\frac{1}{2}}, \boldsymbol{v}_h), \tag{39a}$$

$$(\nabla\cdot\boldsymbol{u}_h^{n+1}, \lambda_h) = 0, \tag{39b}$$

$$\frac{1}{\Delta t}(\boldsymbol{B}_h^{n+1} - \boldsymbol{B}_h^n, \boldsymbol{\chi}_h) + \mu\left(\nabla\boldsymbol{B}_h^{n+\frac{1}{2}}, \nabla\boldsymbol{\chi}_h\right) - ((\boldsymbol{B}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{u}_h^{n+\frac{1}{2}}, \boldsymbol{\chi}_h) + ((\boldsymbol{u}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{B}_h^{n+\frac{1}{2}}, \boldsymbol{\chi}_h) - \left(q_h^{n+\frac{1}{2}}, \nabla\cdot\boldsymbol{\chi}_h\right) = 0, \tag{39c}$$

$$(\nabla\cdot\boldsymbol{B}_h^{n+1}, r_h) = 0, \tag{39d}$$

$$\frac{1}{\Delta t}(\theta_h^{n+1} - \theta_h^n, \phi_h) + \kappa\left(\nabla\theta_h^{n+\frac{1}{2}}, \nabla\phi_h\right) + (\boldsymbol{u}_h^{n+\frac{1}{2}} \cdot \nabla\theta_h^{n+\frac{1}{2}}, \phi_h) = 0. \tag{39e}$$

For the existence of the above scheme, we will prove it by using a simple consequence of Brouwer's fixed point theorem [46], see an application in Lemma 5.3 of [47].

Theorem 4.1. *The scheme* (39) *has a solution for* $\Delta t \le C_{exist}$*, where* $C_{exist} := \kappa(|\boldsymbol{\beta}|^2 C_\Omega^2)^{-1}$ *and* C_Ω *is the Pioncaré constant depending only on* Ω.

Proof. Notice that the existence of $\boldsymbol{u}_h^{n+1}$ is equivalent to the existence of $\boldsymbol{u}_h^{n+\frac{1}{2}}$, for given $\boldsymbol{u}_h^n$. Besides $\boldsymbol{B}^{n+1}$ and θ_h^{n+1} have similar equivalence. We define an operator $G_h : \mathcal{F}_h \longrightarrow \mathcal{F}_h$, where $\mathcal{F}_h = \boldsymbol{X}_h \times \boldsymbol{X}_h \times Q_h$ by

$$\begin{aligned}(G_h(\boldsymbol{F}), \boldsymbol{w}_h) =& \frac{2}{\Delta t}\sum_{i=1}^{3}(F_i - F_{i,h}^n, \boldsymbol{w}_{i,h}) + \frac{2\alpha^2}{\Delta t}(\nabla(F_1 - \boldsymbol{u}_h^n), \nabla \boldsymbol{v}_h) + ((F_1\cdot\nabla)F_1, \boldsymbol{v}_h) \\ &- ((F_2\cdot\nabla)F_2, \boldsymbol{v}_h) - (\boldsymbol{\beta}F_3, \boldsymbol{v}_h) + \mu(\nabla F_2, \nabla \boldsymbol{\chi}_h) - ((F_2\cdot\nabla)F_1, \boldsymbol{\chi}_h) \\ &+ ((F_1\cdot\nabla)F_2, \boldsymbol{\chi}_h) + \kappa(\nabla F_3, \nabla\phi_h) + (F_1\cdot\nabla F_3, \phi_h), \end{aligned} \tag{40}$$

where $\boldsymbol{F} = (F_1, F_2, F_3) := (\boldsymbol{u}_h^{n+\frac{1}{2}}, \boldsymbol{B}_h^{n+\frac{1}{2}}, \theta_h^{n+\frac{1}{2}})$, $\boldsymbol{F}_h^n := (\boldsymbol{u}_h^n, \boldsymbol{B}_h^n, \theta_h^n)$, and $\boldsymbol{w}_h := (\boldsymbol{v}_h, \boldsymbol{\chi}_h, \phi_h)$. It is easy to prove that G_h is continuous for given $\boldsymbol{F}_h^n$.

Next, we define a convex, finite dimensional and closed set

$$\boldsymbol{S}_h = \{\boldsymbol{F} \in \mathcal{F}_h, \|\boldsymbol{F}\|_{\mathcal{F}} \le C_{\mathcal{F}}\},$$

with $C_{\mathcal{F}} > (\sum_{i=1}^{3}\|F_{i,h}^n\|^2 + \alpha^2\|\nabla \boldsymbol{u}_h^n\|^2)^{\frac{1}{2}}$ and the norm

$$\|\boldsymbol{F}\|_{\mathcal{F}} = \left(\sum_{i=2}^{3}\|F_i\|^2 + \alpha^2\|\nabla F_1\|^2 + \mu\Delta t\|\nabla F_2\|^2 + \Delta t(\kappa - \frac{\Delta t|\boldsymbol{\beta}|^2 C_\Omega^2}{4})\|\nabla F_3\|^2\right)^{\frac{1}{2}}.$$

Now, in order to prove that the equation $G_h(\boldsymbol{F}) = 0$ has a solution, i.e., the scheme (39) has a solution, all we need to show is that $G_h(\boldsymbol{F}, \boldsymbol{F}) > 0$ for $\|\boldsymbol{F}\|_{\mathcal{F}} = C_{\mathcal{F}}$ [46]. In fact, let $\boldsymbol{w}_h = \Delta t\boldsymbol{F}$ in (40) we have

$$\begin{aligned}(G_h(\boldsymbol{F}), \boldsymbol{F}) =& 2\sum_{i=1}^{3}(F_i - F_{i,h}^n, F_i) + 2\alpha^2(\nabla(F_1 - \boldsymbol{u}_h^n), \nabla F_1) - \Delta t(\boldsymbol{\beta}F_3, F_1) \\ &+ \mu\Delta t\|\nabla F_2\|^2 + \kappa\Delta t\|\nabla F_3\|^2 \\ \ge& \sum_{i=2}^{3}\|F_i\|^2 - \sum_{i=1}^{3}\|F_{i,h}^n\|^2 + \alpha^2\|\nabla F_1\|^2 - \alpha^2\|\nabla \boldsymbol{u}_h^n\|^2 \\ &+ \Delta t(\kappa - \frac{\Delta t|\boldsymbol{\beta}|^2 C_\Omega^2}{4})\|\nabla F_3\|^2 + \mu\Delta t\|\nabla F_2\|^2 \\ =& \|\boldsymbol{F}\|_{\mathcal{F}}^2 - \sum_{i=1}^{3}\|F_{i,h}^n\|^2 - \alpha^2\|\nabla \boldsymbol{u}_h^n\|^2. \end{aligned}$$

Thus, if $\|\boldsymbol{F}\|_{\mathcal{F}} = C_{\mathcal{F}}$, we have $(G_h(\boldsymbol{F}), \boldsymbol{F}) > 0$. Since the SV elements are a stable finite element pair under corresponding mesh restrictions, $p_h^{n+\frac{1}{2}}$ and $q_h^{n+\frac{1}{2}}$ exist if $\boldsymbol{u}_h^{n+1}, \boldsymbol{B}_h^{n+1}, \theta_h^{n+1}$ are given. □

We prove here that the numerical solution is stable.

Theorem 4.2. *Under assumption of Theorem 4.1, suppose that* $\boldsymbol{u}_0 \in H^1(\Omega)^d$, $\boldsymbol{B}_0 \in L^2(\Omega)^d$ *and* $\theta_0 \in L^2(\Omega)$. *Then there exists a positive constant* $C_g = \max\{|\beta|, 1\}$

such that when $C_g\Delta t < 1$, the solution (39a)-(39e) *is stable, that is*

$$\begin{aligned}
&\|\boldsymbol{u}_h^M\|^2+\alpha^2\|\nabla\boldsymbol{u}_h^M\|^2+\|\boldsymbol{B}_h^M\|^2+\|\theta_h^M\|^2\\
&+\Delta t\sum_{n=0}^{M-1}\Big(\mu\|\nabla\boldsymbol{B}_h^{n+\frac{1}{2}}\|^2+\kappa\|\nabla\theta_h^{n+\frac{1}{2}}\|^2\Big)\\
&\le\exp(C_gT)(\|\boldsymbol{u}_h^0\|^2+\alpha^2\|\nabla\boldsymbol{u}_h^0\|+\|\boldsymbol{B}_h^0\|^2+\|\theta_h^0\|^2).
\end{aligned}\tag{41}$$

Proof. Select $\boldsymbol{v}_h=\boldsymbol{u}_h^{n+\frac{1}{2}}$, $\boldsymbol{\chi}_h=\boldsymbol{B}_h^{n+\frac{1}{2}}$ and $\phi_h=\theta_h^{n+\frac{1}{2}}$ in (39a), (39c) and (39e), respectively. Next, adding the ensuing equations, multiplying through by Δt and summing from $n=0$ to $M-1$ yield

$$\begin{aligned}
&\frac{1}{2}\left(\|\boldsymbol{u}_h^M\|^2+\alpha^2\|\nabla\boldsymbol{u}_h^M\|^2+\|\boldsymbol{B}_h^M\|^2+\|\theta_h^M\|^2\right)\\
&+\Delta t\sum_{n=0}^{M-1}\Big(\mu\|\nabla\boldsymbol{B}_h^{n+\frac{1}{2}}\|^2+\kappa\|\nabla\theta_h^{n+\frac{1}{2}}\|^2\Big)\\
=&\frac{1}{2}\left(\|\boldsymbol{u}_h^0\|^2+\alpha^2\|\nabla\boldsymbol{u}_h^0\|+\|\boldsymbol{B}_h^0\|^2+\|\theta_h^0\|^2\right)+\Delta t\sum_{n=0}^{M-1}(\boldsymbol{\beta}\theta_h^{n+\frac{1}{2}},\boldsymbol{u}_h^{n+\frac{1}{2}}),
\end{aligned}$$

which and the Cauchy-Schwarz-Young inequality imply

$$\begin{aligned}
&\|\boldsymbol{u}_h^M\|^2+\alpha^2\|\nabla\boldsymbol{u}_h^M\|^2+\|\boldsymbol{B}_h^M\|^2+\|\theta_h^M\|^2\\
&+\Delta t\sum_{n=0}^{M-1}\Big(\mu\|\nabla\boldsymbol{B}_h^{n+\frac{1}{2}}\|^2+\kappa\|\nabla\theta_h^{n+\frac{1}{2}}\|^2\Big)\\
\le&\Delta t\sum_{n=0}^{M-1}C_g\Big(\|\boldsymbol{u}_h^{n+1}\|^2+\alpha^2\|\nabla\boldsymbol{u}_h^{n+1}\|^2+\|\theta_h^{n+1}\|^2+\|\boldsymbol{B}_h^{n+1}\|^2\Big)\\
&+\|\boldsymbol{u}_h^0\|^2+\alpha^2\|\nabla\boldsymbol{u}_h^0\|+\|\boldsymbol{B}_h^0\|^2+\|\theta_h^0\|^2.
\end{aligned}$$

Using Grönwall's inequality (6) completes the proof. □

Now we prove the uniqueness of the solution to the scheme (39).

Theorem 4.3. *Under assumption of Theorem 4.2, let that $2\alpha^2-C(\mu^{-1}+\kappa^{-1})>0$ and $\Delta t<C_{unique}$ where $C_{unique}:=\min\{2,\frac{8}{|\boldsymbol{\beta}|^2},\frac{2\alpha^2}{C\alpha^{-2}+1},C\alpha^2(2\alpha^2-C(\mu^{-1}+\kappa^{-1}))\}$. Then the scheme* (39) *has a unique solution.*

Proof. Let $\boldsymbol{u}_{1,h}^{n+1},\boldsymbol{B}_{1,h}^{n+1},\theta_{1,h}^{n+1}$ and $\boldsymbol{u}_{2,h}^{n+1},\boldsymbol{B}_{2,h}^{n+1},\theta_{2,h}^{n+1}$ be two solutions to the scheme (39), and $e_s=s_{1,h}^{n+\frac{1}{2}}-s_{2,h}^{n+\frac{1}{2}}$, where $s=\boldsymbol{u},\boldsymbol{B},\theta$. Then we have

$$\begin{aligned}
&\frac{2}{\Delta t}\Big((e_{\boldsymbol{u}},\boldsymbol{v}_h)+(e_{\boldsymbol{B}},\chi_h)+(e_\theta,\phi_h)\Big)+\frac{2\alpha^2}{\Delta t}(\nabla e_{\boldsymbol{u}},\nabla\boldsymbol{v}_h)+\mu(\nabla e_{\boldsymbol{B}},\nabla\chi_h)\\
&+\kappa(\nabla e_\theta,\nabla\phi_h)+\Big((\boldsymbol{u}_{1,h}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{u}_{1,h}^{n+\frac{1}{2}}-(\boldsymbol{u}_{2,h}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{u}_{2,h}^{n+\frac{1}{2}},\boldsymbol{v}_h\Big)\\
&-\Big((\boldsymbol{B}_{1,h}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{B}_{1,h}^{n+\frac{1}{2}}-(\boldsymbol{B}_{2,h}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{B}_{2,h}^{n+\frac{1}{2}},\boldsymbol{v}_h\Big)\\
&-\Big((\boldsymbol{B}_{1,h}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{u}_{1,h}^{n+\frac{1}{2}}-(\boldsymbol{B}_{2,h}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{u}_{2,h}^{n+\frac{1}{2}},\boldsymbol{\chi}_h\Big)\\
&+\Big((\boldsymbol{u}_{1,h}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{B}_{1,h}^{n+\frac{1}{2}}-(\boldsymbol{u}_{2,h}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{B}_{2,h}^{n+\frac{1}{2}},\boldsymbol{\chi}_h\Big)\\
&+\Big((\boldsymbol{u}_{1,h}^{n+\frac{1}{2}}\cdot\nabla)\theta_{1,h}^{n+\frac{1}{2}}-(\boldsymbol{u}_{2,h}^{n+\frac{1}{2}}\cdot\nabla)\theta_{2,h}^{n+\frac{1}{2}},\phi_h\Big)-(\boldsymbol{\beta}e_\theta,\boldsymbol{v}_h)
\end{aligned}$$

$$
\begin{aligned}
=&\frac{2}{\Delta t}\Big((e_{\boldsymbol{u}},\boldsymbol{v}_h)+(e_{\boldsymbol{B}},\chi_h)+(e_\theta,\phi_h)\Big)+\frac{2\alpha^2}{\Delta t}(\nabla e_{\boldsymbol{u}},\nabla \boldsymbol{v}_h)\\
&+\mu(\nabla e_{\boldsymbol{B}},\nabla\chi_h)+\kappa(\nabla e_\theta,\nabla\phi_h)\\
&+\Big((\boldsymbol{u}_{1,h}^{n+\frac12}\cdot\nabla)e_{\boldsymbol{u}}+(e_{\boldsymbol{u}}\cdot\nabla)\boldsymbol{u}_{2,h}^{n+\frac12},\boldsymbol{v}_h\Big)-\Big((\boldsymbol{B}_{1,h}^{n+\frac12}\cdot\nabla)e_{\boldsymbol{B}}+(e_{\boldsymbol{B}}\cdot\nabla)\boldsymbol{B}_{2,h}^{n+\frac12},\boldsymbol{v}_h\Big)\\
&-\Big((\boldsymbol{B}_{1,h}^{n+\frac12}\cdot\nabla)e_{\boldsymbol{u}}+(e_{\boldsymbol{B}}\cdot\nabla)\boldsymbol{u}_{2,h}^{n+\frac12},\boldsymbol{\chi}_h\Big)+\Big((\boldsymbol{u}_{1,h}^{n+\frac12}\cdot\nabla)e_{\boldsymbol{B}}+(e_{\boldsymbol{u}}\cdot\nabla)\boldsymbol{B}_{2,h}^{n+\frac12},\boldsymbol{\chi}_h\Big)\\
&+\Big((\boldsymbol{u}_{1,h}^{n+\frac12}\cdot\nabla)e_\theta+(e_{\boldsymbol{u}}\cdot\nabla)\theta_{2,h}^{n+\frac12},\phi_h\Big)-(\boldsymbol{\beta}e_\theta,\boldsymbol{v}_h)=0.
\end{aligned}
$$

Selecting $\boldsymbol{v}_h=e_{\boldsymbol{u}},\boldsymbol{\chi}_h=e_{\boldsymbol{B}}$, and $\phi_h=e_\theta$, we have

$$
\begin{aligned}
&\frac{2}{\Delta t}\Big(\|e_{\boldsymbol{u}}\|^2+\|e_{\boldsymbol{B}}\|^2+\|e_\theta\|^2\Big)+\frac{2\alpha^2}{\Delta t}\|\nabla e_{\boldsymbol{u}}\|^2+\mu\|\nabla e_{\boldsymbol{B}}\|^2+\kappa\|\nabla e_\theta\|^2\\
&\quad+\Big((e_{\boldsymbol{u}}\cdot\nabla)\boldsymbol{u}_{2,h}^{n+\frac12},e_{\boldsymbol{u}}\Big)-\Big((e_{\boldsymbol{B}}\cdot\nabla)\boldsymbol{B}_{2,h}^{n+\frac12},e_{\boldsymbol{u}}\Big)-\Big((e_{\boldsymbol{B}}\cdot\nabla)\boldsymbol{u}_{2,h}^{n+\frac12},e_{\boldsymbol{B}}\Big)\\
&\quad+\Big((e_{\boldsymbol{u}}\cdot\nabla)\boldsymbol{B}_{2,h}^{n+\frac12},e_{\boldsymbol{B}}\Big)+\Big((e_{\boldsymbol{u}}\cdot\nabla)\theta_{2,h}^{n+\frac12},e_\theta\Big)-(\boldsymbol{\beta}e_\theta,e_{\boldsymbol{u}})=0,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\frac{2}{\Delta t}\Big(\|e_{\boldsymbol{u}}\|^2+\|e_{\boldsymbol{B}}\|^2+\|e_\theta\|^2\Big)+\frac{2\alpha^2}{\Delta t}\|\nabla e_{\boldsymbol{u}}\|^2+\mu\|\nabla e_{\boldsymbol{B}}\|^2+\kappa\|\nabla e_\theta\|^2\\
&\le C\Big[\|\nabla\boldsymbol{u}_{2,h}^{n+\frac12}\|^2\|\nabla e_{\boldsymbol{u}}\|^2+\|\nabla\boldsymbol{B}_{2,h}^{n+\frac12}\|^2\|\nabla e_{\boldsymbol{u}}\|^2+\|\nabla\boldsymbol{u}_{2,h}^{n+\frac12}\|^2\|\nabla e_{\boldsymbol{B}}\|^2\\
&\quad+\|\nabla\theta_{2,h}^{n+\frac12}\|^2\|\nabla e_{\boldsymbol{u}}\|^2\Big]+\epsilon_1\|\nabla e_{\boldsymbol{B}}\|^2+\epsilon_2\|\nabla e_\theta\|^2+\frac{|\boldsymbol{\beta}|^2}{4}\|e_\theta\|^2+\|e_{\boldsymbol{u}}\|^2,\\
&\le C\Big[\frac{C_{stab}}{\alpha^2}\|\nabla e_{\boldsymbol{u}}\|^2+\frac{C_{stab}}{\mu\Delta t}\|\nabla e_{\boldsymbol{u}}\|^2+\frac{C_{stab}}{\alpha^2}\|\nabla e_{\boldsymbol{u}}\|^2\\
&\quad+\frac{C_{stab}}{\kappa\Delta t}\|\nabla e_{\boldsymbol{u}}\|^2\Big]+\epsilon_1\|\nabla e_{\boldsymbol{B}}\|^2+\epsilon_2\|\nabla e_\theta\|^2+\frac{|\boldsymbol{\beta}|^2}{4}\|e_\theta\|^2+\|e_{\boldsymbol{u}}\|^2,
\end{aligned}
$$

where $C_{stab}=\exp(C_gT)(\|\boldsymbol{u}_h^0\|^2+\alpha^2\|\nabla\boldsymbol{u}_h^0\|+\|\boldsymbol{B}_h^0\|^2+\|\theta_h^0\|^2)$ and $0<\epsilon_1\le\mu$, $0<\epsilon_2\le\kappa$ from Young's inequality. Then

$$
\begin{aligned}
&(2-\Delta t)\|e_{\boldsymbol{u}}\|^2+\|e_{\boldsymbol{B}}\|^2+\Big(2-\frac{\Delta t|\boldsymbol{\beta}|^2}{4}\Big)\|e_\theta\|^2\\
&+\Big(2\alpha^2-C\alpha^{-2}\Delta t-C(\mu^{-1}+\kappa^{-1})\Big)\|\nabla e_{\boldsymbol{u}}\|^2\\
&+\Delta t(\mu-\epsilon_1)\|\nabla e_{\boldsymbol{B}}\|^2+\Delta t(\kappa-\epsilon_2)\|\nabla e_\theta\|^2\le 0.
\end{aligned}
$$

Therefore, the uniqueness of scheme (39) holds with uniqueness condition. □

We now prove convergence of the scheme. In order to obtain the error equations, we first multiply (2a), (2c) and (2e) at $t^{n+\frac12}$ by $\boldsymbol{v}_h\in\boldsymbol{X}_h$, $\boldsymbol{\chi}_h\in\boldsymbol{X}_h$ and $\phi_h\in X_h$, respectively. Then integrate the ensuing equations over the domain. For simplicity, we denote $e_{\boldsymbol{u}}^k=\boldsymbol{u}_h^k-\boldsymbol{u}(t^k)$, $e_{\boldsymbol{B}}^k=\boldsymbol{B}_h^k-\boldsymbol{B}(t^k)$ and $e_\theta^k=\theta_h^k-\theta(t^k)$. Additionally, $s^{n+\frac12}:=\frac{s(t^n)+s(t^{n+1})}{2}$ with $s=\boldsymbol{u}$, $\boldsymbol{B}$ and θ.

$$
\begin{aligned}
&(\boldsymbol{u}_t(t^{n+\frac12}),\boldsymbol{v}_h)+\alpha^2(\nabla\boldsymbol{u}_t(t^{n+\frac12}),\nabla\boldsymbol{v}_h)+((\boldsymbol{u}(t^{n+\frac12})\cdot\nabla)\boldsymbol{u}(t^{n+\frac12}),\boldsymbol{v}_h)\\
&\quad-((\boldsymbol{B}(t^{n+\frac12})\cdot\nabla)\boldsymbol{B}(t^{n+\frac12}),\boldsymbol{v}_h)=(\boldsymbol{\beta}\theta(t^{n+\frac12}),\boldsymbol{v}_h),
\end{aligned}\tag{42}
$$

$$
\begin{aligned}
&(\boldsymbol{B}_t(t^{n+\frac12}),\boldsymbol{\chi}_h)+\mu(\nabla\boldsymbol{B}(t^{n+\frac12}),\nabla\boldsymbol{\chi}_h)-((\boldsymbol{B}(t^{n+\frac12})\cdot\nabla)\boldsymbol{u}(t^{n+\frac12}),\boldsymbol{\chi}_h)\\
&\quad+((\boldsymbol{u}(t^{n+\frac12})\cdot\nabla)\boldsymbol{B}(t^{n+\frac12}),\boldsymbol{\chi}_h)=0,
\end{aligned}\tag{43}
$$

$$
(\theta_t(t^{n+\frac12}),\phi_h)+\kappa(\nabla\theta(t^{n+\frac12}),\nabla\phi_h)+(\boldsymbol{u}(t^{n+\frac12})\cdot\nabla\theta(t^{n+\frac12}),\phi_h)=0.\tag{44}
$$

Subtract (42) from (39a)

$$\begin{aligned}
&\frac{1}{\Delta t}(e_{\boldsymbol{u}}^{n+1}-e_{\boldsymbol{u}}^{n},\boldsymbol{v}_h)+\frac{\alpha^2}{\Delta t}(\nabla(e_{\boldsymbol{u}}^{n+1}-e_{\boldsymbol{u}}^{n}),\nabla\boldsymbol{v}_h)+((\boldsymbol{u}_h^{n+\frac{1}{2}}\cdot\nabla)e_{\boldsymbol{u}}^{n+\frac{1}{2}},\boldsymbol{v}_h)\\
&\quad+((e_{\boldsymbol{u}}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{u}^{n+\frac{1}{2}},\boldsymbol{v}_h)-((\boldsymbol{B}_h^{n+\frac{1}{2}}\cdot\nabla)e_{\boldsymbol{B}}^{n+\frac{1}{2}},\boldsymbol{v}_h)-((e_{\boldsymbol{B}}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{B}^{n+\frac{1}{2}},\boldsymbol{v}_h)\\
&=\left(\boldsymbol{u}_t(t^{n+\frac{1}{2}})-\frac{\boldsymbol{u}(t^{n+1})-\boldsymbol{u}(t^n)}{\Delta t},\boldsymbol{v}_h\right)\\
&\quad+\alpha^2\left(\nabla(\boldsymbol{u}_t(t^{n+\frac{1}{2}})-\frac{\boldsymbol{u}(t^{n+1})-\boldsymbol{u}(t^n)}{\Delta t}),\nabla\boldsymbol{v}_h\right)\\
&\quad+((\boldsymbol{u}(t^{n+\frac{1}{2}})\cdot\nabla)(\boldsymbol{u}(t^{n+\frac{1}{2}})-\boldsymbol{u}^{n+\frac{1}{2}}),\boldsymbol{v}_h)\\
&\quad+(((\boldsymbol{u}(t^{n+\frac{1}{2}})-\boldsymbol{u}^{n+\frac{1}{2}})\cdot\nabla)\boldsymbol{u}^{n+\frac{1}{2}},\boldsymbol{v}_h)\\
&\quad+((\boldsymbol{B}^{n+\frac{1}{2}}\cdot\nabla)(\boldsymbol{B}^{n+\frac{1}{2}}-\boldsymbol{B}(t^{n+\frac{1}{2}})),\boldsymbol{v}_h)\\
&\quad+(((\boldsymbol{B}^{n+\frac{1}{2}}-\boldsymbol{B}(t^{n+\frac{1}{2}}))\cdot\nabla)\boldsymbol{B}(t^{n+\frac{1}{2}}),\boldsymbol{v}_h)\\
&\quad-\boldsymbol{\beta}(\theta(t^{n+\frac{1}{2}})-\theta^{n+\frac{1}{2}},\boldsymbol{v}_h)+(\boldsymbol{\beta}\frac{e_\theta^{n+1}+e_\theta^n}{2},\boldsymbol{v}_h)\\
&=:G_1(t,\boldsymbol{B},\boldsymbol{u},\theta,\boldsymbol{v}_h)+(\boldsymbol{\beta}e_\theta^{n+\frac{1}{2}},\boldsymbol{v}_h). \qquad (45)
\end{aligned}$$

Note that G_1 represents terms associated only with the true solution. Furthermore, one easily gets

$$\begin{aligned}
&G_1(t,\boldsymbol{B},\boldsymbol{u},\theta,\boldsymbol{v}_h)\\
&\le C\Delta t^{\frac{3}{2}}\Big((\int_{t_n}^{t^{n+1}}\|\boldsymbol{u}_{ttt}\|^2dt)^{\frac{1}{2}}+(\int_{t_n}^{t^{n+1}}\|\nabla\boldsymbol{u}_{tt}\|^2dt)^{\frac{1}{2}}+(\int_{t_n}^{t^{n+1}}\|\boldsymbol{u}_{tt}\|^2dt)^{\frac{1}{2}}\\
&\quad+(\int_{t_n}^{t^{n+1}}\|\boldsymbol{B}_{tt}\|^2dt)^{\frac{1}{2}}+(\int_{t_n}^{t^{n+1}}\|\nabla\boldsymbol{B}_{tt}\|^2dt)^{\frac{1}{2}}+(\int_{t_n}^{t^{n+1}}\|\theta_{tt}\|^2dt)^{\frac{1}{2}}\Big)\|\boldsymbol{v}_h\|\\
&\quad+C\alpha^2\Delta t^{3/2}(\int_{t^n}^{t^{n+1}}\|\nabla\boldsymbol{u}_{ttt}\|^2dt)^{\frac{1}{2}}\|\nabla\boldsymbol{v}_h\|\\
&\le C\Delta t^3\int_{t_n}^{t^{n+1}}\Big(\|\boldsymbol{u}_{ttt}\|^2+\|\nabla\boldsymbol{u}_{tt}\|^2+\|\boldsymbol{u}_{tt}\|^2+\|\boldsymbol{B}_{tt}\|^2+\|\nabla\boldsymbol{B}_{tt}\|^2+\|\theta_{tt}\|^2\Big)dt\\
&\quad+\|\boldsymbol{v}_h\|^2+\alpha^2\|\nabla\boldsymbol{v}_h\|^2+C\alpha^2\Delta t^3\int_{t^n}^{t^{n+1}}\|\nabla\boldsymbol{u}_{ttt}\|^2dt. \qquad (46)
\end{aligned}$$

By a similar argument for the magnetic and thermal equations, we have

$$\begin{aligned}
&\frac{1}{\Delta t}(e_{\boldsymbol{B}}^{n+1}-e_{\boldsymbol{B}}^{n},\boldsymbol{\chi}_h)+\mu(\nabla e_{\boldsymbol{B}}^{n+\frac{1}{2}},\nabla\boldsymbol{\chi}_h)-((\boldsymbol{B}_h^{n+\frac{1}{2}}\cdot\nabla)e_{\boldsymbol{u}}^{n+\frac{1}{2}},\boldsymbol{\chi}_h)\\
&\quad-((e_{\boldsymbol{B}}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{u}^{n+\frac{1}{2}},\boldsymbol{\chi}_h)+((\boldsymbol{u}_h^{n+\frac{1}{2}}\cdot\nabla)e_{\boldsymbol{B}}^{n+\frac{1}{2}},\boldsymbol{\chi}_h)+((e_{\boldsymbol{u}}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{B}^{n+\frac{1}{2}},\boldsymbol{\chi}_h)\\
&=(\boldsymbol{B}_t(t^{n+\frac{1}{2}})-\frac{\boldsymbol{B}(t^{n+1})-\boldsymbol{B}(t^n)}{\Delta t},\boldsymbol{\chi}_h)+\mu(\nabla(\boldsymbol{B}(t^{n+\frac{1}{2}})-\boldsymbol{B}^{n+\frac{1}{2}}),\nabla\boldsymbol{\chi}_h)\\
&\quad+((\boldsymbol{B}(t^{n+\frac{1}{2}})\cdot\nabla)(\boldsymbol{u}^{n+\frac{1}{2}}-\boldsymbol{u}(t^{n+\frac{1}{2}})),\boldsymbol{\chi}_h)+(((\boldsymbol{B}^{n+\frac{1}{2}}-\boldsymbol{B}(t^{n+\frac{1}{2}}))\cdot\nabla)\boldsymbol{u}^{n+\frac{1}{2}},\boldsymbol{\chi}_h)\\
&\quad+((\boldsymbol{u}(t^{n+\frac{1}{2}})\cdot\nabla)(\boldsymbol{B}(t^{n+\frac{1}{2}})-\boldsymbol{B}^{n+\frac{1}{2}}),\boldsymbol{\chi}_h)+(((\boldsymbol{u}(t^{n+\frac{1}{2}})-\boldsymbol{u}^{n+\frac{1}{2}})\cdot\nabla)\boldsymbol{B}^{n+\frac{1}{2}},\boldsymbol{\chi}_h)\\
&=:G_2(t,\boldsymbol{B},\boldsymbol{u},\boldsymbol{\chi}_h), \qquad (47)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\Delta t}(e_\theta^{n+1}-e_\theta^n,\phi_h)+\kappa(\nabla e_\theta^{n+\frac12},\nabla\phi_h)+(\boldsymbol{u}_h^{n+\frac12}\cdot\nabla e_\theta^{n+\frac12},\phi_h)+(e_{\boldsymbol{u}}^{n+\frac12}\cdot\nabla\theta^{n+\frac12},\phi_h)\\
&=(\theta_t(t^{n+\frac12})-\frac{\theta(t^{n+1})-\theta(t^n)}{\Delta t},\phi_h)+\kappa(\nabla(\theta(t^{n+\frac12})-\theta^{n+\frac12}),\nabla\phi_h)\\
&\quad+(\boldsymbol{u}(t^{n+\frac12})\cdot\nabla(\theta(t^{n+\frac12})-\theta^{n+\frac12}),\phi_h)+((\boldsymbol{u}(t^{n+\frac12})-\boldsymbol{u}^{n+\frac12})\cdot\nabla\theta^{n+\frac12},\phi_h)\\
&=:G_3(t,\boldsymbol{u},\theta,\phi_h).
\end{aligned}\tag{48}$$

Moreover we bound G_2 and G_3 by

$$\begin{aligned}
&G_2(t,\boldsymbol{B},\boldsymbol{u},\boldsymbol{\chi}_h)\\
&\le C\Delta t^{\frac32}\Big((\int_{t_n}^{t^{n+1}}\|\boldsymbol{B}_{ttt}\|^2dt)^{\frac12}+(\int_{t_n}^{t^{n+1}}\|\nabla\boldsymbol{u}_{tt}\|^2dt)^{\frac12}+(\int_{t_n}^{t^{n+1}}\|\boldsymbol{B}_{tt}\|^2dt)^{\frac12}\\
&\quad+(\int_{t_n}^{t^{n+1}}\|\nabla\boldsymbol{B}_{tt}\|^2dt)^{\frac12}+(\int_{t_n}^{t^{n+1}}\|\boldsymbol{u}_{tt}\|^2dt)^{\frac12}\Big)\|\boldsymbol{\chi}_h\|\\
&\quad+C\Delta t^{\frac32}\|\nabla\boldsymbol{\chi}_h\|(\int_{t_n}^{t^{n+1}}\|\nabla\boldsymbol{B}_{tt}\|^2dt)^{\frac12}\\
&\le C\Delta t^3\int_{t_n}^{t^{n+1}}\Big(\|\boldsymbol{B}_{ttt}\|^2+\|\boldsymbol{B}_{tt}\|^2+\|\nabla\boldsymbol{u}_{tt}\|^2+\|\boldsymbol{u}_{tt}\|^2+\|\nabla\boldsymbol{B}_{tt}\|^2\Big)dt\\
&\quad+\|\boldsymbol{\chi}_h\|^2+\frac{\mu}{8}\|\nabla\boldsymbol{\chi}_h\|^2+C\Delta t^3\int_{t_n}^{t^{n+1}}\|\nabla\boldsymbol{B}_{tt}\|^2dt,
\end{aligned}\tag{49}$$

as well as

$$\begin{aligned}
G_3(t,\boldsymbol{u},\theta,\phi_h)&\le C\Delta t^{\frac32}\Big((\int_{t_n}^{t^{n+1}}\|\theta_{ttt}\|^2dt)^{\frac12}+(\int_{t_n}^{t^{n+1}}\|\nabla\theta_{tt}\|^2dt)^{\frac12}\\
&\quad+(\int_{t_n}^{t^{n+1}}\|\boldsymbol{u}_{tt}\|^2dt)^{\frac12}\Big)\|\phi_h\|+C\Delta t^{\frac32}(\int_{t_n}^{t^{n+1}}\|\nabla\theta_{tt}\|^2dt)^{\frac12}\|\nabla\phi_h\|\\
&\le C\Delta t^3\int_{t_n}^{t^{n+1}}(\|\theta_{ttt}\|^2+\|\nabla\theta_{tt}\|^2+\|\boldsymbol{u}_{tt}\|^2)dt+\|\phi_h\|^2+\frac{\kappa}{8}\|\nabla\phi_h\|^2\\
&\quad+C\Delta t^3\int_{t_n}^{t^{n+1}}\|\nabla\theta_{tt}\|^2dt.
\end{aligned}\tag{50}$$

Furthermore, we spilt the error as $e_{\boldsymbol{u}}^k=\boldsymbol{\xi}_h^k-\boldsymbol{\eta}_{\boldsymbol{u}}^k$ where $\boldsymbol{\xi}_h^k=(\boldsymbol{u}_h^k-\mathscr{U}^k)$ and $\boldsymbol{\eta}^k=(\boldsymbol{u}(t^k)-\mathscr{U}^k)$. Analogously, $e_{\boldsymbol{B}}^k=(\boldsymbol{B}_h^k-\mathscr{B}^k)+(\mathscr{B}^k-\boldsymbol{B}(t^k))=:\boldsymbol{\psi}_h^k-\boldsymbol{\eta}_{\boldsymbol{B}}^k$, and $e_\theta^k=(\theta_h^k-\Theta^k)+(\Theta^k-\theta(t^k))=:\omega_h^k-\eta_\theta^k$ where $\mathscr{U}^k\in\boldsymbol{X}_h$, $\mathscr{B}^k\in\boldsymbol{X}_h$ and $\Theta^k\in X_h$ are the finite element approximations of $\boldsymbol{u}$, $\boldsymbol{B}$ and θ. Substituting into (45), (47) and (48) results in

$$\begin{aligned}
&\frac{1}{\Delta t}(\boldsymbol{\xi}_h^{n+1}-\boldsymbol{\xi}_h^n,\boldsymbol{v}_h)+\frac{\alpha^2}{\Delta t}(\nabla(\boldsymbol{\xi}_h^{n+1}-\boldsymbol{\xi}_h^n),\nabla\boldsymbol{v}_h)+((\boldsymbol{u}_h^{n+\frac12}\cdot\nabla)\boldsymbol{\xi}_h^{n+\frac12},\boldsymbol{v}_h)\\
&\quad+((\boldsymbol{\xi}_h^{n+\frac12}\cdot\nabla\boldsymbol{u}^{n+\frac12},\boldsymbol{v}_h)-((\boldsymbol{B}_h^{n+\frac12}\cdot\nabla)\boldsymbol{\psi}_h^{n+\frac12},\boldsymbol{v}_h)-((\boldsymbol{\psi}_h^{n+\frac12}\cdot\nabla)\boldsymbol{B}^{n+\frac12},\boldsymbol{v}_h)\\
&=\frac{1}{\Delta t}(\boldsymbol{\eta}_{\boldsymbol{u}}^{n+1}-\boldsymbol{\eta}_{\boldsymbol{u}}^n,\boldsymbol{v}_h)+\frac{\alpha^2}{\Delta t}(\nabla(\boldsymbol{\eta}_{\boldsymbol{u}}^{n+1}-\boldsymbol{\eta}_{\boldsymbol{u}}^n),\nabla\boldsymbol{v}_h)+((\boldsymbol{u}_h^{n+\frac12}\cdot\nabla)\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac12},\boldsymbol{v}_h)\\
&\quad+((\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac12}\cdot\nabla)\boldsymbol{u}^{n+\frac12},\boldsymbol{v}_h)-((\boldsymbol{B}_h^{n+\frac12}\cdot\nabla)\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac12},\boldsymbol{v}_h)-((\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac12}\cdot\nabla)\boldsymbol{B}^{n+\frac12},\boldsymbol{v}_h)
\end{aligned}$$

$$+(\boldsymbol{\beta} e_\theta^{n+\frac{1}{2}}, \boldsymbol{v}_h) + G_1(t, \boldsymbol{u}, \boldsymbol{B}, \theta, \boldsymbol{v}_h), \tag{51}$$

as well as

$$\begin{aligned}
&\frac{1}{\Delta t}(\boldsymbol{\psi}_h^{n+1} - \boldsymbol{\psi}_h^n, \boldsymbol{\chi}_h) + \mu(\nabla \boldsymbol{\psi}_h^{n+\frac{1}{2}}, \nabla \boldsymbol{\chi}_h) - ((\boldsymbol{B}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{\xi}_h^{n+\frac{1}{2}}, \boldsymbol{\chi}_h) \\
&\quad - ((\boldsymbol{\psi}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{u}^{n+\frac{1}{2}}, \boldsymbol{\chi}_h) + ((\boldsymbol{u}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{\psi}_h^{n+\frac{1}{2}}, \boldsymbol{\chi}_h) + ((\boldsymbol{\xi}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{B}^{n+\frac{1}{2}}, \boldsymbol{\chi}_h) \\
&= \frac{1}{\Delta t}(\boldsymbol{\eta}_{\boldsymbol{B}}^{n+1} - \boldsymbol{\eta}_{\boldsymbol{B}}^n, \boldsymbol{\chi}_h) + \mu(\nabla \boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}, \nabla \boldsymbol{\chi}_h) - ((\boldsymbol{B}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}, \boldsymbol{\chi}_h) \\
&\quad - ((\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{u}^{n+\frac{1}{2}}, \boldsymbol{\chi}_h) \\
&\quad + ((\boldsymbol{u}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}, \boldsymbol{\chi}_h) + ((\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{B}^{n+\frac{1}{2}}, \boldsymbol{\chi}_h) + G_2(t, \boldsymbol{u}, \boldsymbol{B}, \boldsymbol{\chi}_h), \qquad (52)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\Delta t}(\omega_h^{n+1} - \omega_h^n, \phi_h) + \kappa(\nabla \omega_h^{n+\frac{1}{2}}, \nabla \phi_h) \\
&\quad + (\boldsymbol{u}_h^{n+\frac{1}{2}} \cdot \nabla \omega_h^{n+\frac{1}{2}}, \phi_h) + (\boldsymbol{\xi}_h^{n+\frac{1}{2}} \cdot \nabla \theta^{n+\frac{1}{2}}, \phi_h) \\
&= \frac{1}{\Delta t}(\eta_\theta^{n+1} - \eta_\theta^n, \phi_h) + \kappa(\nabla \eta_\theta^{n+\frac{1}{2}}, \nabla \phi_h) + (\boldsymbol{u}_h^{n+\frac{1}{2}} \cdot \nabla \eta_\theta^{n+\frac{1}{2}}, \phi_h) \\
&\quad + (\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}} \cdot \nabla \theta^{n+\frac{1}{2}}, \phi_h) + G_3(t, \boldsymbol{u}, \theta, \phi_h). \qquad (53)
\end{aligned}$$

Theorem 4.4. *Under assumption of Theorem 4.3, assume that* $(\boldsymbol{u}, \boldsymbol{B}, \theta)$ *solves* (2) *and satisfies regularity conditions:*

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(0, T, W_3^{k+2}(\Omega)^d), \ \boldsymbol{B}, \theta \in L^\infty(0, T, W_3^{k+1}(\Omega)^s), \ \boldsymbol{u}_{ttt} \in L^2(0, T, H^1(\Omega)^d), \\
&\qquad \boldsymbol{B}_{tt}, \theta_{tt} \in L^2(0, T, H^1(\Omega)^s), \ \boldsymbol{B}_{ttt}, \theta_{ttt} \in L^2(0, T, L^2(\Omega)^s),
\end{aligned}$$

where $s = 1$ *or* d. *Then the solution to* (39) *converges to the true solution with rate*

$$\begin{aligned}
&\|\boldsymbol{u}(t^M) - \boldsymbol{u}_h^M\|^2 + \alpha^2 \|\nabla(\boldsymbol{u}(t^M) - \boldsymbol{u}_h^M)\|^2 + \|\boldsymbol{B}(t^M) - \boldsymbol{B}_h^M\|^2 + \|\theta(t^M) - \theta_h^M\|^2 \\
&\quad + \Delta t \sum_{n=0}^{M-1} (\|\nabla(\boldsymbol{B}(t^{n+\frac{1}{2}}) - \boldsymbol{B}_h^{n+\frac{1}{2}})\|^2 + \|\nabla(\theta(t^{n+\frac{1}{2}}) - \theta_h^{n+\frac{1}{2}})\|^2) \\
&\le C(\alpha^2 + 1)(\Delta t^4 + h^{2k}).
\end{aligned}$$

Proof. Choose $\boldsymbol{v}_h = \boldsymbol{\xi}_h^{n+\frac{1}{2}}$, $\boldsymbol{\chi}_h = \boldsymbol{\psi}_h^{n+\frac{1}{2}}$ and $\phi_h = \omega_h^{n+\frac{1}{2}}$ in (51), (52) and (53), respectively. Rewrite the ensuing equations

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\boldsymbol{\xi}_h^{n+1}\|^2 - \|\boldsymbol{\xi}_h^n\|^2) + \frac{\alpha^2}{2\Delta t}(\|\nabla \boldsymbol{\xi}_h^{n+1}\|^2 - \|\nabla \boldsymbol{\xi}_h^n\|^2) + ((\boldsymbol{\xi}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{u}^{n+\frac{1}{2}}, \boldsymbol{\xi}_h^{n+\frac{1}{2}}) \\
&\quad - ((\boldsymbol{B}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{\psi}_h^{n+\frac{1}{2}}, \boldsymbol{\xi}_h^{n+\frac{1}{2}}) - ((\boldsymbol{\psi}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{B}^{n+\frac{1}{2}}, \boldsymbol{\xi}_h^{n+\frac{1}{2}}) \\
&= \frac{1}{\Delta t}(\boldsymbol{\eta}_{\boldsymbol{u}}^{n+1} - \boldsymbol{\eta}_{\boldsymbol{u}}^n, \boldsymbol{\xi}_h^{n+\frac{1}{2}}) + \frac{\alpha^2}{\Delta t}(\nabla \boldsymbol{\eta}_{\boldsymbol{u}}^{n+1} - \nabla \boldsymbol{\eta}_{\boldsymbol{u}}^n, \nabla \boldsymbol{\xi}_h^{n+\frac{1}{2}}) \\
&\quad + ((\boldsymbol{u}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}, \boldsymbol{\xi}_h^{n+\frac{1}{2}}) + ((\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{u}^{n+\frac{1}{2}}, \boldsymbol{\xi}_h^{n+\frac{1}{2}}) \\
&\quad - ((\boldsymbol{B}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}, \boldsymbol{\xi}_h^{n+\frac{1}{2}}) - ((\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{B}^{n+\frac{1}{2}}, \boldsymbol{\xi}_h^{n+\frac{1}{2}}) \\
&\quad + (\boldsymbol{\beta}(\omega_h^{n+\frac{1}{2}} - \eta_\theta^{n+\frac{1}{2}}), \boldsymbol{\xi}_h^{n+\frac{1}{2}}) + G_1(t, \boldsymbol{u}, \boldsymbol{B}, \theta, \boldsymbol{\xi}_h^{n+\frac{1}{2}}), \qquad (54)
\end{aligned}$$

as well as

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\boldsymbol{\psi}_h^n\|^2 - \|\boldsymbol{\psi}_h^n\|^2) + \mu \|\nabla \boldsymbol{\psi}_h^{n+\frac{1}{2}}\|^2 - ((\boldsymbol{B}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{\xi}_h^{n+\frac{1}{2}}, \boldsymbol{\psi}_h^{n+\frac{1}{2}}) \\
&\quad - ((\boldsymbol{\psi}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{u}^{n+\frac{1}{2}}, \boldsymbol{\psi}_h^{n+\frac{1}{2}}) + ((\boldsymbol{\xi}_h^{n+\frac{1}{2}} \cdot \nabla)\boldsymbol{B}^{n+\frac{1}{2}}, \boldsymbol{\psi}_h^{n+\frac{1}{2}})
\end{aligned}$$

$$=\frac{1}{\Delta t}(\boldsymbol{\eta}_{\boldsymbol{B}}^{n+1}-\boldsymbol{\eta}_{\boldsymbol{B}}^{n},\boldsymbol{\psi}_h^{n+\frac{1}{2}})+\mu(\nabla\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}},\nabla\boldsymbol{\psi}_h^{n+\frac{1}{2}})-((\boldsymbol{B}_h^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}},\boldsymbol{\psi}_h^{n+\frac{1}{2}})$$
$$-((\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{u}^{n+\frac{1}{2}},\boldsymbol{\psi}_h^{n+\frac{1}{2}})+((\boldsymbol{u}_h^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}},\boldsymbol{\psi}_h^{n+\frac{1}{2}})$$
$$+((\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{B}^{n+\frac{1}{2}},\boldsymbol{\psi}_h^{n+\frac{1}{2}})+G_2(t,\boldsymbol{u},\boldsymbol{B},\boldsymbol{\psi}_h^{n+\frac{1}{2}}),\tag{55}$$

and

$$\frac{1}{2\Delta t}(\|\omega_h^{n+1}\|^2-\|\omega_h^n\|^2)+\kappa\|\nabla\omega_h^{n+\frac{1}{2}}\|^2+(\boldsymbol{\xi}_h^{n+\frac{1}{2}}\cdot\nabla\theta^{n+\frac{1}{2}},\omega_h^{n+\frac{1}{2}})$$
$$=\frac{1}{\Delta t}(\eta_\theta^{n+1}-\eta_\theta^n,\omega_h^{n+\frac{1}{2}})+\kappa(\nabla\eta_\theta^{n+\frac{1}{2}},\nabla\omega_h^{n+\frac{1}{2}})+(\boldsymbol{u}_h^{n+\frac{1}{2}}\cdot\nabla\eta_\theta^{n+\frac{1}{2}},\omega_h^{n+\frac{1}{2}})$$
$$+(\boldsymbol{\eta}_u^{n+\frac{1}{2}}\cdot\nabla\theta^{n+\frac{1}{2}},\omega_h^{n+\frac{1}{2}})+G_3(t,\boldsymbol{u},\theta,\omega_h^{n+\frac{1}{2}}).\tag{56}$$

For (54), apply the Cauchy-Schwarz-Young inequality to get

$$\frac{1}{\Delta t}(\boldsymbol{\eta}_{\boldsymbol{u}}^{n+1}-\boldsymbol{\eta}_{\boldsymbol{u}}^{n},\boldsymbol{\xi}_h^{n+\frac{1}{2}})\le\frac{1}{2\Delta t}\int_{t^n}^{t^{n+1}}\|\partial_t(\boldsymbol{\eta}_{\boldsymbol{u}})\|^2dt+\frac{1}{2}\|\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|^2,$$
$$\frac{\alpha^2}{\Delta t}(\nabla\boldsymbol{\eta}_{\boldsymbol{u}}^{n+1}-\nabla\boldsymbol{\eta}_{\boldsymbol{u}}^{n},\nabla\boldsymbol{\xi}_h^{n+\frac{1}{2}})\le\frac{\alpha^2}{2\Delta t}\int_{t^n}^{t^{n+1}}\|\partial_t(\nabla\boldsymbol{\eta}_{\boldsymbol{u}})\|^2dt+\frac{\alpha^2}{2}\|\nabla\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|^2.$$

Next, with the help of Hölder's inequality, we rewrite (54) as

$$\frac{1}{2\Delta t}(\|\boldsymbol{\xi}_h^{n+1}\|^2-\|\boldsymbol{\xi}_h^n\|^2)+\frac{\alpha^2}{2\Delta t}(\|\nabla\boldsymbol{\xi}_h^{n+1}\|^2-\|\nabla\boldsymbol{\xi}_h^n\|^2)$$
$$\le\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}(\|\partial_t(\boldsymbol{\eta}_{\boldsymbol{u}})\|^2+\alpha^2\|\partial_t(\nabla\boldsymbol{\eta}_{\boldsymbol{u}})\|^2)dt+\frac{1}{2}\|\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|^2+\frac{\alpha^2}{2}\|\nabla\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|^2$$
$$+\|\nabla\boldsymbol{u}^{n+\frac{1}{2}}\|_{L^\infty}\|\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|^2+((\boldsymbol{B}_h^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{\psi}_h^{n+\frac{1}{2}},\boldsymbol{\xi}_h^{n+\frac{1}{2}})$$
$$+\|\nabla\boldsymbol{B}^{n+\frac{1}{2}}\|_{L^\infty}\|\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|\|\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|$$
$$+\|\boldsymbol{u}_h^{n+\frac{1}{2}}\|\|\nabla\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\|_{L^\infty}\|\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|+\|\nabla\boldsymbol{u}^{n+\frac{1}{2}}\|_{L^\infty}\|\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\|\|\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|$$
$$+C\|\nabla\boldsymbol{B}_h^{n+\frac{1}{2}}\|\|\nabla\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\|_{L^3}\|\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|+\|\nabla\boldsymbol{B}^{n+\frac{1}{2}}\|_{L^\infty}\|\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\|\|\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|$$
$$+(\boldsymbol{\beta}(\omega_h^{n+\frac{1}{2}}-\eta_\theta^{n+\frac{1}{2}}),\boldsymbol{\xi}_h^{n+\frac{1}{2}})+G_1(t,\boldsymbol{u},\boldsymbol{B},\theta,\boldsymbol{\xi}_h^{n+\frac{1}{2}}).$$

Then, according to the Cauchy-Schwarz-Young, Poincaré's inequalities and the assumption of the regularity of the solution, we discover

$$\frac{1}{2\Delta t}(\|\boldsymbol{\xi}_h^{n+1}\|^2-\|\boldsymbol{\xi}_h^n\|^2)+\frac{\alpha^2}{2\Delta t}(\|\nabla\boldsymbol{\xi}_h^{n+1}\|^2-\|\nabla\boldsymbol{\xi}_h^n\|^2)$$
$$\le\frac{1}{2\Delta t}\int_{t^n}^{t^{n+1}}(\|\partial_t(\boldsymbol{\eta}_{\boldsymbol{u}})\|^2+\alpha^2\|\nabla\partial_t(\boldsymbol{\eta}_{\boldsymbol{u}})\|^2)dt+\frac{\alpha^2}{2}\|\nabla\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|^2$$
$$+((\boldsymbol{B}_h^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{\psi}_h^{n+\frac{1}{2}},\boldsymbol{\xi}_h^{n+\frac{1}{2}})+C\Big(\|\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|^2+\|\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|^2$$
$$+\|\boldsymbol{u}_h^{n+\frac{1}{2}}\|^2\|\nabla\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\|_{L^\infty}^2+\|\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2+\|\nabla\boldsymbol{B}_h^{n+\frac{1}{2}}\|^2\|\nabla\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\|_{L^3}^2$$
$$+\|\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\|^2+\|\omega_h^{n+\frac{1}{2}}\|^2+\|\eta_\theta^{n+\frac{1}{2}}\|^2\Big)+G_1(t,\boldsymbol{u},\boldsymbol{B},\theta,\boldsymbol{\xi}_h^{n+\frac{1}{2}}).\tag{57}$$

We now suspend (57) and return to (55) and (56), which can be estimated by proceeding as the momentum system, yielding

$$\frac{1}{2\Delta t}(\|\boldsymbol{\psi}_h^{n+1}\|^2-\|\boldsymbol{\psi}_h^n\|^2)+\frac{\mu}{4}\|\nabla\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|^2$$

$$\begin{aligned}
&\leq \frac{1}{2\Delta t}\int_{t^n}^{t^{n+1}} \|\partial_t(\boldsymbol{\eta_B})\|^2 dt + ((\boldsymbol{B}_h^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{\xi}_h^{n+\frac{1}{2}}, \boldsymbol{\psi}_h^{n+\frac{1}{2}}) \\
&\quad + C\Big(\|\nabla\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\|^2 + \|\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|^2 + \|\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|^2 + \|\nabla\boldsymbol{B}_h^{n+\frac{1}{2}}\|^2\|\nabla\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2 \\
&\quad + \|\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\|^2 + \|\boldsymbol{u}_h^{n+\frac{1}{2}}\|^2\|\nabla\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\|_{L^3}^2 + \|\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2\Big) + G_2(t,u,\boldsymbol{B},\boldsymbol{\psi}_h^{n+\frac{1}{2}}),
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\omega_h^{n+1}\|^2 - \|\omega_h^n\|^2) + \frac{\kappa}{4}\|\nabla\omega_h^{n+\frac{1}{2}}\|^2 \\
&\quad \leq \frac{1}{2\Delta t}\int_{t^n}^{t^{n+1}} \|\partial_t(\eta_\theta)\|^2 dt + C\Big(\|\omega_h^{n+\frac{1}{2}}\|^2 + \|\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|^2 + \|\nabla\eta_\theta^{n+\frac{1}{2}}\|^2 \\
&\quad\quad + \|\boldsymbol{u}_h^{n+\frac{1}{2}}\|^2\|\nabla\eta_\theta^{n+\frac{1}{2}}\|_{L^3}^2 + \|\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2\Big) + G_3(t,\boldsymbol{u},\theta,\omega_h^{n+\frac{1}{2}}).
\end{aligned} \tag{59}$$

Moreover, combining (58), (59) and (57) and then noticing the fact that

$$((\boldsymbol{B}_h^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{\xi}_h^{n+\frac{1}{2}}, \boldsymbol{\psi}_h^{n+\frac{1}{2}}) = -((\boldsymbol{B}_h^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{\psi}_h^{n+\frac{1}{2}}, \boldsymbol{\xi}_h^{n+\frac{1}{2}}),$$

we gain

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\boldsymbol{\xi}_h^{n+1}\|^2 - \|\boldsymbol{\xi}_h^n\|^2) + \frac{1}{2\Delta t}(\|\boldsymbol{\psi}_h^{n+1}\|^2 - \|\boldsymbol{\psi}_h^n\|^2) + \frac{1}{2\Delta t}(\|\omega_h^{n+1}\|^2 - \|\omega_h^n\|^2) \\
&+ \frac{\alpha^2}{2\Delta t}(\|\nabla\boldsymbol{\xi}_h^{n+1}\|^2 - \|\nabla\boldsymbol{\xi}_h^n\|^2) + \frac{\mu}{4}\|\nabla\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|^2 + \frac{\kappa}{4}\|\nabla\omega_h^{n+\frac{1}{2}}\|^2 \\
\leq &\frac{1}{2\Delta t}\int_{t^n}^{t^{n+1}} (\|\partial_t(\boldsymbol{\eta_u})\|^2 + \alpha^2\|\nabla\partial_t(\boldsymbol{\eta_u})\|^2 + \|\partial_t(\boldsymbol{\eta_B})\|^2 + \|\partial_t(\eta_\theta)\|^2) dt \\
&+ \frac{\alpha^2}{2}\|\nabla\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|^2 + C\Big(\|\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|^2 + \|\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|^2 + \|\omega_h^{n+\frac{1}{2}}\|^2 + \|\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2 + \|\nabla\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\|^2 \\
&+ \|\nabla\eta_\theta^{n+\frac{1}{2}}\|^2 + \|\boldsymbol{u}_h^{n+\frac{1}{2}}\|^2\|\nabla\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\|_{L^\infty}^2 + \|\nabla\boldsymbol{B}_h^{n+\frac{1}{2}}\|^2\|\nabla\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\|_{L^3}^2 \\
&+ \|\nabla\boldsymbol{B}_h^{n+\frac{1}{2}}\|^2\|\nabla\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2 + \|\boldsymbol{u}_h^{n+\frac{1}{2}}\|^2\|\nabla\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\|_{L^3}^2 + \|\boldsymbol{u}_h^{n+\frac{1}{2}}\|^2\|\nabla\eta_\theta^{n+\frac{1}{2}}\|_{L^3}^2\Big) \\
&+ G_1(t,\boldsymbol{u},\boldsymbol{B},\theta,\boldsymbol{\xi}_h^{n+\frac{1}{2}}) + G_2(t,\boldsymbol{u},\boldsymbol{B},\boldsymbol{\psi}_h^{n+\frac{1}{2}}) + G_3(t,\boldsymbol{u},\theta,\omega_h^{n+\frac{1}{2}}).
\end{aligned} \tag{60}$$

Multiply (60) by $2\Delta t$, then recall (46), (49) and (50), and finally sum the ensuing inequality over time steps.

$$\begin{aligned}
&\|\boldsymbol{\xi}_h^M\|^2 + \alpha^2\|\nabla\boldsymbol{\xi}_h^M\|^2 + \|\boldsymbol{\psi}_h^M\|^2 + \|\omega_h^M\|^2 + \frac{\Delta t}{4}\sum_{n=0}^{M-1}(\mu\|\nabla\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|^2 + \kappa\|\nabla\omega_h^{n+\frac{1}{2}}\|^2) \\
\leq &\int_0^T (\|\partial_t(\boldsymbol{\eta_u})\|^2 + \alpha^2\|\nabla\partial_t(\boldsymbol{\eta_u})\|^2 + \|\partial_t(\boldsymbol{\eta_B})\|^2 + \|\partial_t(\eta_\theta)\|^2) dt \\
&+ C\Delta t\sum_{n=0}^{M-1}\Big(\|\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2 + \|\boldsymbol{u}_h^{n+\frac{1}{2}}\|^2\big(\|\nabla\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\|_{L^\infty}^2 + \|\nabla\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\|_{L^3}^2 + \|\nabla\eta_\theta^{n+\frac{1}{2}}\|_{L^3}^2\big) \\
&+ \|\nabla\boldsymbol{B}_h^{n+\frac{1}{2}}\|^2(\|\nabla\boldsymbol{\eta}_{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2 + \|\nabla\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\|_{L^3}^2) + \|\nabla\boldsymbol{\eta}_{\boldsymbol{B}}^{n+\frac{1}{2}}\|^2 + \|\nabla\eta_\theta^{n+\frac{1}{2}}\|^2\Big) \\
&+ C\Delta t\sum_{n=0}^{M-1}\Big(\|\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|^2 + \alpha^2\|\nabla\boldsymbol{\xi}_h^{n+\frac{1}{2}}\|^2 + \|\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|^2 + \|\omega_h^{n+\frac{1}{2}}\|^2\Big) \\
&+ C(\Delta t^4 + \alpha^2\Delta t^4).
\end{aligned}$$

Finally, based on the finite element approximation (see, e.g., p.108 in [48]) and Theorem 4.2, we deduce that

$$\begin{aligned}
&\|\boldsymbol{\xi}_h^M\|^2 + \alpha^2\|\nabla\boldsymbol{\xi}_h^M\|^2 + \|\boldsymbol{\psi}_h^M\|^2 + \|\omega_h^M\|^2 + \frac{\Delta t}{4}\sum_{n=0}^{M-1}(\mu\|\nabla\boldsymbol{\psi}_h^{n+\frac{1}{2}}\|^2 + \kappa\|\nabla\omega_h^{n+\frac{1}{2}}\|^2)\\
\leq & C\Delta t\sum_{n=0}^{M-1}(\|\boldsymbol{\xi}_h^{n+1}\|^2 + \alpha^2\|\nabla\boldsymbol{\xi}_h^{n+1}\|^2 + \|\boldsymbol{\psi}_h^{n+1}\|^2 + \|\omega_h^{n+1}\|^2)\\
&+ C(\Delta t^4 + \alpha^2\Delta t^4 + \alpha^2 h^{2k} + h^{2k}).
\end{aligned}$$

Applying Grönwall's inequality (6) completes the proof with the help of the triangle inequality. □

5. Numerical experience

In this section, we present some numerical examples to test the established theoretical findings, and show the performances of the fully discrete scheme (39) for the Voigt-regularization of the thermally coupled inviscid, resistive MHD equations (2). A linearization of (39a)-(39e) is derived by using explicit second-order extrapolation of the five nonlinear terms via the substitution

$$((\boldsymbol{u}_h^{n+\frac{1}{2}}\cdot\nabla)\boldsymbol{u}_h^{n+\frac{1}{2}}, \boldsymbol{v}_h) \to (((\frac{3}{2}\boldsymbol{u}_h^n - \frac{1}{2}\boldsymbol{u}_h^{n-1})\cdot\nabla)\boldsymbol{u}_h^{n+\frac{1}{2}}, \boldsymbol{v}_h).$$

In all tests, the SV element $((P_2)^2, P_1^{dc})$ is used for the velocity-pressure, magnetic field-Lagrange multiplier systems and P_2 element is applied for the temperature.

5.1. Stability test. In order to demonstrate the stability of the presented scheme showed in Theorem 4.2, we test it with different time step for the considered problem (2) in the unit cube $[0,1]^2$. Set the parameters $\mu = 1$, $\kappa = 1.0$, $\boldsymbol{\beta} = (0,-1)^\top$, $\alpha = 0.2$ and the final time $T = 5.0$. The initial values are taken as follows:

$$\begin{aligned}
&\boldsymbol{u}_0(x,y) = (10x^2(x-1)^2y(y-1)(2y-1),\ -10y^2(y-1)^2x(x-1)(2x-1))^\top,\\
&\boldsymbol{B}_0(x,y) = (\sin(\pi x)\cos(\pi y), -\sin(\pi y)\cos(\pi x))^\top, \quad p_0(x,y) = 0,\\
&\theta_0(x,y) = 10x^2(x-1)^2y(y-1)(2y-1) - 10y^2(y-1)^2x(x-1)(2x-1).
\end{aligned}$$

Due to the zero source functions and the homogeneous boundary conditions in the considered domain, the numerical solutions are expected to remain bounded over time.

We denote $E_h^n = \|\boldsymbol{u}_h^n\|^2 + \alpha^2\|\nabla\boldsymbol{u}_h^n\|^2 + \|\boldsymbol{B}_h^n\|^2 + \|\theta_h^n\|^2$ and fix mesh size $h = \frac{1}{8}$. Figure 1 shows the values of E_h^n versus time evolution with different time steps $\Delta t = 0.1,\ 0.05$ and 0.02. We observe that it is monotonic decay for all time steps, which shows that the proposed scheme is stable.

5.2. Convergence test. This example is to test the convergence rate for the presented scheme showed in Theorem 4.4. The RHSs $\boldsymbol{f}, \boldsymbol{g}, \Psi$ and boundary conditions are chosen such that the exact solution in the domain $\Omega = [0,1]^2$ is given as

$$\begin{aligned}
&\boldsymbol{u} = (k\pi\sin^2(\pi x)\sin(\pi y)\cos(\pi y)\cos(t), -k\pi\sin(\pi x)\sin^2(\pi y)\cos(\pi x)\cos(t))^\top,\\
&\boldsymbol{B} = (k\sin(\pi x)\cos(\pi y)\cos(t), -k\cos(\pi x)\sin(\pi y)\cos(t))^\top,\\
&\theta = k\pi\sin^2(\pi x)\sin(\pi y)\cos(\pi y)\cos(t) - k\pi\sin(\pi x)\sin^2(\pi y)\cos(\pi x)\cos(t)\\
&p = (1-x)(1-y)(1-z)\sin(t),
\end{aligned}$$

where $k = 0.01$. Set $\mu = 1$, $\kappa = 1$, $\boldsymbol{\beta} = (0,-1)^\top$, and $\alpha = 0.2$. On the one hand, when studying the convergence order with respect to the spatial mesh size h, we

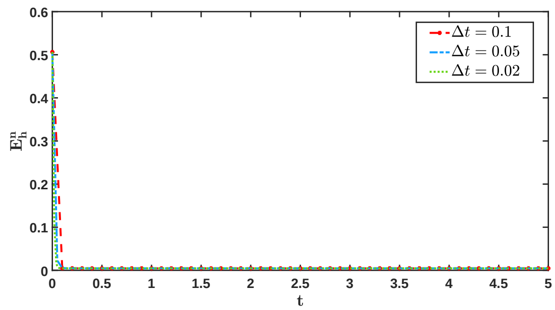

FIGURE 1. The values of E_h^n with different time steps.

fix a small $\Delta t = 0.001$ to reduce impact of the time step on spatial convergence order. The errors and convergence rates for corresponding norms at the final time $T = 0.01$ with varying h are displayed in Table 1. On the other hand, when it comes to the convergence order with respect to Δt, we select $\Delta t = h$ and $T = 1$. The corresponding results are listed in Table 2. From these tables we can see that both of them are approximately 2. Hence, the proposed scheme works well and keeps the convergence rates just like the theoretical analysis.

TABLE 1. Errors and convergence rates with respect to h.

h	$\lVert\nabla e_{\boldsymbol{u}}^N\rVert$	Rate	$\lVert\nabla e_{\boldsymbol{B}}^N\rVert$	Rate	$\lVert\nabla e_{\theta}^N\rVert$	Rate
1/4	$1.803E-02$	–	$4.246E-03$	–	$4.206E-03$	–
1/8	$6.184E-03$	1.54	$1.216E-03$	1.80	$1.131E-03$	1.89
1/16	$1.892E-03$	1.71	$3.271E-04$	1.89	$3.123E-04$	1.86
1/32	$5.170E-04$	1.87	$8.394E-05$	1.96	$9.047E-05$	1.79
1/40	$3.353E-04$	1.94	$5.392E-05$	1.98	$5.962E-04$	1.87

TABLE 2. Errors and convergence order with respect to Δt.

Δt	$\lVert\nabla e_{\boldsymbol{u}}^N\rVert$	Rate	$\lVert\nabla e_{\boldsymbol{B}}^N\rVert$	Rate	$\lVert\nabla e_{\theta}^N\rVert$	Rate
1/5	$6.984E-03$	–	$1.540E-03$	–	$3.186E-03$	–
1/10	$2.314E-03$	1.59	$4.335E-04$	1.83	$7.900E-04$	2.01
1/20	$6.801E-04$	1.77	$1.145E-04$	1.92	$1.968E-04$	2.00
1/40	$1.811E-04$	1.91	$2.914E-05$	1.97	$4.916E-05$	2.00

5.3. Driven cavity flow. The following numerical simulations are carried out with the 2D lid-driven flow as the experimental model [49]. The external body force is zero for this problem and the computational domain is $\Omega = [0,1]^2$. The initial values are given by $\boldsymbol{B}_0 = \boldsymbol{u}_0 = \boldsymbol{0}$, and $\theta_0 = x$. Then, the boundary conditions are given as follows:

$$\begin{cases} \boldsymbol{u} = (1,0)^\top \quad \text{on the top wall,} \\ \boldsymbol{u} = \boldsymbol{0}, \quad \text{on the rest of the wall,} \\ \boldsymbol{B} \times \boldsymbol{n} = (1,0)^\top \times \boldsymbol{n}, \quad \theta = \theta_0, \quad \text{on } \partial\Omega, \end{cases}$$

where $\boldsymbol{n}$ is the outer normal vector on $\partial\Omega$. Define $Er_h^{n+1} := \|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 + \|\boldsymbol{B}_h^{n+1} - \boldsymbol{B}_h^n\|^2 + \|\theta_h^{n+1} - \theta_h^n\|^2$, and set $\mu = 1$, $\kappa = 1$ and $\Delta t = 0.01$.

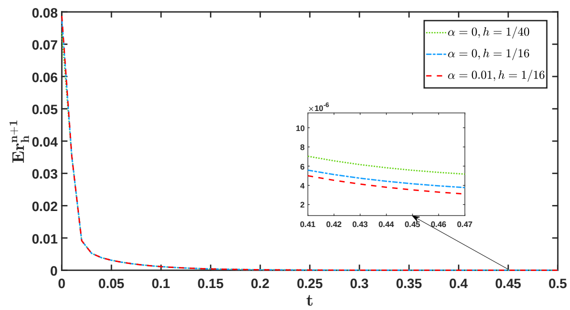

FIGURE 2. The evolution of Er_h^{n+1}.

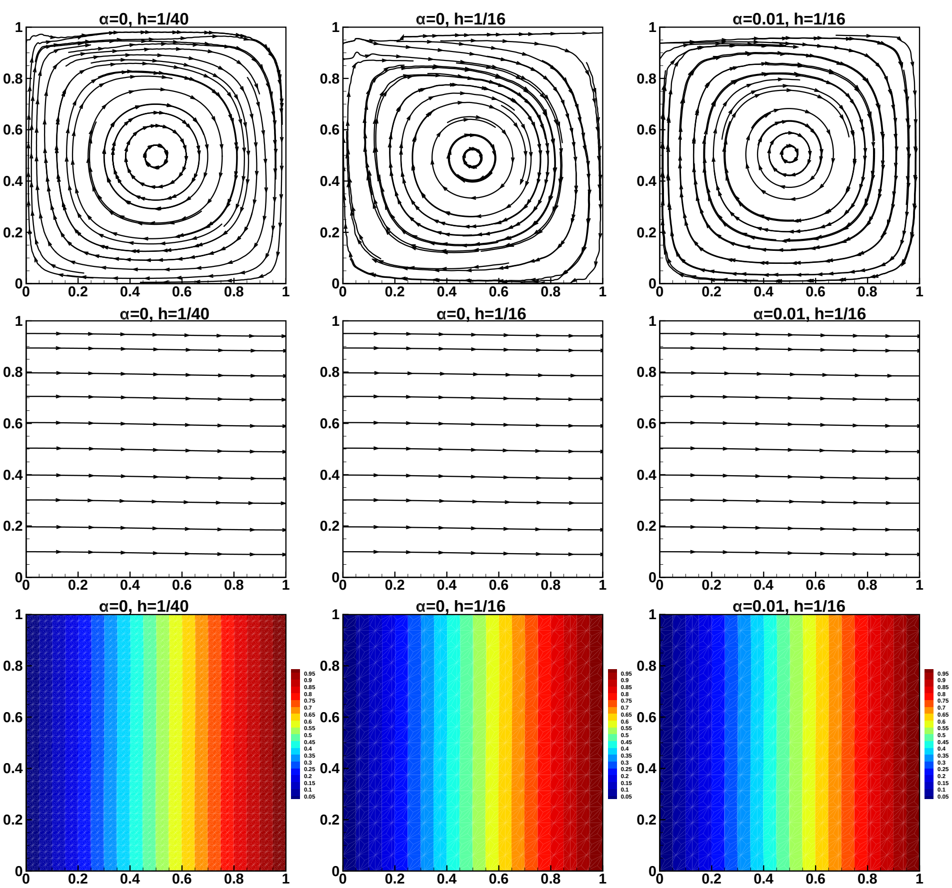

FIGURE 3. The streamlines of velocity and magnetic field, and the contours of temperature (from top to bottom) with different α and h.

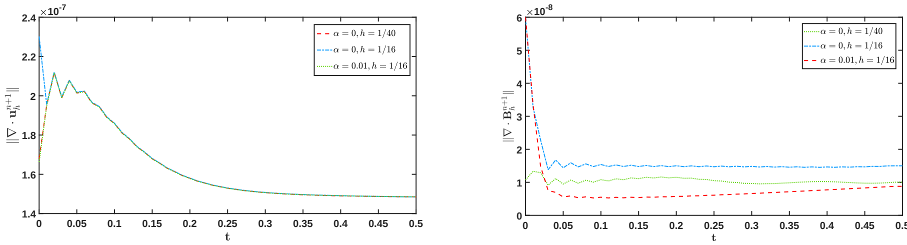

FIGURE 4. The evolution of $\|\nabla \cdot \boldsymbol{u}_h^{n+1}\|$ and $\|\nabla \cdot \boldsymbol{B}_h^{n+1}\|$.

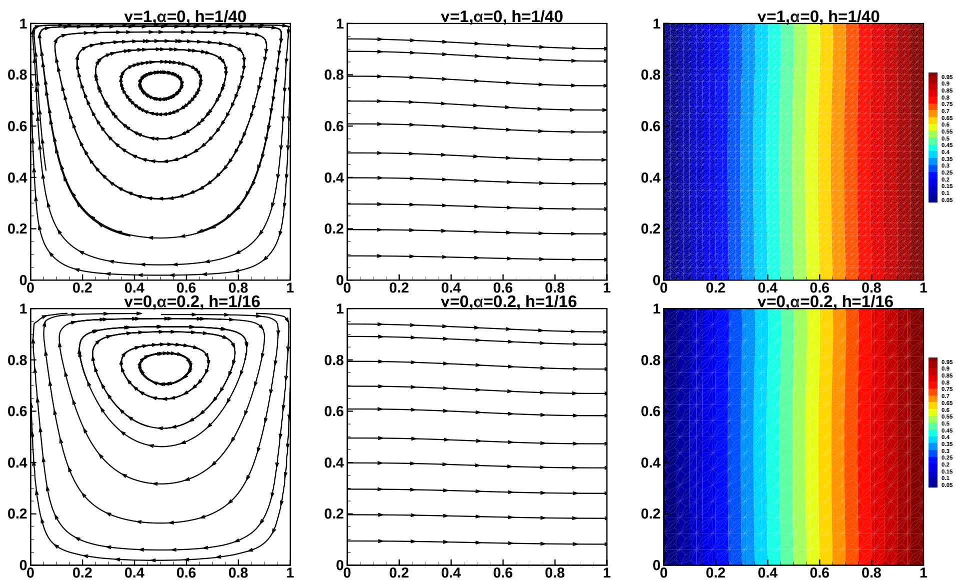

FIGURE 5. The streamlines of velocity and magnetic field, and the contours of temperature (from left to right).

We first compute the problem with $\alpha = 0$ at $h = 1/40$ as a reference solution. Then we compare the numerical results of (39) with $\alpha = 0.01$ and (39) with $\alpha = 0$ (the usual thermally coupled inviscid, resistive MHD) on a coarse mesh $h = 1/16$. Figure 2 gives the evolution of Er_h^{n+1} versus time for different cases. We can see that all cases get almost the same results.Further, we plot the streamlines of velocity, magnetic field and the contours of the temperature in Figure 3. Based on the reference solution, we find that the case of (39) with $\alpha = 0.01$ (using Voigt-regularization) has a better approximation in the streamline of velocity than that of (39) with $\alpha = 0$.

Furthermore, we compare the computational divergence values, which are shown in Figure 4. It is clear that the evolution of $\|\nabla \cdot \boldsymbol{u}_h^{n+1}\|$ is almost the same for (39) with $\alpha = 0.01$ and the reference solution. Besides, (39) with $\alpha = 0.01$ has the least value of $\|\nabla \cdot \boldsymbol{B}_h^{n+1}\|$ among these three cases.

Finally, we simulate the problem (2) with $\alpha = 0.2$ and $h = 1/16$, and the problem (1) $\nu = 1$ and $h = 1/40$ (the thermally coupled viscid, resistive MHD). Figure 5 plots the computational results of both cases, which show almost the same results. To further explain it, let us take a closer look at the form of the Voigt-regularization and the viscosity term in the numerical scheme as follows:

$$\frac{\alpha^2}{\Delta t}(\nabla(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n),\nabla\boldsymbol{v}_h) \quad \text{and} \quad \nu(\nabla\boldsymbol{u}_h^{n+\frac{1}{2}},\nabla\boldsymbol{v}_h).$$

Clearly, both terms contribute to the main diagonal of the stiffness matrix and the right-hand side term of the linear system. Therefore, for this experiment, with appropriate parameters, the Voigt-regularization acts as a viscous term in the momentum equation.

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