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A STABILIZER-FREE WEAK GALERKIN FINITE ELEMENT METHOD FOR THE DARCY-STOKES EQUATIONS

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Abstract. In this paper, we propose a new method for the Darcy-Stokes equations based on the stabilizer-free weak Galerkin finite element method. In the proposed method, we remove the stabilizer term by increasing the degree of polynomial approximation space of the weak gradient operator. Compared with the classical weak Galerkin finite element method, it will not increase the size of global stiffness matrix. We show that the new algorithm not only has a simpler formula, but also reduces the computational complexity. Optimal order error estimates are established for the corresponding numerical approximation in various norms. Finally, we numerically illustrate the accuracy and convergence of this method.

Key words. Stabilizer-free, weak Galerkin finite element method, Darcy-Stokes equations, weak gradient operator, optimal order error estimates.

1. Introduction

Darcy-Stokes equations has widely applications in groundwater protection, oil extraction, geophysics and etc. The research on the numerical solution of Darcy-Stokes equations has been attracting much more attention in recent years. The mixed finite element method[5, 13] is a common numerical method for solving fluid problems. However, the mixed finite element method must satisfy the inf-sup condition[1], which restricts the choice of finite element space pairs. For example, a finite element space pair of equal order will not satisfy this condition. The stabilized finite element methods have been proposed to solve the above problem. These stabilized methods are constructed by adding stabilizer terms to the classical mixed finite element formulations, for example, a least squares terms, which obtained by bubble condensation contain coefficients related to the mesh size and equation parameters. On the other hand, untraditional finite element methods are developed for solving PDEs, which used discontinuous functions as approximation functions. The Discontinuous Galerkin finite element method (DG) was first proposed by Reed and Hill in 1973. Based on this method, various DG methods have been proposed: the Local Discontinuous Galerkin finite element method[2, 4], the Hybrid Discontinuous Galerkin finite element method[7, 30], the selective immersed discontinuous Galerkin method[16], etc. These above methods have some good features like local conservation of physical quantities and flexibility in meshes.

The weak Galerkin(WG) finite element method was first proposed by Wang and Ye for second-order elliptic problems. Subsequently, they have been used for solving Stokes[19, 20, 32], Darcy-Stokes[21, 28], Brinkman[14, 29], convection-diffusion-reaction equations[25] and so on. It is well known that the WG finite element method has two main features. One is the use of discontinuous piecewise polynomials in the finite element space. Therefore, this method is suitable for general polygonal or polyhedral meshes, and it is easy to construct finite element spaces. The other feature is that the differential operators are approximated by weak forms

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which are locally-defined on each element. For example, gradient operator ∇ and divergence operator $\nabla\cdot$ are approximated by weak gradient operator ∇_w and weak divergence operator $\nabla_w\cdot$ respectively. Thus, it is straightforward for building the discretization scheme. Due to these advantages, the WG finite element method has been widely developed, various WG finite element methods have been proposed, such as modified WG finite element method [18, 26], hybridized WG finite element method [22, 39], and so on. It is worth to mention, all the above methods contain a stabilizer which enforces a certain weak continuity in their numerical schemes. Recently, a new WG finite element method, called the stabilizer-free weak Galerkin (SFWG) finite element method, was derived for second elliptic equations on polytopal mesh[33]. The main idea of this SFWG method is raise the degree of polynomials used to compute weak gradient ∇_w. Removing stabilizers from the WG finite element method simplifies fomulations and reduces progrmming complexity. Based on this idea, in [36, 37], a new SFWG method was employed for the Stokes equations. Especially, the method in [37] not only achieved exact divergence free velocity field, but also obtained pressure-robustness.

Now, let us turn to the Darcy-Stokes model which is established for free fluids and coexisting flow system. In this paper, we consider the following problem: seek unknown functions $\boldsymbol{u}$ and p satisfying

$$\begin{cases} -\epsilon^2\Delta\boldsymbol{u}+\boldsymbol{u}+\nabla p=\boldsymbol{f} & in\ \Omega, \\ \nabla\cdot\boldsymbol{u}=0 & in\ \Omega, \\ \boldsymbol{u}=\boldsymbol{g} & on\ \partial\Omega, \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^d$ is a polygonal ($d=2$) or a polyhedral domain ($d=3$). Here ϵ is the fluid viscosity coefficient, $\boldsymbol{f}$ and $\boldsymbol{g}$ are some given data. Throughout this paper, we assume that $\boldsymbol{g}=\boldsymbol{0}$.

The weak formulation for the Darcy-Stokes equations (1) is finding $\boldsymbol{u}\in[H_0^1(\Omega)]^d$ and $p\in L_0^2(\Omega)$ that satisfying

$$\epsilon^2(\nabla\boldsymbol{u},\nabla\boldsymbol{v})+(\boldsymbol{u},\boldsymbol{v})-(\nabla\cdot\boldsymbol{v},p) = (\boldsymbol{f},\boldsymbol{v}), \tag{2}$$

$$(\nabla\cdot\boldsymbol{u},q) = 0, \tag{3}$$

for all $\boldsymbol{v}\in[H_0^1(\Omega)]^d$ and $q\in L_0^2(\Omega)$.

In [28], a WG finite element method was proposed for the Darcy-Stokes equations (1). They proved the inf-sup condition and derived the optimal order error estimates. However, the new scheme contained a stabilizer. The goal of this paper is to continue the investigation of SFWG finite element methods for the Darcy-Stokes equations. By using high order polynomials to compute the weak gradient operator ∇_w and weak divergence operator $\nabla_w\cdot$, we introduce a new WG finite element method without stabilizers. Although raising the degree of polynomials in the computation of weak gradient operator and weak divergence operator, it will not increase the size of global stiffness matrix. In addition, the proposed method will reduce the computational complexity of programming and achieve the optimal order of convergence.

This paper is organised as follows. In the next section, it presents a SFWG finite element method for the Darcy-Stokes equations (1). In Section 3, we give some essential lemmas and prove the existence and uniqueness of the solution of the new method. In Section 4, some error equations are derived. In Section 5, we derive the error estimates of the $|||\cdot|||$ norm and L^2 norm of the velocity function and the L^2 norm of the pressure function. The sixth Section shows some numerical experiments.

2. Preliminary and a SFWG finite element scheme

In this section, we will introduce some definitions of the standard *Sobolev* space, and define the finite element spaces, the definitions of weak operators and the SFWG finite element formulation.

2.1. Sobolev space. In this paper, we denote the standard Sobolev spaces by $H^m(\Lambda)$ and $H^m_0(\Lambda)$, and the norm and semi-norm in these spaces by $\|\cdot\|_{m,\Lambda}$ and $|\cdot|_{m,\Lambda}$ for any open bounded domain $\Lambda \in \mathbb{R}^s$ $(s = d, d-1)$. The inner product of $H^m(\Lambda)$ is denoted by $(\cdot,\cdot)_{m,\Lambda}$. In particular, the subscript Λ in the norm, semi-norm and the inner product can be omitted when $\Lambda = \Omega$. We will also use the following spaces:

$$L^2_0(\Lambda) := \{q \in L^2(\Lambda), \int_\Lambda q dx = 0\},$$

and the space $H(div,\Lambda)$ with its associated norm $\|\cdot\|_{H(div,\Lambda)}$

$$H(div,\Lambda) := \{\boldsymbol{v} : \boldsymbol{v} \in [L^2(\Lambda)]^d, \nabla\cdot\boldsymbol{v} \in L^2(\Lambda)\},$$

$$\|\boldsymbol{v}\|_{H(div,\Lambda)} := (\|\boldsymbol{v}\|^2_\Lambda + \|\nabla\cdot\boldsymbol{v}\|^2_\Lambda)^{1/2}.$$

2.2. A SFWG finite element scheme. Let $\mathcal{T}_h = \bigcup\{T\}$ be a triangular or tetrahedral partition of the domain Ω satisfying a set of conditions[20]. For each element T, ∂T and h_T denote its boundary and diameter, respectively. In particular, the size of the mesh h is defined by $h := \max_{T\in\mathcal{T}_h} h_T$. Denote by $\mathcal{E}_h$ the set of all edges or flat faces and $\mathcal{E}^0_h := \mathcal{E}_h \setminus \partial\Omega$.

The finite element spaces are given by

$$V_h{:=}\{\boldsymbol{v} = \{\boldsymbol{v}_0,\boldsymbol{v}_b\} : \boldsymbol{v}_0 \in [P_k(T)]^d, \boldsymbol{v}_b \in [P_k(e)]^d, e \in \partial T\}, \tag{4}$$

$$W_h{:=}\{q \in L^2_0(\Omega) : q\,|_T \in P_{k-1}(T)\}. \tag{5}$$

Let V^0_h be the subspace of V_h with value 0 on $\partial\Omega$,

$$V^0_h{:=}\{\boldsymbol{v} : \boldsymbol{v} \in V_h, \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\}. \tag{6}$$

The weak gradient operator and weak divergence operator are defined as follows.

Definition 2.1. *For any $T \in \mathcal{T}_h$ and $\boldsymbol{v} = \{\boldsymbol{v}_0,\boldsymbol{v}_b\} \in V_h + [H^1(\Omega)]^d$, the weak gradient $\nabla_w\boldsymbol{v}$ is the unique polynomial in $[P_j(T)]^{d\times d}$ satisfying*

$$(\nabla_w\boldsymbol{v},\boldsymbol{\tau})_T{:=} -(\boldsymbol{v}_0,\nabla\cdot\boldsymbol{\tau})_T + \langle\boldsymbol{v}_b,\boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_{\partial T}, \quad \forall\boldsymbol{\tau} \in [P_j(T)]^{d\times d}, (j > k), \tag{7}$$

where $\boldsymbol{n}$ is the outward normal direction to ∂T.

Definition 2.2. *For any $T \in \mathcal{T}_h$ and $\boldsymbol{v} = \{\boldsymbol{v}_0,\boldsymbol{v}_b\} \in V_h + [H^1(\Omega)]^d$, the weak divergence $\nabla_w\cdot\boldsymbol{v}$ is the unique polynomial in $P_{k-1}(T)$ satisfying*

$$(\nabla_w\cdot\boldsymbol{v},q)_T{:=} -(\boldsymbol{v}_0,\nabla q)_T + \langle\boldsymbol{v}_b,q\cdot\boldsymbol{n}\rangle_{\partial T}, \quad \forall q \in P_{k-1}(T), \tag{8}$$

where $\boldsymbol{n}$ is the outward normal direction to ∂T.

Then, define two bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ in the following

$$a(\boldsymbol{v},\boldsymbol{w}){:=}\epsilon^2\sum_{T\in\mathcal{T}_h}(\nabla_w\boldsymbol{v},\nabla_w\boldsymbol{w})_T + \sum_{T\in\mathcal{T}_h}(\boldsymbol{v}_0,\boldsymbol{w}_0)_T,$$

$$b(\boldsymbol{v},q){:=}\sum_{T\in\mathcal{T}_h}(\nabla_w\cdot\boldsymbol{v},q)_T.$$

For any $\boldsymbol{v}_h \in V_h + [H^1(\Omega)]^d$, $|||\boldsymbol{v}_h|||$ and $|\boldsymbol{v}_h|_h$ are given by

$$|||\boldsymbol{v}_h\,|||^2 := \epsilon^2 \sum_{T\in\mathcal{T}_h} (\nabla_w \boldsymbol{v}, \nabla_w \boldsymbol{v})_T + \sum_{T\in\mathcal{T}_h} (\boldsymbol{v}_0, \boldsymbol{v}_0)_T, \tag{9}$$

$$|\boldsymbol{v}_h|_h^2 := \sum_{T\in\mathcal{T}_h} h_T^{-1} \|\boldsymbol{v}_0 - \boldsymbol{v}_b\|_{\partial T}^2. \tag{10}$$

Next, denote $a \le Cb$ ($a \ge Cb$) by $a \lesssim b$ ($a \gtrsim b$), where C is a generic constant independent of the fluid viscosity coefficient ϵ and the grid size h.

Lemma 2.3. *$|||\cdot|||$ defines a norm in the space V_h^0.*

Proof. According to the definition of $|||\cdot|||$, we just need to prove its positive-length property, i.e., for any $\boldsymbol{v} \in V_h^0$, $|||\boldsymbol{v}||| = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$.

Let $\boldsymbol{v} \in V_h^0$ and $|||\boldsymbol{v}||| = 0$. From (9), it follows that

$$\epsilon^2(\nabla_w \boldsymbol{v}, \nabla_w \boldsymbol{v}) + (\boldsymbol{v}_0, \boldsymbol{v}_0) = 0,$$

which implies that $\nabla_w \boldsymbol{v} = 0$, $\boldsymbol{v}_0 = 0$ in each $T \in \mathcal{T}_h$. And according to the definition of ∇_w, we get

$$0 = (\nabla_w \boldsymbol{v}, \boldsymbol{\tau})_T = -(\boldsymbol{v}_0, \nabla\cdot\boldsymbol{\tau})_T + \langle \boldsymbol{v}_b, \boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_{\partial T} \quad \forall \boldsymbol{\tau} \in [P_j(T)]^{d\times d}, (j > k).$$

We know that $\langle \boldsymbol{v}_b, \boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_{\partial T} = 0$, thus $\boldsymbol{v}_b = \boldsymbol{0}$, i.e., $\boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} = \boldsymbol{0}$. □

Lemma 2.4. *For any $\boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in V_h^0$, $T \in \mathcal{T}_h$, we have*

$$|\boldsymbol{v}|_h^2 \lesssim |||\boldsymbol{v}\,|||^2\,. \tag{11}$$

Proof. For any $\boldsymbol{v} \in V_h$, $\tau \in [P_j(T)]^{d\times d}$, using the definition of ∇_w and integration by parts

$$\begin{aligned}(\nabla_w \boldsymbol{v}, \tau)_T &= -(\boldsymbol{v}_0, \nabla\cdot\tau)_T + \langle \boldsymbol{v}_b, \tau\cdot\boldsymbol{n}\rangle_{\partial T} \\ &= (\nabla \boldsymbol{v}_0, \tau)_T - \langle \boldsymbol{v}_0 - \boldsymbol{v}_b, \tau\cdot\boldsymbol{n}\rangle_{\partial T}.\end{aligned} \tag{12}$$

According to the references[31], suppose T be a convex m-polygon/polyhedron of size h_T with edges/faces $e, e_1, ...,$ and e_{m-1}. There exists $\tau_0 \in [P_j(T)]^{d\times d}$, $j = m + k - 1$, such that

$$\begin{aligned}(\nabla \boldsymbol{v}_0, \tau_0)_T &= 0, \\ \langle \boldsymbol{v}_0 - \boldsymbol{v}_b, \tau_0\cdot\boldsymbol{n}\rangle_{\partial T\backslash e} &= 0, \\ \langle \boldsymbol{v}_b - \boldsymbol{v}_0, \tau_0\cdot\boldsymbol{n}\rangle_e &= \|\boldsymbol{v}_b - \boldsymbol{v}_0\|_e^2, \\ \|\tau_0\|_T &\lesssim h_T^{\frac{1}{2}} \|\boldsymbol{v}_b - \boldsymbol{v}_0\|_e.\end{aligned} \tag{13}$$

In (12), let $\tau = \tau_0$ we get

$$\|\boldsymbol{v}_b - \boldsymbol{v}_0\|_e^2 = (\nabla_w \boldsymbol{v}, \tau_0)_T. \tag{14}$$

Applying the Cauchy-Schwarz inequality to the above equation and then following (13) yields

$$\|\boldsymbol{v}_b - \boldsymbol{v}_0\|_e^2 = (\nabla_w \boldsymbol{v}, \tau_0)_T \lesssim \|\nabla_w \boldsymbol{v}\|_T \|\tau_0\|_T \lesssim h_T^{\frac{1}{2}} \|\nabla_w \boldsymbol{v}\|_T \|\boldsymbol{v}_b - \boldsymbol{v}_0\|_e.$$

Thus, we have

$$\|\boldsymbol{v}_b - \boldsymbol{v}_0\|_{\partial T} \lesssim h_T^{\frac{1}{2}} \|\nabla_w \boldsymbol{v}\|_T.$$

Then sum all the elements, we get

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\boldsymbol{v}_b-\boldsymbol{v}_0\|_{\partial T}^2 &\lesssim \sum_{T\in\mathcal{T}_h}\|\nabla_w\boldsymbol{v}\|_T^2\\
&\lesssim \sum_{T\in\mathcal{T}_h}(\|\nabla_w\boldsymbol{v}\|_T^2+\|\boldsymbol{v}_0\|_T^2)\\
&= |||\boldsymbol{v}|||^2 . \qquad (15)
\end{aligned}
$$

□

Finally, our SFWG finite element method for the Darcy-Stokes equations (1) is as follows.

Algorithm 2.1. Seek $\boldsymbol{u}_h=\{\boldsymbol{u}_0,\boldsymbol{u}_b\}\in V_h^0$, $p_h\in W_h$, satisfying

$$
\begin{aligned}
a(\boldsymbol{u}_h,\boldsymbol{v}_h)-b(\boldsymbol{v}_h,p_h) &= (\boldsymbol{f},\boldsymbol{v}_0), \qquad (16)\\
b(\boldsymbol{u}_h,q_h) &= 0, \qquad (17)
\end{aligned}
$$

for any $\boldsymbol{v}_h\in V_h^0+[H_0^1(\Omega)]^d$ and $q_h\in W_h$.

3. Existence and uniqueness

Here, we will give some crucial lemmas and prove the existence and uniqueness of the solution of the SFWG finite element method (16)-(17).

Firstly, we will introduce three projection operators. For each element $T\in\mathcal{T}_h$, let Q_h' be the projection operator from $L^2(T)$ onto $P_{k-1}(T)$. Denote the L^2 projection of $\boldsymbol{u}$ by $Q_h\boldsymbol{u}=\{Q_0\boldsymbol{u},Q_b\boldsymbol{u}\}$. Q_0 is the L^2 projection operator from $[L^2(T)]^d$ onto $[P_k(T)]^d$. Similarly, for each boundary ∂T, Q_b is the L^2 projection operator from $[L^2(\partial T)]^d$ onto $[P_k(\partial T)]^d$. Denote by Q_h'' the L^2 projection from $[L^2(T)]^{d\times d}$ onto $[P_j(T)]^{d\times d}$.

Lemma 3.1. [28] *The projection operators Q_h', Q_h, Q_h'' satisfy the following commutative properties*

$$
\begin{aligned}
\nabla_w(Q_h\boldsymbol{u}) &= Q_h''(\nabla\boldsymbol{u}), \quad \forall\boldsymbol{u}\in[H^1(\Omega)]^d. \qquad (18)\\
\nabla_w\cdot(Q_h\boldsymbol{u}) &= Q_h'(\nabla\cdot\boldsymbol{u}), \quad \forall\boldsymbol{u}\in[H(div;\Omega)]^d. \qquad (19)
\end{aligned}
$$

Lemma 3.2. [12] *For any $\boldsymbol{v},\boldsymbol{w}\in V_h^0$, the boundedness and coercivity of $a(\cdot,\cdot)$ as follows*

$$
\begin{aligned}
a(\boldsymbol{v},\boldsymbol{w}) &\le |||\boldsymbol{v}|||\cdot|||\boldsymbol{w}|||,\\
a(\boldsymbol{v},\boldsymbol{v}) &= |||\boldsymbol{v}|||^2 .
\end{aligned}
$$

Lemma 3.3. *For any $q\in W_h$, we have*

$$
\sup_{\boldsymbol{v}\in V_h^0}\frac{b(\boldsymbol{v},q)}{|||\boldsymbol{v}|||}\gtrsim\|q\|. \qquad (20)
$$

Proof. In fact, for any $q\in W_h\subset L_0^2(\Omega)$, one can find a $\tilde{\boldsymbol{v}}\in[H_0^1(\Omega)]^d$ such that

$$
\frac{(\nabla\cdot\tilde{\boldsymbol{v}},q)}{\|\tilde{\boldsymbol{v}}\|_1}\gtrsim\|q\|. \qquad (21)
$$

Let $\boldsymbol{v}=Q_h\tilde{\boldsymbol{v}}$, using the definition of Q_h' and (19), we get

$$
b(\boldsymbol{v},q)=(\nabla_w\cdot(Q_h\tilde{\boldsymbol{v}}),q)=(Q_h'(\nabla\cdot\tilde{\boldsymbol{v}}),q)=(\nabla\cdot\tilde{\boldsymbol{v}},q).
$$

Use the property (18)

$$\sum_{T\in\mathcal{T}_h}\|\nabla_w\boldsymbol{v}\|_T^2 = \sum_{T\in\mathcal{T}_h}\|\nabla_w(Q_h\tilde{\boldsymbol{v}})\|_T^2$$
$$= \sum_{T\in\mathcal{T}_h}\|Q_h''\nabla\tilde{\boldsymbol{v}}\|_T^2 \lesssim \|\nabla\tilde{\boldsymbol{v}}\|^2.$$

According to the definition of $|||\boldsymbol{v}|||$, we have

$$|||\boldsymbol{v}|||^2 = \epsilon^2\sum_{T\in\mathcal{T}_h}\|\nabla_w\boldsymbol{v}\|_T^2 + \sum_{T\in\mathcal{T}_h}\|\boldsymbol{v}_0\|_T^2.$$

And then, we get $|||\boldsymbol{v}||| \lesssim \|\tilde{\boldsymbol{v}}\|_1$. From (21), it follows that

$$\frac{b(\boldsymbol{v},q)}{|||\boldsymbol{v}|||} \gtrsim \frac{(\nabla\cdot\tilde{\boldsymbol{v}},q)}{\|\tilde{\boldsymbol{v}}\|_1} \gtrsim \|q\|.$$

□

Theorem 3.4. *The SFWG finite element scheme (16)-(17) has one and only one solution.*

Proof. Let $\boldsymbol{v}=\boldsymbol{u_h}, \boldsymbol{f}=\boldsymbol{0}$ and $q=p_h$ in (16)-(17),we have

$$a(\boldsymbol{u_h},\boldsymbol{u_h}) = \epsilon^2(\nabla_w\boldsymbol{u_h},\nabla_w\boldsymbol{u_h}) + (\boldsymbol{u_0},\boldsymbol{u_0}) = 0,$$

implying that $\boldsymbol{u_h}=\boldsymbol{0}$ on each T. Then by $\boldsymbol{u_h}=0$, $b(\boldsymbol{v},p_h)=0$, the definitions of $b(\cdot,\cdot)$ and $\nabla_w\cdot$, we have

$$b(\boldsymbol{v},p_h) = (\nabla_w\cdot\boldsymbol{v},p_h) = -\sum_{T\in\mathcal{T}_h}(\boldsymbol{v}_0,\nabla p_h)_T + \sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_b,p_h\boldsymbol{n}\rangle_{\partial T} = 0.$$

Let $\boldsymbol{v}=\{\boldsymbol{v}_0,\boldsymbol{v}_b\}=\{\nabla p_h,\boldsymbol{0}\}$ in above equation, we get $\nabla p_h=\boldsymbol{0}$ for any $T\in\mathcal{T}_h$, which implies p_h is a constant for any $T\in\mathcal{T}_h$.

Similarly, letting $\boldsymbol{v}_0=0, \boldsymbol{v}_b=[p_h]\equiv p_h|_{T_1}\boldsymbol{n}_1+p_h|_{T_2}\boldsymbol{n}_2$, and $\boldsymbol{v}_b=0$ on $e\in\mathcal{E}_h^0$ which shares by two elements T_1 and T_2 in the above equation, respectively, we get that p_h is a constant for any $T\in\mathcal{T}_h$.

Finally, from the fact $p_h\in L_0^2(\Omega)$, we get $p_h=0$. □

4. Error equations

The error equations for the SFWG finite element schemes (16)-(17) are derived in this section. Let $(\boldsymbol{u},p)$ and $(\boldsymbol{u}_h,p_h)$ be the exact solution of the Darcy-Stokes equations (1) and the numerical solution of the SFWG finite element method (16)-(17), respectively. Let $\boldsymbol{e}_h$ and ε_h be the errors of the velocity function and pressure function, respectively, defined as follows

$$\boldsymbol{e}_h:=\{\boldsymbol{e}_0,\boldsymbol{e}_b\}=\{Q_0\boldsymbol{u}-\boldsymbol{u}_0,Q_b\boldsymbol{u}-\boldsymbol{u}_b\},\quad \varepsilon_h=Q_h'p-p_h.$$

Lemma 4.1. *Let $\boldsymbol{e}_h$ and ε_h be the error of the SFWG finite element solution arising from(16)-(17). For any $(\boldsymbol{u}_h,p_h)\in V_h^0\times W_h$, the following equations hold true*

$$a(\boldsymbol{e}_h,\boldsymbol{v}) - b(\boldsymbol{v},\varepsilon_h) = \eta(\boldsymbol{v},\boldsymbol{u}) - \theta(\boldsymbol{v},p), \tag{22}$$
$$b(\boldsymbol{e}_h,q) = 0, \tag{23}$$

where

$$\eta(\boldsymbol{v},\boldsymbol{u}) = \epsilon^2\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_0-\boldsymbol{v}_b,\nabla\boldsymbol{u}\cdot\boldsymbol{n}-Q_h''(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T},$$
$$\theta(\boldsymbol{v},p) = \sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_0-\boldsymbol{v}_b,(p-Q_h'p)\cdot\boldsymbol{n}\rangle_{\partial T}.$$

Proof. From (18), (7) and the integration by parts, we have

$$\begin{aligned}(\nabla_w(Q_h\boldsymbol{u}),\nabla_w\boldsymbol{v})_T &= (Q_h''(\nabla\boldsymbol{u}),\nabla_w\boldsymbol{v})_T\\ &= -(\boldsymbol{v}_0,\nabla\cdot Q_h''(\nabla\boldsymbol{u}))_T+\langle\boldsymbol{v}_b,Q_h''(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}\\ &= (\nabla\boldsymbol{v}_0,Q_h''(\nabla\boldsymbol{u}))_T-\langle\boldsymbol{v}_0-\boldsymbol{v}_b,Q_h''(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}\\ &= (\nabla\boldsymbol{u},\nabla\boldsymbol{v}_0)_T-\langle\boldsymbol{v}_0-\boldsymbol{v}_b,Q_h''(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}.\end{aligned}\tag{24}$$

By (8), the integration by parts, and the fact $\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_b,p\cdot\boldsymbol{n}\rangle_{\partial T}=0$, we arrived at

$$\begin{aligned}(\nabla_w\cdot\boldsymbol{v},Q_h'p) &= -\sum_{T\in\mathcal{T}_h}(\boldsymbol{v}_0,\nabla(Q_h'p))_T+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_b,(Q_h'p)\cdot\boldsymbol{n}\rangle_{\partial T}\\ &= \sum_{T\in\mathcal{T}_h}(\nabla\cdot\boldsymbol{v}_0,Q_h'p)_T-\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_0-\boldsymbol{v}_b,(Q_h'p)\cdot\boldsymbol{n}\rangle_{\partial T}\\ &= -\sum_{T\in\mathcal{T}_h}(\boldsymbol{v}_0,\nabla p)_T+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_0,p\cdot\boldsymbol{n}\rangle_{\partial T}\\ &\quad -\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_0-\boldsymbol{v}_b,(Q_h'p)\cdot\boldsymbol{n}\rangle_{\partial T}\\ &= -\sum_{T\in\mathcal{T}_h}(\boldsymbol{v}_0,\nabla p)_T+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_0-\boldsymbol{v}_b,p\cdot\boldsymbol{n}\rangle_{\partial T}\\ &\quad -\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_0-\boldsymbol{v}_b,(Q_h'p)\cdot\boldsymbol{n}\rangle_{\partial T}\\ &= -(\boldsymbol{v}_0,\nabla p)_T+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_0-\boldsymbol{v}_b,(p-Q_h'p)\cdot\boldsymbol{n}\rangle_{\partial T},\end{aligned}$$

i.e.

$$(\boldsymbol{v}_0,\nabla p)=-(\nabla_w\cdot\boldsymbol{v},Q_h'p)+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_0-\boldsymbol{v}_b,(p-Q_h'p)\cdot\boldsymbol{n}\rangle_{\partial T}.\tag{25}$$

Testing (1) by $\boldsymbol{v}_0$ we obtain

$$-\epsilon^2(\nabla\cdot(\nabla\boldsymbol{u}),\boldsymbol{v}_0)+(\boldsymbol{u},\boldsymbol{v}_0)+(\nabla p,\boldsymbol{v}_0)=(\boldsymbol{f},\boldsymbol{v}_0).\tag{26}$$

According to the integration by parts and $\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_b,\nabla\boldsymbol{u}\cdot\boldsymbol{n}\rangle=0$, it follows that

$$-\epsilon^2(\nabla\cdot(\nabla\boldsymbol{u}),\boldsymbol{v}_0)=\epsilon^2\sum_{T\in\mathcal{T}_h}(\nabla\boldsymbol{u},\nabla\boldsymbol{v}_0)_T-\epsilon^2\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_0-\boldsymbol{v}_b,\nabla\boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\partial T}.\tag{27}$$

From (24) and (27), we have

$$\begin{aligned}-\epsilon^2(\nabla\cdot(\nabla\boldsymbol{u}),\boldsymbol{v}_0) &= \epsilon^2(\nabla_w(Q_h\boldsymbol{u}),\nabla_w\boldsymbol{v})\\ &\quad -\epsilon^2\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_0-\boldsymbol{v}_b,\nabla\boldsymbol{u}\cdot\boldsymbol{n}-Q_h''(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}.\end{aligned}\tag{28}$$

Combining (25)-(28), it follows

$$\begin{aligned}&\epsilon^2(\nabla_w(Q_h\boldsymbol{u}),\nabla_w\boldsymbol{v})+(\boldsymbol{u},\boldsymbol{v}_0)-(\nabla_w\cdot\boldsymbol{v},Q_h'p)\\ &= (\boldsymbol{f},\boldsymbol{v}_0)+\eta(\boldsymbol{v},\boldsymbol{u})-\theta(\boldsymbol{v},p).\end{aligned}\tag{29}$$

According to the equation above and (2) we arrive at

$$a(\boldsymbol{e}_h,\boldsymbol{v})-b(\boldsymbol{v},\varepsilon_h) = \eta(\boldsymbol{v},\boldsymbol{u})-\theta(\boldsymbol{v},p).\tag{30}$$

Similarly, we have

$$b(\boldsymbol{e}_h,q) = 0.\tag{31}$$

Thus, we complete the proof. □

5. Error estimate

In this section, we will establish optimal order error estimates for the velocity approximation $\boldsymbol{u}_h$ in $|||\cdot|||$ norm, and for the pressure approximation p_h in the standard L^2 norm. In addition, we will use the routine duality argument to get an L^2 error estimate for the velocity.

Before that, we present some preparation work.

Lemma 5.1. [20] *Assume $\mathcal{T}_h$ is a partition of Ω that satisfies the shape regularity assumption and $\boldsymbol{w} \in [H^{r+1}(\Omega)]^d$, $\rho \in H^r(\Omega)$, we obtain*

$$\sum_{T\in\mathcal{T}_h} h^{2s}\|\boldsymbol{w} - Q_0\boldsymbol{w}\|^2_{T,s} \lesssim h^{2(r+1)}\|\boldsymbol{w}\|^2_{r+1}, \tag{32}$$

$$\sum_{T\in\mathcal{T}_h} h^{2s}\|\nabla\boldsymbol{w} - Q_h''(\nabla\boldsymbol{w})\|^2_{T,s} \lesssim h^{2r}\|\boldsymbol{w}\|^2_{r+1}, \tag{33}$$

$$\sum_{T\in\mathcal{T}_h} h^{2s}\|\rho - Q_h'\rho\|^2_{T,s} \lesssim h^{2r}\|\rho\|^2_{r}. \tag{34}$$

where $1 \le r \le k$, $0 \le s \le 1$.

Lemma 5.2. [28] *Assume $\boldsymbol{w} \in [H^{r+1}(\Omega)]^d$,$\rho \in H^r(\Omega)$, we have*

$$\sum_{T\in\mathcal{T}_h} h_T\|Q_0\boldsymbol{w} - \boldsymbol{w}\|^2_{\partial T} \lesssim h^{2(r+1)}\|\boldsymbol{w}\|^2_{r+1}, \tag{35}$$

$$\sum_{T\in\mathcal{T}_h} h_T\|\nabla\boldsymbol{w} - Q_h''(\nabla\boldsymbol{w})\|^2_{\partial T} \lesssim h^{2r}\|\boldsymbol{w}\|^2_{r+1}, \tag{36}$$

$$\sum_{T\in\mathcal{T}_h} h_T\|Q_h'\rho - \rho\|^2_{\partial T} \lesssim h^{2r}\|\rho\|^2_{r+1}, \tag{37}$$

$$\sum_{T\in\mathcal{T}_h} h^{-1}\|Q_0\boldsymbol{w} - Q_b\boldsymbol{w}\|^2_{\partial T} \lesssim h^{2r}\|\boldsymbol{w}\|^2_{r+1}. \tag{38}$$

Theorem 5.3. *Let $(\boldsymbol{u}, p) \in [H_0^1(\Omega) \cap H^{k+1}(\Omega)]^d \times [L_0^2(\Omega) \cap H^k(\Omega)]$ with $k \ge 1$ be the solutions of (1), and $(\boldsymbol{u}_h, p_h) \in V_h^0 \times W_h$ is the numerical solution of (16)-(17), respectively. Then we have*

$$|||Q_h\boldsymbol{u} - \boldsymbol{u}_h\,||| + \|Q_h'p - p_h\| \lesssim h^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k). \tag{39}$$

Proof. Letting $\boldsymbol{v} = \boldsymbol{e}_h$ in (22) and $q = \varepsilon_h$ in (23), then we arrive at

$$|||\boldsymbol{e}_h|||^2 = \eta(\boldsymbol{e}_h, \boldsymbol{u}) - \theta(\boldsymbol{e}_h, p). \tag{40}$$

It follows from the Cauchy-Schwarz inequality, (36) and (11) that

$$\begin{aligned} |\eta(\boldsymbol{e}_h, \boldsymbol{u})| &= \epsilon^2 \Big|\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{e}_0 - \boldsymbol{e}_b, \nabla\boldsymbol{u}\cdot\boldsymbol{n} - Q_h''(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}\Big| \\ &\lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T\|\nabla\boldsymbol{u}\cdot\boldsymbol{n} - Q_h''(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\|^2_{\partial T}\Big)^{\frac12}|\boldsymbol{e}_h|_h \\ &\lesssim h^k\|\boldsymbol{u}\|_{k+1}\,|||\,\boldsymbol{e}_h\,|||\,. \end{aligned} \tag{41}$$

Similarly, from the Cauchy-Schwarz inequality, (37) and (11) we also have

$$\begin{aligned} |\theta(\boldsymbol{v}, p)| &= \Big|\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{e}_0 - \boldsymbol{e}_b, (p - Q_h'p)\cdot\boldsymbol{n}\rangle_{\partial T}\Big| \\ &\lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T\|(Q_h'p - p)\cdot\boldsymbol{n}\|^2_{\partial T}\Big)^{\frac12}|\boldsymbol{e}_h|_h \\ &\lesssim h^k\|p\|_k\,|||\,\boldsymbol{e}_h\,|||\,. \end{aligned} \tag{42}$$

Substituting (41)-(42) into (40) yields the desired error estimate for $\boldsymbol{u}_h$

$$|||\boldsymbol{e}_h|||^2 \lesssim h_T^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k) \, ||| \, \boldsymbol{e}_h \, ||| \, .$$

Next, consider $\|\varepsilon_h\|$, from (22) for any $\boldsymbol{v} \in V_h$ we have

$$b(\boldsymbol{v}, \varepsilon_h) = a(\boldsymbol{e}_h, \boldsymbol{v}) - \eta(\boldsymbol{v}, \boldsymbol{u}) + \theta(\boldsymbol{v}, p),$$

It is similar to the proof of $|||\boldsymbol{e}_h|||$. According to the Cauchy-Schwarz inequality, lemma 3.2 and lemma 5.2, we get

$$\begin{aligned}
|a(\boldsymbol{e}_h, \boldsymbol{v})| &\leq |||\boldsymbol{e}_h \, ||| \cdot ||| \, \boldsymbol{v}||| \lesssim h^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k) \, ||| \, \boldsymbol{v}|||, \\
|\eta(\boldsymbol{v}, \boldsymbol{u})| &\lesssim h^k\|\boldsymbol{u}\|_{k+1} \, ||| \, \boldsymbol{v}|||, \\
|\theta(\boldsymbol{v}, p)| &\lesssim h^k\|p\|_k \, ||| \, \boldsymbol{v} \, ||| \, .
\end{aligned}$$

Combining above inequalities, we have

$$b(\boldsymbol{v}, \varepsilon_h) \lesssim h^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k) \, ||| \, \boldsymbol{v} \, ||| \, .$$

From the inequality above and (20) we obtain

$$\|\varepsilon_h\| \lesssim h^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k).$$

By the all above, we can get that

$$|||Q_h\boldsymbol{u} - \boldsymbol{u}_h \, ||| + \|Q_h'p - p_h\| \lesssim h^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k).$$

□

In the last part of this section, we will use the routine duality argument to get an L^2 error estimate for the velocity. Consider the dual problem that seeks (Φ, ξ) satisfying

$$\begin{aligned}
-\epsilon^2\Delta\Phi + \Phi + \nabla\xi &= \boldsymbol{e}_0 \quad \textit{in } \Omega, && (43)\\
\nabla \cdot \Phi &= 0 \quad \textit{in } \Omega, && (44)\\
\Phi &= 0 \quad \textit{on } \partial\Omega. && (45)
\end{aligned}$$

Assume that the following $[H^2(\Omega)]^d \times H^1(\Omega)$-regularity property holds true

$$\|\Phi\|_2 + \|\xi\|_1 \lesssim \|\boldsymbol{e}_0\|. \tag{46}$$

Theorem 5.4. *Let $(\boldsymbol{u}, p) \in [H_0^1(\Omega) \cap H^{k+1}(\Omega)]^d \times [L_0^2(\Omega) \cap H^k(\Omega)]$ with $k \geq 1$ be the solution of (1) and $(\boldsymbol{u}_h, p_h) \in V_h^0 \times W_h$ be the numerical solution of (16)-(17). Then we have the following estimate*

$$\|Q_0\boldsymbol{u} - \boldsymbol{u}_0\| \lesssim h^{k+1}(\|\boldsymbol{u}\|_{k+1} + \|p\|_k). \tag{47}$$

Proof. Testing (43) by $\boldsymbol{e}_0$, we have

$$-\epsilon^2(\Delta\Phi, \boldsymbol{e}_0) + (\Phi, \boldsymbol{e}_0) + (\nabla\xi, \boldsymbol{e}_0) = (\boldsymbol{e}_0, \boldsymbol{e}_0). \tag{48}$$

With (26) and (29), we get

$$\begin{aligned}
& -\epsilon^2(\Delta\boldsymbol{u}, \boldsymbol{v}_0) + (\boldsymbol{u}, \boldsymbol{v}_0) + (\nabla p, \boldsymbol{v}_0) \\
&= \epsilon^2(\nabla_w(Q_h\boldsymbol{u}), \nabla_w\boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}_0) - (\nabla_w \cdot \boldsymbol{v}, Q_h'p) - \eta(\boldsymbol{v}, \boldsymbol{u}) + \theta(\boldsymbol{v}, p).
\end{aligned} \tag{49}$$

In (49), let $\boldsymbol{u} = \Phi$, $\boldsymbol{v} = \boldsymbol{e}_0$ and $p = \xi$, we obtain

$$\begin{aligned}
\|Q_0\boldsymbol{u} - \boldsymbol{u}_0\|^2 &= -\epsilon^2(\Delta\Phi, \boldsymbol{e}_0) + (\Phi, \boldsymbol{e}_0) + (\nabla\xi, \boldsymbol{e}_0) \\
&= \epsilon^2(\nabla_w(Q_h\Phi), \nabla_w\boldsymbol{e}_h) + (\Phi, \boldsymbol{e}_0) - (\nabla_w \cdot \boldsymbol{e}_h, Q_h'\xi) \\
&\quad -\eta(\boldsymbol{e}_h, \Phi) + \theta(\boldsymbol{e}_h, \xi) \\
&= \epsilon^2(\nabla_w(Q_h\Phi), \nabla_w\boldsymbol{e}_h) + (Q_0\Phi, \boldsymbol{e}_0) - (\nabla_w \cdot \boldsymbol{e}_h, Q_h'\xi) \\
&\quad -\eta(\boldsymbol{e}_h, \Phi) + \theta(\boldsymbol{e}_h, \xi) \\
&= a(\boldsymbol{e}_h, Q_h\Phi) - b(\boldsymbol{e}_h, Q_h'\xi) - \eta(\boldsymbol{e}_h, \Phi) + \theta(\boldsymbol{e}_h, \xi). \qquad (50)
\end{aligned}$$

Consider the fact that $b(Q_h\Phi, \varepsilon_h) = 0$ and $b(\boldsymbol{e}_h, Q_h'\xi) = 0$, the above equation becomes

$$\|Q_0\boldsymbol{u} - \boldsymbol{u}_0\|^2 = a(\boldsymbol{e}_h, Q_h\Phi) - b(Q_h\Phi, \varepsilon_h) - \eta(\boldsymbol{e}_h, \Phi) + \theta(\boldsymbol{e}_h, \xi). \qquad (51)$$

Then, following the equation (22), we have

$$\|Q_0\boldsymbol{u} - \boldsymbol{u}_0\|^2 = \eta(Q_h\Phi, \boldsymbol{u}) - \theta(Q_h\Phi, p) - \eta(\boldsymbol{e}_h, \Phi) + \theta(\boldsymbol{e}_h, \xi). \qquad (52)$$

From (33), (10) and the trace inequality

$$\begin{aligned}
\eta(Q_h\Phi, \boldsymbol{u}) &= \sum_{T\in\mathcal{T}_h} \langle Q_0\Phi - Q_b\Phi, \nabla\boldsymbol{u}\cdot\boldsymbol{n} - Q_h''(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T} \\
&\le \Big(\sum_{T\in\mathcal{T}_h} h\|\nabla\boldsymbol{u}\cdot\boldsymbol{n} - Q_h''(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\|_{\partial T}^2\Big)^{\frac{1}{2}}|\Phi|_h \\
&\lesssim h^{k+1}\|\boldsymbol{u}\|_{k+1}\|\Phi\|_2.
\end{aligned}$$

Similarly, according to (34), (10) and trace inequality, we have

$$\begin{aligned}
\theta(Q_h\Phi, p) &= \sum_{T\in\mathcal{T}_h} \langle Q_0\Phi - Q_b\Phi, (p - Q_h'p)\boldsymbol{n}\rangle_{\partial T} \\
&\le \Big(\sum_{T\in\mathcal{T}_h} h\|p - L_hp\|_{\partial T}^2\Big)^{\frac{1}{2}}|\Phi|_h \\
&\lesssim h^{k+1}\|p\|_k\|\Phi\|_2.
\end{aligned}$$

Next, using lemma 5.1 and (39)

$$\begin{aligned}
\eta(\boldsymbol{e}_h, \Phi) &\lesssim h\|\Phi\|_2 \,|\!|\!|\, \boldsymbol{e}_h|\!|\!| \lesssim h^{k+1}(\|\boldsymbol{u}\|_{k+1} + \|p\|_k)\|\Phi\|_2, \\
\theta(\boldsymbol{e}_h, \xi) &\lesssim h\|\xi\|_1 \,|\!|\!|\, \boldsymbol{e}_h|\!|\!| \lesssim h^{k+1}(\|\boldsymbol{u}\|_{k+1} + \|p\|_k)\|\xi\|_1.
\end{aligned}$$

Substituting two equations above into (51), we can get

$$\|Q_0\boldsymbol{u} - \boldsymbol{u}_0\|^2 \lesssim h^{k+1}(\|\boldsymbol{u}\|_{k+1} + \|p\|_k)(\|\Phi\|_2 + \|\xi\|_1).$$

Finally, we combine the above equation with the regularity estimate (46) to obtain

$$\|Q_0\boldsymbol{u} - \boldsymbol{u}_0\| \lesssim h^{k+1}(\|\boldsymbol{u}\|_{k+1} + \|p\|_k).$$

□

6. Numerical experiment

We will use some numerical results to demonstrate the effectiveness of our S-FWG finite element algorithm (16)-(17) in this section. In the following numerical experiments, the finite element spaces (4)-(5) mentioned in Section 2 are used. The weak gradient is defined in $[P_{k+1}(T)]^2$, and the weak divergence is defined in $P_k(T)$.

Table 6.1. Comparison between our SFWG method and method in [28] as $\epsilon = 1$.

(a) History of errors and convergence rates for $\|Q_0\boldsymbol{u} - \boldsymbol{u}_0\|$.

h	SFWG		method in [28]	
	Error	Rate	Error	Rate
1/16	2.32e-2		5.72e-2	
1/24	1.04e-2	1.97	2.53e-2	2.01
1/32	5.89e-3	1.98	1.42e-2	2.01
1/40	3.78e-3	1.99	9.08e-3	2.00
1/48	2.63e-3	1.99	6.30e-3	2.00
1/56	1.93e-3	2.00	4.63e-3	2.00

(b) History of errors and convergence rates for $|||Q_h\boldsymbol{u} - \boldsymbol{u}_h|||$.

h	SFWG		method in [28]	
	Error	Rate	Error	Rate
1/16	1.31		2.68e-1	
1/24	8.81e-1	0.982	1.79e-1	1.00
1/32	6.62e-1	0.991	1.34e-1	1.00
1/40	5.31e-1	0.995	1.07e-1	1.00
1/48	4.42e-1	0.996	8.96e-2	1.00
1/56	3.79e-1	0.997	7.68e-2	1.00

(c) History of errors and convergence rates for $\|Q_h'p - p_h\|$.

h	SFWG		method in [28]	
	Error	Rate	Error	Rate
1/16	5.18e-1		7.71e-1	
1/24	3.52e-1	0.956	5.11e-1	1.01
1/32	2.66e-1	0.977	3.83e-1	1.00
1/40	2.13e-1	0.986	3.06e-1	1.00
1/48	1.78e-1	0.991	2.55e-1	1.00
1/56	1.53e-1	0.993	2.18e-1	1.00

We use triangular meshes for all examples. The error of the SFWG finite element method is measured by the following norms:

$$\begin{aligned}
\|q\| &= (\sum_{T\in\mathcal{T}_h}\int |q|^2dx)^{\frac{1}{2}},\\
\|\boldsymbol{v}\| &= (\sum_{T\in\mathcal{T}_h}\int |\boldsymbol{v}_0|^2dx)^{\frac{1}{2}},\\
|||\boldsymbol{v}_h||| &= (\sum_{T\in\mathcal{T}_h}\epsilon^2\int |\nabla_w\boldsymbol{v}|^2dx + \int |\boldsymbol{v}_0|^2dx)^{\frac{1}{2}}.
\end{aligned}$$

6.1. example 1. In this example, we compare the new SFWG finite element method (16)-(17) with the classical WG finite element method proposed in [28].

TABLE 6.2. Comparison between our SFWG method and method in [28] as $\epsilon = 0.25$.

(a) History of errors and convergence rates for $\|Q_0\boldsymbol{u} - \boldsymbol{u}_0\|$.

h	SFWG		method in [28]	
	Error	Rate	Error	Rate
1/16	2.24e-2		9.72e-3	
1/24	1.01e-2	1.97	4.27e-3	2.03
1/32	5.70e-3	1.99	2.39e-3	2.01
1/40	3.65e-3	1.99	1.53e-3	2.01
1/48	2.54e-3	1.99	1.06e-3	2.01
1/56	1.87e-3	2.00	7.78e-3	2.00

(b) History of errors and convergence rates for $|||Q_h\boldsymbol{u} - \boldsymbol{u}_h|||$.

h	SFWG		method in [28]	
	Error	Rate	Error	Rate
1/16	3.63e-1		3.05e-1	
1/24	2.44e-1	0.98	2.08e-1	0.94
1/32	1.83e-1	0.99	1.58e-1	0.97
1/40	1.47e-1	1.00	1.27e-1	0.98
1/48	1.22e-1	1.00	1.06e-1	0.99
1/56	1.05e-1	1.00	9.08e-2	0.99

(c) History of errors and convergence rates for $\|Q_h' p - p_h\|$.

h	SFWG		method in [28]	
	Error	Rate	Error	Rate
1/16	5.73e-2		1.32e-1	
1/24	3.82e-2	1.00	8.94e-2	0.97
1/32	2.86e-2	1.00	6.73e-2	0.98
1/40	2.29e-2	1.00	5.40e-2	0.99
1/48	1.91e-2	1.00	4.50e-2	0.99
1/56	1.63e-2	1.00	3.86e-2	1.00

Let the square domain $\Omega = (0,1)^2$ and $g = 0$. The exact solutions are given by

$$\boldsymbol{u} = \begin{pmatrix} -2\pi \sin^2(\pi x)\sin(\pi y)\cos(\pi y) \\ 2\pi \sin(\pi x)\cos(\pi x)\sin^2(\pi y) \end{pmatrix}, \quad p = \sin(\pi x) + \sin(\pi y) - 4/\pi.$$

We employ the finite element space with $k = 1$. So the right hand side function $\boldsymbol{f}$ can be calculated. All calculation results are shown in Table 6.1 and 6.2. From the tables, we get the following observations:

1) Table 6.1 and 6.2 show the errors and the convergence rates of the WG finite element algorithm and our SFWG finite element algorithm (16)-(17) as $\epsilon = 1$ and $\epsilon = 0.25$. From these two tables, it is clear that the convergence effect of our method is almost the same as the classical WG finite element method proposed in [28].

2) From Table 6.1, we can obtain the convergence rates for $\boldsymbol{u}$ in $|||\cdot|||$ norm and p in L^2 norm are of order $\mathcal{O}(h)$. The convergence rates for $\boldsymbol{u}$ in L^2 norm is of order $\mathcal{O}(h^2)$. These results are in agreement with Theorem 5.3 and 5.4, respectively.

TABLE 6.3. Errors and convergence rates for $|||Q_h\boldsymbol{u} - \boldsymbol{u}_h|||$.

h	$\epsilon = 8$		$\epsilon = 4$		$\epsilon = 2$		$\epsilon = 1/2$		$\epsilon = 1/4$		$\epsilon = 1/8$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
1/8	4.51e-2		2.65e-2		3.09e-2		1.15e-1		2.29e-1		4.52e-1	
1/12	3.08e-2	0.940	1.82e-2	0.946	2.09e-2	0.959	7.81e-2	0.960	1.56e-1	0.955	3.09e-1	0.934
1/16	2.33e-2	0.969	1.37e-2	0.972	1.58e-2	0.978	5.89e-2	0.978	1.18e-1	0.976	2.34e-1	0.965
1/20	1.87e-2	0.981	1.10e-2	0.983	1.27e-2	0.986	4.73e-2	0.986	9.44e-2	0.985	1.88e-1	0.978
1/24	1.56e-2	0.987	9.16e-3	0.988	1.06e-2	0.990	3.95e-2	0.990	7.89e-2	0.989	1.57e-1	0.985
1/28	1.34e-2	0.991	7.86e-3	0.991	9.09e-3	0.993	3.39e-2	0.993	6.77e-2	0.992	1.35e-1	0.989

TABLE 6.4. Errors and convergence rates for $\|Q_0\boldsymbol{u} - \boldsymbol{u}_0\|$.

h	$\epsilon = 8$		$\epsilon = 4$		$\epsilon = 2$		$\epsilon = 1/2$		$\epsilon = 1/4$		$\epsilon = 1/8$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
1/8	1.88e-4		2.07e-4		4.13e-4		5.93e-3		2.35e-2		9.05e-2	
1/12	8.66e-5	1.91	9.53e-5	1.91	1.89e-4	1.93	2.71e-3	1.93	1.08e-2	1.92	4.23e-2	1.87
1/16	4.94e-5	1.95	5.43e-5	1.95	1.07e-4	1.96	1.54e-3	1.96	6.14e-3	1.96	2.43e-2	1.93
1/20	3.18e-5	1.97	3.50e-5	1.97	6.91e-5	1.98	9.91e-4	1.98	3.95e-3	1.97	1.57e-2	1.95
1/24	2.22e-5	1.98	2.44e-5	1.98	4.81e-5	1.98	6.91e-4	1.98	2.76e-3	1.98	1.09e-2	1.97
1/28	1.63e-5	1.98	1.80e-5	1.99	3.54e-5	1.99	5.08e-4	1.99	2.03e-3	1.99	8.06e-3	1.98

TABLE 6.5. Errors and convergence rates for $\|Q_h' p - p_h\|$.

h	$\epsilon = 8$		$\epsilon = 4$		$\epsilon = 2$		$\epsilon = 1/2$		$\epsilon = 1/4$		$\epsilon = 1/8$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
1/8	1.46e-1		7.60e-2		6.89e-2		6.83e-2		6.85e-2		6.94e-2	
1/12	1.01e-1	0.919	5.14e-2	0.968	4.64e-2	0.974	4.60e-2	0.974	4.61e-2	0.977	4.64e-2	0.994
1/16	7.64e-2	0.956	3.87e-2	0.982	3.49e-2	0.988	3.46e-2	0.988	3.47e-2	0.989	3.48e-2	0.999
1/20	6.15e-2	0.972	3.11e-2	0.989	2.80e-2	0.993	2.78e-2	0.993	2.78e-2	0.994	2.78e-2	1.00
1/24	5.15e-2	0.981	2.59e-2	0.992	2.34e-2	0.996	2.31e-2	0.996	2.32e-2	0.996	2.32e-2	1.00
1/28	4.42e-2	0.986	2.22e-2	0.995	2.00e-2	0.997	1.98e-2	0.997	1.99e-2	0.998	1.99e-2	0.999

3) Similarly, we can obtain the same conclusion when the fluid viscosity coefficient $\epsilon = 0.25$.

6.2. example 2. Let the square domain $\Omega = (0,1)^2$ and $g = 0$. The exact solutions are

$$\boldsymbol{u} = \begin{pmatrix} -x^2(x-1)^2 y(y-1)(2y-1) \\ x(x-1)(2x-1)y^2(y-1)^2 \end{pmatrix}, \quad p = x^6 - y^6.$$

The second example illustrates the uniform convergence of our SFWG finite element method for the Darcy-Stokes equations. Specifically, the fluid viscosity coefficient ϵ changes from 1/8 to 8. For convenience, we also use the finite element space with $k = 1$. Tables 6.3-6.4 show all the results. From this two tables, we can obtain:

1) Table 6.3 presents the errors and the convergence rates of our SFWG finite element method when parameter ϵ changes. From Table 6.3, we can see that the rates of convergence for $\boldsymbol{u}$ in $|||\cdot|||$ norm is of order $\mathcal{O}(h)$.

2) Table 6.4 presents the errors and the convergence rates of scheme (16)-(17). From Table 6.4, we get that the rates of convergence for $\boldsymbol{u}$ in L^2 norm is of order $\mathcal{O}(h^2)$, which verify the theoretical results in Theorem 5.4.

3) Similarly, we can see that from Table 6.5 the rates of convergence for p in L^2 norm is of order $\mathcal{O}(h)$, which are consistent with the theory developed preciously.

6.3. example 3. Let the square domain $\Omega = (0,1)^2$. The exact solutions are

$$\boldsymbol{u} = \begin{pmatrix} -x(x-1)(2y-1) \\ y(y-1)(2x-1) \end{pmatrix}, \quad p = x^2 - y^2 - 2/3.$$

TABLE 6.6. Errors and convergence rates for $|||Q_h\boldsymbol{u} - \boldsymbol{u}_h|||$.

h	$\epsilon = 10$		$\epsilon = 1$		$\epsilon = 0.1$		$\epsilon = 0.01$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
1/8	1.47e-1		1.47e-2		1.49e-3		2.70e-4	
1/12	6.55e-2	2.00	6.55e-3	2.00	6.58e-4	2.02	9.36e-5	2.61
1/16	3.68e-2	2.00	3.68e-3	2.00	3.69e-4	2.01	4.64e-5	2.44
1/20	2.36e-2	2.00	2.36e-3	2.00	2.36e-4	2.00	2.76e-5	2.32
1/24	1.64e-2	2.00	1.64e-3	2.00	1.64e-4	2.00	1.84e-5	2.24
1/28	1.20e-2	2.00	1.20e-3	2.00	1.20e-4	2.00	1.31e-5	2.18

TABLE 6.7. Errors and convergence rates for $\|Q_0\boldsymbol{u} - \boldsymbol{u}_0\|$.

h	$\epsilon = 10$		$\epsilon = 1$		$\epsilon = 0.1$		$\epsilon = 0.01$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
1/8	2.26e-4		2.26e-4		2.26e-4		2.26e-4	
1/12	6.69e-5	3.00	6.69e-5	3.00	6.69e-5	3.00	6.69e-5	3.00
1/16	2.82e-5	3.00	2.82e-5	3.00	2.82e-5	3.00	2.82e-5	3.00
1/20	1.44e-5	3.00	1.44e-5	3.00	1.44e-5	3.00	1.44e-5	3.00
1/24	8.36e-6	3.00	8.36e-6	3.00	8.36e-6	3.00	8.36e-6	3.00
1/28	5.26e-6	3.00	5.26e-6	3.00	5.26e-6	3.00	5.26e-6	3.00

TABLE 6.8. Errors and convergence rates for $\|Q'_h p - p_h\|$.

h	$\epsilon = 10$		$\epsilon = 1$		$\epsilon = 0.1$		$\epsilon = 0.01$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
1/8	3.40e0		3.41e-2		1.59e-3		1.55e-3	
1/12	1.51e0	2.00	1.51e-2	2.00	7.07e-4	2.00	6.90e-4	2.00
1/16	8.51e-1	2.00	8.51e-3	2.00	4.52e-4	1.56	3.88e-4	2.00
1/20	5.44e-1	2.00	5.45e-3	2.00	2.54e-4	2.57	2.49e-4	1.99
1/24	3.78e-1	2.00	3.78e-3	2.00	1.77e-4	2.00	1.73e-4	2.01
1/28	2.78e-1	2.00	2.78e-3	2.00	1.39e-4	1.57	1.28e-4	1.95

In this numerical example, we consider the finite element space with $k = 2$. So the weak gradient operator and weak divergence operator are defined in the spaces $[P_3(T)]^2$ and $P_2(T)$, respectively. From Table 6.6-6.8, we get the following observations:

1) Tables 6.6 and 6.7 illustrate the errors and convergence rates in $|||\cdot|||$ norm and L^2 norm for $\boldsymbol{u}$ by the SFWG finite element method (16)-(17) with the case $\epsilon = 10, 1, 0.1, 0.01$. From Table 6.6 and Table 6.7, we can see that the rates of convergence for $\boldsymbol{u}$ in $|||\cdot|||$ norm and L^2 norm are $\mathcal{O}(h^2)$ and $\mathcal{O}(h^3)$, respectively. The results are consistent with the theoretical analysis. And when the value of ϵ is smaller, the convergence order for the velocity function in $|||\cdot|||$ norm can achieve superconvergence.

2) Table 6.8 shows the errors and convergence rates in L^2 norm for p by the proposed method (16)-(17) with the case $\epsilon = 10, 1, 0.1, 0.01$. From Table 6.8, we get that the rates of convergence for p in L^2 norm is of order $\mathcal{O}(h^2)$.

3) It can be seen from the Table 6.6-6.8 that when ϵ takes different values from 10 to 0.01, the SFWG method in this paper can obtain excellent numerical simulation results.

7. Conclusion

In this paper, we have developed a new WG scheme for solving Darcy-Stokes equations. The new method has no stabilizer term. The idea of getting rid of the stabilizer term is to increase the degree of polynomials used to compute weak gradient operator ∇_w. The size of the global stiffness matrix does not increase when using higher degree polynomials to calculate the weak gradient operator. Conversely, the complexity of the program is reduced by omitting the stabilizer term. And from the numerical results, this method can almost achieve the same effect as the WG finite element method.

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References

[1] I. Babŭska, The finite element method with Lagrangian multipliers, Numerische Mathematik, 20, 179-192, 1973.
[2] B. Cockburn, G. Kanschat, D. Schötzau, C. Schwab, Local discontinuous Galerkin methods for the Stokes system, SIAM Journal on Numerical Analysis, 40, 319-343, 2002.
[3] K. Mardal, X. Tai, R. Winther, A robust finite element method for Darcy–Stokes flow, SIAM Journal on Numerical Analysis, 40 1605-1631, 2002.
[4] B. Cockburn, G. Kanschat, D. Schötzau, A locally conservative LDG method for the incompressible Navier-Stokes equations, Mathematics of Computation, 74, 1067-1095, 2004.
[5] F. Brezzi, D. Boffi, L. Demkowicz, et al. Mixed finite elements, compatibility conditions, and applications, Springer, 2: 4-2, 2008.
[6] L. Chen, M. Holst, J. Xu, Convergence and optimality of adaptive mixed finite element methods, Mathematics of Computation, 78, 35-53, 2009.
[7] N.C. Nguyen, J. Peraire, B. Cockburn, A hybridizable discontinuous Galerkin method for Stokes flow, Computer Methods in Applied Mechanics and Engineering, 199, 582-597, 2010.
[8] X. He, T. Lin, Y. Lin, Interior penalty bilinear IFE discontinuous Galerkin methods for elliptic equations with discontinuous coefficient, Journal of Systems Science and Complexity, 23, 467-483, 2010.
[9] W. Peng, G. Cao, Darcy-Stokes equations with finite difference and natural boundary element coupling method, Computer Modeling in Engineering & Sciences, 75, 173-188, 2011.
[10] X. Xie, J. Xu, New mixed finite elements for plane elasticity and Stokes equations, Science China Mathematics, 54, 1499-1519, 2011.
[11] Z. Yin, Z. Jiang, Q. Xu, A discontinuous finite volume method for the Darcy-Stokes equations, Journal of Applied Mathematics, 3, 401-430, 2012.
[12] J. Wang, X. Ye, A weak Galerkin finite element method for second-order elliptic problems, Journal of Computational and Applied Mathematics, 241, 103-115, 2012.
[13] D. Boffi, F. Brezzi, M. Fortin, Mixed finite element methods and applications, Heidelberg: Springer, 2013.
[14] L. Mu, J. Wang, X. Ye, A stable numerical algorithm for the Brinkman equations by weak Galerkin finite element methods, Journal of Computational Physics, 273, 327-342, 2014.
[15] J. Wang, X. Ye, A weak Galerkin mixed finite element method for second order elliptic problems, Mathematics of Computation, 83, 2101-2126, 2014.
[16] X. He, T. Lin, Y. Lin, A selective immersed discontinuous Galerkin method for elliptic interface problems, Mathematical Methods in the Applied Sciences, 37, 983-1002, 2014.
[17] T. Zhang, L. Tang, A stabilized finite volume method for Stokes equations using the lowest order P1-P0 element pair, Advances in Computational Mathematics, 41, 781-798, 2015.
[18] L. Mu, X. Wang, X. Ye, A modified weak Galerkin finite element method for the Stokes equations, Journal of Computational and Applied Mathematics, 275, 79-90, 2015.

[19] G. Chen, M. Feng, X. Xie, Robust globally divergence-free weak Galerkin methods for Stokes equations, Journal of Computational Mathematics, 34, 549-572, 2016.

[20] J. Wang, X. Ye, A weak Galerkin finite element method for the Stokes equations, Advances in Computational Mathematics, 42, 155-174, 2016.

[21] W. Chen, F. Wang, Y. Wang, Weak Galerkin method for the coupled Darcy-Stokes flow, IMA Journal of Numerical Analysis, 36, 897-921, 2016.

[22] L. Mu, J. Wang, X. Ye, A hybridized formulation for the weak Galerkin mixed finite element method, Journal of Computational and Applied Mathematics, 307, 335-345, 2016.

[23] X. Wang, Q. Zhai, R. Zhang, The weak Galerkin method for solving the incompressible Brinkman flow, Journal of Computational and Applied Mathematics, 307, 13-24, 2016.

[24] L. Chen, J. Hu, X. Huang, Stabilized mixed finite element methods for linear elasticity on simplicial grids in $\mathbb{R}^n$, Computational Methods in Applied Mathematics, 17, 17-31, 2017.

[25] G. Chen, M. Feng, X. Xie, A robust WG finite element method for convection-diffusion-reaction equations, Journal of Computational and Applied Mathematics, 315, 107-125, 2017.

[26] R. Wang, X. Wang, R. Zhang, A Modified Weak Galerkin Finite Element Method for the Poroelasticity Problems, Numerical Mathematics: Theory, Methods and Applications, 11, 518-539, 2018.

[27] T. Tian, Q. Zhai, R. Zhang, A new modified weak Galerkin finite element scheme for solving the stationary Stokes equations, Journal of Computational and Applied Mathematics, 329, 268-279, 2018.

[28] X. Wang, Q. Zhai, R. Wang, R. Jari, An absolutely stable weak Galerkin finite element method for the Darcy-Stokes problem, Applied Mathematics and Computation, 331, 20-32, 2018.

[29] L. Zhang, M. Feng, J. Zhang, A globally divergence-free weak Galerkin method for Brinkman equations, Applied Numerical Mathematics, 137, 213-229, 2019.

[30] Y. Han, H. Chen, X. Wang, X. Xie, EXtended HDG methods for second order elliptic interface problems, Journal of Scientific Computing, 84, 22, 2020.

[31] X. Ye, S. Zhang, A conforming discontinuous Galerkin finite element method: Part II, International Journal of Numerical Analysis and Modeling, 17, 281-296, 2020.

[32] L. Mu, Pressure robust weak Galerkin finite element methods for Stokes problems, SIAM Journal on Scientific Computing, 42, B608-B629, 2020.

[33] X. Ye, S. Zhang, A stabilizer-free weak Galerkin finite element method on polytopal meshes, Journal of Computational and Applied Mathematics, 371, 112699, 2020.

[34] R. Zhang, Weak Galerkin finite element method for linear elasticity problems, Mathematica Numerica Sinica, 42, 1-17, 2020.

[35] A. Al-Taweel, S. Hussain, X. Wang, A stabilizer free weak Galerkin finite element method for parabolic equation, Journal of Computational and Applied Mathematics, 392, 113373, 2021.

[36] Y. Feng, Y. Liu, R. Wang, S. Zhang, A Stabilizer-Free Weak Galerkin Finite Element Method for the Stokes Equations, Advances in Applied Mathematics and Mechanics, 14, 181-201, 2022.

[37] L. Mu, X.Ye, S.Zhang, A Stabilizer-Free, Pressure-Robust, and Superconvergence Weak Galerkin Finite Element Method for the Stokes Equations on Polytopal Mesh, Society for Industrial and Applied Mathematics, 4, A2614-2637, 2021.

[38] H. Peng, Q. Zhai, R. Zhang, S. Zhang, A weak Galerkin-mixed finite element method for the Stokes-Darcy problem, Science China Mathematics, 64, 2357-2380, 2021.

[39] A. Zaghdani, S. Sayari, M. ELHajji, A new hybridized mixed weak Galerkin method for second-order elliptic problems, Journal of Computational Mathematics, 40, 501-518, 2022.

[40] H. Yuan, X. Xie, Semi-discrete and fully discrete mixed finite element methods for Maxwell viscoelastic model of wave propagation, Advances in Applied Mathematics and Mechanics, 14, 344-364, 2022.

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