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A DIFFERENCE FINITE ELEMENT METHOD FOR CONVECTION-DIFFUSION EQUATIONS IN CYLINDRICAL DOMAINS

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Abstract. In this paper, we consider 3D steady convection-diffusion equations in cylindrical domains. Instead of applying the finite difference methods (FDM) or the finite element methods (FEM), we propose a difference finite element method (DFEM) that can maximize good applicability and efficiency of both FDM and FEM. The essence of this method lies in employing the centered difference discretization in the z-direction and the finite element discretization based on the P_1 conforming elements in the (x, y) plane. This allows us to solve partial differential equations on complex cylindrical domains at lower computational costs compared to applying the 3D finite element method. We derive stability estimates for the difference finite element solution and establish the explicit dependence of H_1 error bounds on the diffusivity, convection field modulus, and mesh size. Finally, we provide numerical examples to verify the theoretical predictions and showcase the accuracy of the considered method.

Key words. Convection-diffusion equation, difference finite element method, cylindrical domain, error estimates.

1. Introduction

In this paper, we consider the difference finite element method (DFEM) to the following convection-diffusion equation with the homogeneous Dirichlet boundary condition:

$$-\alpha\widehat{\Delta}u(\mathbf{x},z) + \widehat{\boldsymbol{\beta}}\cdot\widehat{\boldsymbol{\nabla}}\,u(\mathbf{x},z) = f(\mathbf{x},z), \quad (\mathbf{x},z)\in\Omega, \tag{1a}$$

$$u(\mathbf{x},z) = 0, \quad (\mathbf{x},z)\in\partial\Omega. \tag{1b}$$

Here, and in what follows, frequently we use the notation $\mathbf{x} = (x, y)$. The unknown is a function $u : \overline{\Omega} \to \mathbb{R}$, $\overline{\Omega}$ is the closure of the open set $\Omega = \omega \times [a_3, b_3]$, $\alpha > 0$ is the constant diffusivity, $\widehat{\boldsymbol{\beta}} = (\boldsymbol{\beta}, \beta_3) = (\beta_1, \beta_2, \beta_3)$ is the given convection field satisfying that the components are constants and the RHS function $f : \Omega \to \mathbb{R}$ is the given source function. In a quest for greater clarity, we use the following notation $\widehat{\Delta} = \partial_{xx} + \partial_{yy} + \partial_{zz}$ and $\widehat{\boldsymbol{\nabla}} = (\partial_x, \partial_y, \partial_z)^\top$.

The finite element method (FEM) and the finite difference method (FDM) are two traditional important methods to solve partial differential equations (PDEs) using computers. FEMs are more adequate to handle PDEs with irregular coefficients and boundary conditions prescribed on complex geometric shapes, and thus can be used for modeling complex physical problems, but more expensive computation costs are needed especially for high-dimensional problems. On the other hand, FDMs have clear advantages in their implementation and low computing cost, but FDMs that require high regularity of solutions to the governing PDEs have certain

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limitation in direct application. For references, the reader may suggested to consult [2, 4, 9, 22, 21, 19, 3, 6, 1, 7, 15, 18], and the references therein.

Based on these, it is natural to combine these two methods to maximize applicability and efficiency to solve a certain suitable class of problems that bear both benefits of FEM and FDMs. Such cases occur, for instance, in dealing most problems with cylidrical domains whose underlying base geometries are complicate. In this spirit, the idea of difference finite element methods (DFEM) have been developed in recent years [14].

In [14], the authors proposed the Difference Finite Element Method (DFEM) for solving the 3D Poisson equation. The method utilizes a combination of the finite difference discretization in the z-direction and the finite element discretization in the (x, y)-domain ω using the P_1-conforming elements. In DFEM, the numerical solution of the 3D Poisson equation is obtained by solving a series of $2D$ elliptic equations, thereby reducing the computational complexity. Specifically, the coefficient matrix only needs to be computed in a $2D$ domain ω, making the overall computation more efficient. In this paper, our work are to discretize the convection-diffusion equation in a 3D domain using the Difference Finite Element Method (DFEM) and explicitly provide the matrix representation of the DFE discretization of the gradient term. This allows us to use the finite element method in the (x, y) plane where high flexibility and strong adaptability are required, and use the finite difference method in the z-direction to save computation cost and reduce implementation difficulty. Superconvergence in H_1 norm of this approach was studied in [10]. Since then, the idea of DFEM has been applied to solve 3D steady state Stokes and Navier-Stokes problems [17, 16, 11, 12].

We are interested in further development of DFEM for the convection-diffusion equation particularly in cylindrical domains. FDM is applied in the lateral direction while FEM is applied in the longitudinal 2D domain.

The remaining part of this paper is structured as follows. In Section 2, we recall the FE methods and establish the stability and error estimates for the 2D steady convection-diffusion problems. In Section 3, we present the DFE method based on the P_1-element for the z-direction discretization of the 3D steady convection-diffusion problems and perform stability and error estimates. In Section 4, we define the DFE solution pair $u_{h\tau}$ based on the $P_1 \times P_1$-element of the 3D steady convection-diffusion equation and prove the first order H_1-error bound of the DFE solution pair $u_{h\tau}$ with respect to the solution u of the 3D steady convection-diffusion equation. In Section 5, several numerical examples are presented to illustrate the effectiveness of the proposed method. Finally, the conclusions are drawn in Section 6.

1.1. Notations. For measurable set S in $\mathbb{R}^d$, by $(\cdot, \cdot)_S$ we denote the $L^2(S)$ inner product. For $k \in \mathbb{Z}$, standard notations for Sobolev spaces $H^k(S)$ will be employed. By $\|v\|_{k,S}$ and $|v|_{k,S}$ we mean the standard Sobolev norms and seminorms for $H^k(S)$. $\langle \cdot, \cdot \rangle_{X',X}$ will mean the duality paring between the topological vector space X and its dual X'. However, wherever there is no confusion, the subscripts may be omitted.

2. Finite element method based on the conforming P_1-element for 2D convection-diffusion problems

Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with boundary $\partial\omega$. Given $f \in H^{-1}(\omega) = (H_0^1(\omega))'$, consider the convection-diffusion equation with the homogeneous Dirichlet boundary condition as follows.

$$\begin{aligned} -\alpha\Delta u(\mathbf{x}) + \boldsymbol{\beta}\cdot\boldsymbol{\nabla}\, u(\mathbf{x}) &= f(\mathbf{x}), \quad \mathbf{x}\in\omega, &\text{(2a)}\\ u(\mathbf{x}) &= 0, \quad \mathbf{x}\in\partial\omega, &\text{(2b)}\end{aligned}$$

where $\alpha > 0$ denotes a constant diffusion coefficient, $\boldsymbol{\beta} = (\beta_1, \beta_2)$ is a given constant convection field. Using Green's formula, the weak formulation of the 2D convective-diffusion equations (2a)-(2b) is given as follows: find $u \in H_0^1(\omega)$ such that

$$A\,(u, v) = \langle f, v\rangle_{H^{-1}(\omega), H_0^1(\omega)} \quad \forall v \in H_0^1(\omega), \tag{3}$$

where the bilinear form $A(\cdot,\cdot) : H_0^1(\omega)\times H_0^1(\omega) \to \mathbb{R}$ is defined as

$$A(u, v) := \alpha\,(\boldsymbol{\nabla}\, u, \boldsymbol{\nabla}\, v)_\omega + (\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\, u, v)_\omega\,. \tag{4}$$

Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulation of ω into triangles such that the maximum diameter of triangles $K \in \mathcal{T}_h$ are bounded by h. Denote by $\mathcal{E}_h$ and $\mathcal{V}_h$ the sets of all edges and vertices in $\mathcal{T}_h$, respectively. Also by $\mathcal{E}_h^{(i)}, \mathcal{V}_h^{(i)}\mathcal{E}_h^{(b)}$, and $\mathcal{V}_h^{(b)}$ those in $\mathcal{T}_h$ which are in the interior of the domain and on the boundary of the domain, respectively.

Associated with $\mathbf{x}_j \in \mathcal{V}_h$, denote by φ_j the conforming P_1 basis function defined on ω such that

$$\varphi_j(\mathbf{x}_k) = \delta_{jk} \quad \forall\,\mathbf{x}_k \in \mathcal{V}_h, \quad \varphi_j\,|_K \in P_1(K) \quad \forall\, K \in \mathcal{T}_h.$$

Denote by X_h the conforming P_1 element spanned by φ_j's associated with interior vertices $\mathbf{x}_j \in \mathcal{V}_h^{(i)}$. That is,

$$X_h = \mathrm{Span}\left\{\varphi_j \;\mid\; \mathbf{x}_j \in \mathcal{V}_h^{(i)}\right\} = \mathrm{Span}\left\{\varphi_j, j = 1, \cdots, J = |\mathcal{V}_h^{(i)}|\right\},$$

where $|S|$ denotes the cardinality of set S.

The Galerkin approximation to u of (3) is to find $u_h(\mathbf{x}) = \sum_{j=1}^{J}\sum u_j\varphi_j(\mathbf{x}) \in X_h$ such that

$$A(u_h, v_h) = (f, v_h)_\omega \quad \forall v_h \in X_h. \tag{5}$$

The following Poincaré lemma is useful [13].

Lemma 2.1. *Assume that $v \in H_0^1(\omega)$. Then*

$$\|v\|_{0,\omega} \le P\|\boldsymbol{\nabla}\, v\|_{0,\omega}, \quad \textit{where } P = \sqrt{\frac{|\omega|}{\pi}}. \tag{6}$$

The following lemma is trivial, but essential.

Lemma 2.2. *The following estimates hold: for all $v \in H_0^1(\omega)$, we have*

$$\begin{aligned} A(v, v) &= \alpha\,\|\boldsymbol{\nabla}\, v\|_{0,\omega}^2\,, &\text{(7a)}\\ A(u, v) &\le (\alpha + P\|\boldsymbol{\beta}\|_\infty)\,\|\boldsymbol{\nabla}\, u\|_{0,\omega}\|\boldsymbol{\nabla}\, v\|_{0,\omega}, &\text{(7b)}\end{aligned}$$

where $\|\boldsymbol{\beta}\|_\infty := \max_{i=1,2}|\beta_i|$.

Proof. Using Green's formula, we can have

$$A(v,v) = \alpha\left(\nabla v, \nabla v\right)_\omega + \left(\boldsymbol{\beta}\cdot\nabla v, v\right)_\omega = \alpha\left(\nabla v, \nabla v\right)_\omega + \frac{1}{2}\left(\boldsymbol{\beta}, \nabla |v|^2\right)_\omega$$
$$= \alpha\left\|\nabla v\right\|_{0,\omega}^2 .$$

By Lemma 2.1, it is easy to obtain

$$\begin{aligned} A(u,v) &\le \alpha\|\nabla u\|_{0,\omega}\|\nabla v\|_{0,\omega} + \|\boldsymbol{\beta}\|_\infty \|\nabla u\|_{0,\omega}\|v\|_{0,\omega} \\ &\le \alpha\|\nabla u\|_{0,\omega}\|\nabla v\|_{0,\omega} + P\|\boldsymbol{\beta}\|_\infty \|\nabla u\|_{0,\omega}\|\nabla v\|_{0,\omega} \\ &= \left(\alpha + P\|\boldsymbol{\beta}\|_\infty\right)\|\nabla u\|_{0,\omega}\|\nabla v\|_{0,\omega}, \end{aligned}$$

where $\|\boldsymbol{\beta}\|_\infty := \max_{i=1}^2 \beta_i$. □

Lemma 2.2 ensures the existence of unique solutions for (3) and (5) by the Lax-Milgram lemma, and the stability results hold:

$$\|\nabla u\|_{0,\omega} \le \left(1 + \frac{P\|\boldsymbol{\beta}\|_\infty}{\alpha}\right)\|f\|_{-1,\omega}, \tag{8a}$$

$$\|\nabla u_h\|_{0,\omega} \le \left(1 + \frac{P\|\boldsymbol{\beta}\|_\infty}{\alpha}\right)\|f\|_{-1,\omega}. \tag{8b}$$

Next, we will propose the following stability estimate of the finite element method (5).

Besides, it can be found that (5) is satisfied when the finite element solution u_h is replaced by the exact solution u of (2a). Thus, (5) is consistent. As a result, we have the following error equation:

$$A(u-u_h, v_h) = 0 \quad \forall v_h \in X_h, \tag{9}$$

which plays an important role in estimating errors of (5).

Next, we define the interpolation operator $I_h : H_0^1(\omega) \to X_h$ such that

$$I_h u(\mathbf{x}) = \sum_{j=1}^{J} u(\mathbf{x}_j)\varphi_j(\mathbf{x}). \tag{10}$$

By the Bramble-Hilbert lemma [3, 5, 8, 20], one has for $u \in H_0^1(\omega)\cap H^2(\omega)$,

$$\left\|u - I_h u\right\|_{0,\omega} + h\left\|\nabla\ (u - I_h u)\right\|_{0,\omega} \le c_1 h^l \left\|u\right\|_{l,\omega}, \quad l = 1,2, \tag{11}$$

where c_1 is a positive constant independent of h.

We recall the following elliptic regularity result: If ω is convex or C^2 and $f \in L^2(\omega)$, then the solution $u \in H_0^1(\omega)$ of (3) belongs to $H^2(\omega)$ and the elliptic regularity holds:

$$\|u\|_{2,\omega} \le c_0\|f\|_{0,\omega}. \tag{12}$$

Also, we define the Galerkin projection operator $R_h : H_0^1(\omega) \to X_h$ such that

$$A\left(R_h u, v_h\right) = A\left(u, v_h\right) \quad \forall\, v_h \in X_h. \tag{13}$$

which satisfies the following approximation properties.

Theorem 2.1. *Assume that ω is convex or C^2 and $f \in L^2(\omega)$. Then the Galerkin projection $R_h u$ satisfies, for $l = 1, 2$,*

$$\text{(14a)} \quad \|\boldsymbol{\nabla}\ (u - R_h u)\|_{0,\omega} \leq c_1 h^{l-1}\left(2 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)\|u\|_{l,\omega}\,,$$

$$\text{(14b)} \quad \|u - R_h u\|_{0,\omega} \leq \alpha^2 c_0 c_1^2 h^l \left(1 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)\left(2 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)\|u\|_{l,\omega}\,.$$

Proof. Set $\eta_h = R_h u - I_h u$. Then, by (13), we have

$$A(\eta_h, v_h) = A(u - I_h u, v_h) \qquad \forall v_h \in X_h.$$

Taking $v_h = \eta_h$ and using (7a) and (11), we obtain

$$\begin{aligned}\alpha\|\boldsymbol{\nabla}\,\eta_h\|_{0,\omega}^2 &\leq \alpha\|\boldsymbol{\nabla}\ (u - I_h u)\|_{0,\omega}\|\boldsymbol{\nabla}\,\eta_h\|_{0,\omega} + \|\boldsymbol{\beta}\|_\infty\|u - I_h u\|_{0,\omega}\|\boldsymbol{\nabla}\,\eta_h\|_{0,\omega}\\ &\leq c_1 h^{l-1}(\alpha + \|\boldsymbol{\beta}\|_\infty h)\|\boldsymbol{\nabla}\,\eta_h\|_{0,\omega}\|u\|_{l,\omega}\,, \quad l = 1, 2,\end{aligned}$$

which implies

$$\text{(15)} \qquad \|\boldsymbol{\nabla}\,\eta_h\|_{0,\omega} \leq c_1 h^{l-1}\left(1 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)\|u\|_{l,\omega}\,, \quad l = 1, 2.$$

Writing $u - R_h u = (u - I_h u) + (I_h u - R_h u) = (u - I_h u) - \eta_h$, using the triangle inequality, (15) and (11), we get (14a).

Let $w \in X$ be the solution of

$$\text{(16)} \qquad A(w, v) = (u - R_h u, v) \qquad \forall v \in X.$$

Due to (12),

$$\text{(17)} \qquad \|w\|_{2,\omega} \leq c_0\|u - R_h u\|_{0,\omega}\,.$$

Choosing $v = u - R_h u$ in (16), and using (11) and (17), one has

$$\begin{aligned}\|u - R_h u\|_{0,\omega}^2 &= A(w, u - R_h u) = A(w - I_h w, u - R_h u)\\ &\leq \alpha\|\boldsymbol{\nabla}\ (w - I_h w)\|_{0,\omega}\|\boldsymbol{\nabla}\ (u - R_h u)\|_{0,\omega}\\ &\qquad + \|\boldsymbol{\beta}\|_\infty\|w - I_h w\|\,\|\boldsymbol{\nabla}\ (u - R_h u)\|_{0,\omega}\\ &\leq c_1(\alpha + \|\boldsymbol{\beta}\|_\infty h)h|w|_{2,\omega}\|\boldsymbol{\nabla}(u - R_h u)\|_{0,\omega}\\ &\leq c_0 c_1(\alpha + \|\boldsymbol{\beta}\|_\infty h)h\|u - R_h u\|_{0,\omega}\|\boldsymbol{\nabla}(u - R_h u)\|_{0,\omega}\,.\end{aligned}$$

By dividing the above inequalities by $\|u - R_h u\|_{0,\omega}$, one obtains

$$\|u - R_h u\|_{0,\omega} \leq c_0 c_1(\alpha + \|\boldsymbol{\beta}\|_\infty h)h\|\boldsymbol{\nabla}(u - R_h u)\|_{0,\omega}\,.$$

Then, using (14a), we deduce that

$$\begin{aligned}\|u - R_h u\|_{0,\omega} &\leq \alpha c_0 c_1 h\left(1 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)\|\boldsymbol{\nabla}\ (u - R_h u)\|_{0,\omega}\\ &\leq \alpha^2 c_0 c_1^2 h^l\left(1 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)\left(2 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)\|u\|_{l,\omega}\,, \quad l = 1, 2.\end{aligned}$$

This completes the proof of (14b). □

Then, we will explicitly establish the dependence of H_1 error bounds on the diffusivity α, the modulus of the flow field $\|\boldsymbol{\beta}\|_\infty$, and the mesh size h.

Theorem 2.2. *Assume that* $u \in H_0^1(\omega) \cap H^2(\omega)$ *is the solution of* (2a) *and* $u_h \in X_h \subset H_0^1(\omega)$ *is the finite element solution of* (5). *Then there exists a constant* $c_1 > 0$ *(see* (11)*) such that*

$$\|\nabla (u - u_h)\|_{0,\omega} \le c_1 h \left(2 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right) \|u\|_{2,\omega}. \tag{18}$$

Proof. By (9) and (11), we have

$$\begin{aligned}
&\alpha \|\nabla\ (u_h - I_h u)\|_{0,\omega}^2 = A(u_h - I_h u, u_h - I_h u) \\
&=A(u - I_h u, u_h - I_h u) + A(u_h - u, u_h - I_h u) = A(u - I_h u, u_h - I_h u) \\
&\le \alpha \|\nabla\ (u - I_h u)\|_{0,\omega} \|\nabla\ (u_h - I_h u)\|_{0,\omega} + \|\boldsymbol{\beta}\|_\infty \|u - I_h u\|_{0,\omega} \|\nabla\ (u_h - I_h u)\|_{0,\omega},
\end{aligned}$$

which implies

$$\|\nabla\ (u_h - I_h u)\|_{0,\omega} \le c_1 h \left(1 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right) \|u\|_{2,\omega}. \tag{19}$$

Finally, combining (19) and (11), and using the triangle inequality

$$\|\nabla\ (u - u_h)\|_{0,\omega} \le \|\nabla\ (u - I_h u)\|_{0,\omega} + \|\nabla\ (u_h - I_h u)\|_{0,\omega},$$

one completes the proof. □

Theorem 2.2 indicates that the behavior of error in the H_1 norm is determined by the diffusivity α, the modulus of the convection field $\|\boldsymbol{\beta}\|_\infty$ and the mesh size h.

3. Finite difference discretization in the direction of z for the 3D con -vection-diffusion problems

In this section, we show the finite difference method in the direction of z for the Dirichlet boundary problem of the 3D steady convection-diffusion equation (1). Here and hereafter, any function $v(\mathbf{x}, z)$ is noted as $v(z)$.

We consider the finite difference discretization of (1a)-(1b) in the z direction. For a positive integer K, let $\tau = (b_3 - a_3)/K$, and set $z_k = a_3 + k\tau, k = -1, 0, \cdots, K+1$.

Then, we define the piecewise linear basis functions, for $k = 0, \cdots, K$,

$$\psi_k(z) = \left[\frac{z - z_{k-1}}{\tau} \chi_{[z_{k-1}, z_k)}(z) + \frac{z_{k+1} - z}{\tau} \chi_{[z_k, z_{k+1})}(z)\right] \chi_{[a_3, b_3]}(z).$$

Then define a semidiscrete subspace $\mathcal{X}_\tau \subset H_0^1(\Omega)$ as follows:

$$\mathcal{X}_\tau = \left\{ v_\tau = \sum_{k=1}^{K-1} v^k \psi_k = \sum_{k=0}^{K} v^k \psi_k \ \middle|\ v^k \in H_0^1(\omega), k = 0, \ldots, K \right\},$$

with the discrete L_2-inner product and H_1-inner product with mass lumping defined by, assuming $u^0 = u^K = v^0 = v^K = 0$,

$$(u_\tau, v_\tau)_{L_\tau^2} = \sum_{k=1}^{K-1} \tau \left(u^k, v^k\right)_\omega, \tag{20a}$$

$$\begin{aligned}
(u_\tau, v_\tau)_{H_\tau^1} &= \left(\widehat{\nabla}\, u_\tau, \widehat{\nabla}\, v_\tau\right)_{L_\tau^2} = (\nabla\, u_\tau, \nabla\, v_\tau)_{L_\tau^2} + (\partial_z u_\tau, \partial_z v_\tau)_{L^2(\Omega)}, \\
&= \sum_{k=1}^{K-1} \tau \left(\nabla\, u^k, \nabla\, v^k\right)_\omega + \sum_{k=1}^{K} \tau \left(\mathrm{d}_z u^k, \mathrm{d}_z v^k\right)_\omega,
\end{aligned} \tag{20b}$$

where $\mathrm{d}_z u^k = \frac{u^k - u^{k-1}}{\tau} = \frac{\partial u_\tau}{\partial z}\,|_{(z_{k-1},z_k)}$ denotes the backward difference. Also, define the following notations $\mathrm{d}_z^+ u^k$ and $\mathrm{d}^* u^k$ for the forward and central differences to $\partial_z u^k$ and $\mathrm{d}_{zz} u^k$ for the central difference to $\partial_{zz} u^k$, for $u_\tau = \sum_{k=1}^{K-1} u^k \psi_k \in \mathcal{X}_\tau$,

$$\mathrm{d}_z^+ u^{k-1} = \mathrm{d}_z u^k,\ \mathrm{d}_z^* u^k = \frac{\mathrm{d}_z^+ + \mathrm{d}_z}{2} u^k,\ \mathrm{d}_{zz} u^k = \mathrm{d}_z^+ \mathrm{d}_z u^k,$$
$$\widehat{\Delta}_\tau = \partial_{xx} + \partial_{yy} + \mathrm{d}_{zz},\ \widehat{\boldsymbol{\nabla}}_\tau = (\partial_x, \partial_y, \mathrm{d}_z^*),$$

so that $\mathrm{d}_z u_\tau = \sum_{k=1}^{K} \mathrm{d}_z u^k \psi_k$, $\mathrm{d}_z^+ u_\tau = \sum_{k=0}^{K-1} \mathrm{d}_z^+ u^k \psi_k$, $\mathrm{d}_z^* u_\tau = \sum_{k=1}^{K-1} \mathrm{d}_z^* u^k \psi_k$, and $\mathrm{d}_{zz} u_\tau = \sum_{k=1}^{K-1} \mathrm{d}_{zz} u^k \psi_k \in \mathcal{X}_\tau$.

The following discrete Green formula is useful: if X is an inner product space, then

$$-\sum_{i=1}^{l-1} (a^{i+1} - a^i, b^i) = \sum_{i=1}^{l} (a^i, b^i - b^{i-1}), \tag{21}$$

provided $a^i, b^i \in X$ for $i = 1, \cdots, l$ with $(a^l, b^l) = (a^1, b^0) = 0$.

The following lemma establishes the inequality relationship about the discrete L_2 norm and H_1 norm.

Lemma 3.1. *The following inequalities hold:*

$$\|v_\tau\|_{L^2_\tau} \le P \|\boldsymbol{\nabla}\, v_\tau\|_{L^2_\tau}, \quad \textit{where } P = \sqrt{\frac{|\omega|}{\pi}}, \tag{22}$$

$$\|v_\tau\|_{L^2_\tau} \le (b_3 - a_3) \|\partial_z v_\tau\|_{L^2_\tau}, \tag{23}$$

and

$$\|v_\tau\|_{L^2_\tau} \le M \|v_\tau\|_{H^1_\tau}, \ \textit{ where } M = \sqrt{\frac{P^2 + (b_3 - a_3)^2}{2}}. \tag{24}$$

Proof. The inequality (22) is obtained from (6) and the definition of v_τ. For (23), recalling that $v^0 = 0$ and $v^K = 0$, we deduce

$$\begin{aligned}
2\|v_\tau\|_{L^2_\tau}^2 &= 2\sum_{k=0}^{K} \tau \|v^k\|_{0,\omega}^2 = \sum_{k=1}^{K}\sum_{i=1}^{k} \tau\big(\|v^i\|_{0,\omega}^2 - \|v^{i-1}\|_{0,\omega}^2\big) \\
&\quad + \sum_{k=1}^{K}\sum_{i=k+1}^{K} \tau\big(\|v^{i-1}\|_{0,\omega}^2 - \|v^i\|_{0,\omega}^2\big) \\
&\le \sum_{k=1}^{K} \tau \left(\sum_{i=1}^{K} \tau \|v^i + v^{i-1}\|^2{}_{0,\omega}\right)^{\frac12} \left(\sum_{i=1}^{K} \tau \|\mathrm{d}_z v^i\|^2{}_{0,\omega}\right)^{\frac12} \\
&\le 2(b_3 - a_3) \left(\sum_{k=1}^{K} \tau \|v^k\|^2{}_{0,\omega}\right)^{\frac12} \left(\sum_{k=1}^{K} \tau \|\mathrm{d}_z v^k\|^2{}_{0,\omega}\right)^{\frac12} \\
&= 2(b_3 - a_3) \|v_\tau\|_{L^2_\tau} \|\partial_z v_\tau\|_{L^2_\tau},
\end{aligned}$$

which implies (23). A combination of (22) and (23) yields (24). □

The following Lemma follows by a direct calculation:

Lemma 3.2. *For any $v_\tau \in \mathcal{X}_\tau$, the following hold*

$$\frac{1}{4}\left\|v_\tau\right\|_{L^2_\tau}^2 \le \left\|v_\tau\right\|_{0,\Omega}^2 \le \left\|v_\tau\right\|_{L^2_\tau}^2,$$

and

$$\frac{1}{4}\left\|v_\tau\right\|_{H^1_\tau}^2 \le \left\|\widehat{\boldsymbol{\nabla}}\, v_\tau\right\|_{0,\Omega}^2 \le \left\|v_\tau\right\|_{H^1_\tau}^2.$$

Next, define $u_\tau \in \mathcal{X}_\tau$ as the finite difference solution given by

$$u_\tau(\mathbf{x}, z) = \sum_{k=1}^{K-1} u^k(\mathbf{x})\psi_k(z),$$

where $u^k = u^k(\mathbf{x}) \in H^1_0(\omega)$ fulfills

$$-\alpha\widehat{\Delta}_\tau u^k + \widehat{\boldsymbol{\beta}}\cdot\widehat{\boldsymbol{\nabla}}_\tau u^k = \overline{f}(\cdot, z_k) := \frac{1}{\tau}\int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} f(\cdot, z_k)\mathrm{d}z, \quad k = 1, \ldots, K-1. \tag{25}$$

Set

$$f_\tau = \sum_{k=1}^{K-1} \overline{f}(z_k)\psi_k(z).$$

By Green's formula, we obtain the weak formulation of (25): find $u_\tau \in \mathcal{X}_\tau$, such that

$$B\left(u_\tau, v_\tau\right) = \left(f_\tau, v_\tau\right)_{L^2_\tau} \quad \forall v_\tau \in \mathcal{X}_\tau, \tag{26}$$

where the bilinear form $B_\tau(\cdot,\cdot) : \mathcal{X}_\tau \times \mathcal{X}_\tau \to \mathbb{R}$ is defined by

$$\begin{aligned} B_\tau(u_\tau, v_\tau) := {} & \alpha\left(\boldsymbol{\nabla}\, u_\tau, \boldsymbol{\nabla}\, v_\tau\right)_{L^2_\tau} + \left(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\, u_\tau, v_\tau\right)_{L^2_\tau} - \alpha\left(\mathrm{d}_{zz}u_\tau, v_\tau\right)_{L^2_\tau} \\ & + \beta_3\left(\mathrm{d}^*_z u_\tau, v_\tau\right)_{L^2_\tau} \qquad \forall\, u_\tau, v_\tau \in \mathcal{X}_\tau, \end{aligned} \tag{27}$$

invoked with the definition of the inner product $(\cdot,\cdot)_{L^2_\tau}$ on $\mathcal{X}_\tau$ defined in (20a).

Now, we state and prove the following stability estimate of the finite difference method based on the conforming P_1 element (26):

Theorem 3.1. *The following stability estimates hold: for all $u_\tau, v_\tau \in \mathcal{X}_\tau$, we have*

$$B_\tau(v_\tau, v_\tau) = \alpha\left\|v_\tau\right\|_{H^1_\tau}^2, \tag{28a}$$

$$B_\tau(u_\tau, v_\tau) \le \left(\alpha + M\left\|\widehat{\boldsymbol{\beta}}\right\|_\infty\right)\left\|u_\tau\right\|_{H^1_\tau}\left\|v_\tau\right\|_{H^1_\tau}, \tag{28b}$$

where M is given as in (24).

Proof. For all $u_\tau, v_\tau \in \mathcal{X}_\tau$, we begin with putting

$$\begin{aligned} B_\tau(u_\tau, v_\tau) &= \left[\alpha\left(\boldsymbol{\nabla}\, u_\tau, \boldsymbol{\nabla}\, v_\tau\right)_{L^2_\tau} + \left(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\, u_\tau, v_\tau\right)_{L^2_\tau}\right] \\ &\qquad - \alpha\left(\mathrm{d}_{zz}u_\tau, v_\tau\right)_{L^2_\tau} + \beta_3\left(\mathrm{d}^*_z u_\tau, v_\tau\right)_{L^2_\tau} \\ &:= I_1(u_\tau, v_\tau) + I_2(u_\tau, v_\tau) + I_3(u_\tau, v_\tau). \end{aligned}$$

We estimate each term I_j as follows. Thanks to Lemma 2.2, we have

$$\begin{aligned} |I_1(u_\tau, v_\tau)| &= \left|\sum_{k=1}^{K-1}\tau A(u^k, v^k)\right| \le \sum_{k=1}^{K-1}\tau\left(\alpha + P\|\boldsymbol{\beta}\|_\infty\right)\|\boldsymbol{\nabla}\, u^k\|_{0,\omega}\|\boldsymbol{\nabla}\, v^k\|_{0,\omega} \\ &= \left(\alpha + P\|\boldsymbol{\beta}\|_\infty\right)\left\|\boldsymbol{\nabla}\, u_\tau\right\|_{L^2_\tau}\left\|\boldsymbol{\nabla}\, v_\tau\right\|_{L^2_\tau}. \end{aligned}$$

Using the discrete Green's formula (21) and $v^0 = v^K = 0$, we deduce

$$\begin{aligned}|I_2(u_\tau, v_\tau)| &= \left|(-\alpha \mathrm{d}_{zz} u_\tau, v_\tau)_{L^2_\tau}\right| \le \alpha \left|\sum_{k=1}^{K-1} \left(\mathrm{d}_z u^{k+1} - \mathrm{d}_z u^k, v^k\right)_\omega\right| \\ &= \alpha \left|\sum_{k=1}^{K} \tau \left(\mathrm{d}_z u^k, \mathrm{d}_z v^k\right)_\omega\right| \le \alpha \left\|\mathrm{d}_z u_\tau\right\|_{L^2_\tau} \left\|\mathrm{d}_z v_\tau\right\|_{L^2_\tau},\end{aligned}$$

and

$$\begin{aligned}|I_3(u_\tau, v_\tau)| &= \left|(\beta_3 \mathrm{d}^*_z u_\tau, v_\tau)_{L^2_\tau}\right| \\ &= \left|\frac{\beta_3}{2}\sum_{k=1}^{K-1}\left(u^{k+1} - u^k, v^k\right)_\omega + \frac{\beta_3}{2}\sum_{k=1}^{K-1}\left(u^k - u^{k-1}, v^k\right)_\omega\right| \\ &\le |\beta_3| \left|\left(\sum_{k=1}^{K} \tau \|\mathrm{d}_z u^k\|^2_{0,\omega}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{K-1} \tau \|v^k\|^2_{0,\omega}\right)^{\frac{1}{2}}\right| \\ &= |\beta_3| \|\mathrm{d}_z u_\tau\|_{L^2_\tau} \|v_\tau\|_{L^2_\tau} \\ &\le |\beta_3| (b_3 - a_3) \|\mathrm{d}_z u_\tau\|_{L^2_\tau} \|\mathrm{d}_z v_\tau\|_{L^2_\tau}.\end{aligned}$$

Summing the above estimates for $|I_1(u_\tau, v_\tau)|$, $|I_2(u_\tau, v_\tau)|$ and $|I_3(u_\tau, v_\tau)|$, we see that (28b) holds. Choosing $u_\tau = v_\tau$, we get

$$|I_1(v_\tau, v_\tau)| = \sum_{k=1}^{K-1} \tau |A(v^k, v^k)| = \alpha \left\|\boldsymbol{\nabla}\, v_\tau\right\|^2_{L^2_\tau},$$

and

$$|I_2(v_\tau, v_\tau)| = \alpha \sum_{k=1}^{K-1} \tau \left(\mathrm{d}_z v^k, \mathrm{d}_z v^k\right)_\omega = \alpha \left\|\mathrm{d}_z v_\tau\right\|^2_{L^2_\tau},$$

and obviously we get

$$I_3(v_\tau, v_\tau) = 0.$$

Summing the equalities for $|I_1(v_\tau, v_\tau)|$, $|I_2(v_\tau, v_\tau)|$ and $|I_3(v_\tau, v_\tau)|$, we see that (28b) holds. This complete the proof. $\square$

Based on Theorem 3.1, we can ensure the unique solvability of (26). Now, we proceed to estimate the errors of the finite difference solution u_τ of (26).

Define the interpolation operator $I_\tau : H^1_0(\Omega) \to \mathcal{X}_\tau$ by

$$I_\tau v(\mathbf{x}, z) = \sum_{k=1}^{K-1} v(\mathbf{x}, z_k) \psi_k(z) \quad \forall\, v \in H^1_0(\Omega). \tag{29}$$

Similarly to (11), the following interpolation estimates hold.

Theorem 3.2. *If $v \in H^1_0(\Omega) \cap H^2(\Omega)$, the interpolnt $I_\tau v$ defined by (29) satisfies*

$$\|v - I_\tau v\|_{0,\Omega} + \tau \left\|\partial_z \left(v - I_\tau v\right)\right\|_{0,\Omega} \le c_2 \tau^2 \left\|\partial_{zz} v\right\|_{0,\Omega}, \tag{30a}$$

$$\left\|\widehat{\boldsymbol{\nabla}}\, (v - I_\tau v)\right\|_{0,\Omega} \le c_3 \tau \left\|v\right\|_{2,\Omega}, \tag{30b}$$

where c_2 and c_3 are positive constants independent of mesh size τ.

Then, applying the integral operator $\frac{1}{\tau}\int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} \cdot \mathrm{d}z$ to (1a) we obtain

$$-\alpha\Delta u(z_k) + \boldsymbol{\beta}\cdot\boldsymbol{\nabla}\, u(z_k) - \alpha \mathrm{d}_{zz}u(z_k) + \beta_3 \mathrm{d}_z^* u(z_k) = \overline{f}(z_k) + E^k, \tag{31}$$

for $k = 1, \dots, K-1$, where

$$\begin{aligned} E^k =& -\alpha\Delta\left(u(z_k) - \overline{u}(z_k)\right) + \boldsymbol{\beta}\cdot\boldsymbol{\nabla}\,\left(u(z_k) - \overline{u}(z_k)\right) \\ & - \alpha\left(\mathrm{d}_{zz}u(z_k) - \frac{\partial_z u(z_{k+\frac{1}{2}}) - \partial_z u(z_{k-\frac{1}{2}})}{\tau}\right) \\ & + \beta_3\left(\mathrm{d}_z^* u(z_k) - \frac{u(z_{k+\frac{1}{2}}) - u(z_{k-\frac{1}{2}})}{\tau}\right). \end{aligned}$$

We then set

$$E_\tau = \sum_{k=1}^{K-1} E^k \psi_k(z), \tag{32}$$

and

$$e_\tau = \sum_{k=1}^{K-1} e^k \psi_k(z), \text{ where } e^k = u(z_k) - u^k. \tag{33}$$

It is obvious that

$$B_\tau(I_\tau u, v_\tau) = (f_\tau, v_\tau)_{L^2_\tau} + (E_\tau, v_\tau)_{L^2_\tau}, \quad \forall\, v_\tau \in \mathcal{X}_\tau \tag{34}$$

and

$$B_\tau(e_\tau, v_\tau) = (E_\tau, v_\tau)_{L^2_\tau}, \quad \forall\, v_\tau \in \mathcal{X}_\tau. \tag{35}$$

Now, we are ready to state and prove error estimates of u_τ with respect to u.

Theorem 3.3. *Assume that $u \in H^1_0(\Omega) \cap H^2(\Omega)$ and $u_\tau \in \mathcal{X}_\tau$ are the solution of (1a)-(1b) and the finite difference solution defined by (26), respectively. Then the following error estimates hold:*

$$\begin{aligned} \|\boldsymbol{\nabla}\,(u - u_\tau)\|_{L^2_\tau} &\le \tau\left(c_3 + \frac{\sqrt{3} + 2\sqrt{3}\|\widehat{\boldsymbol{\beta}}\|_\infty M}{6\alpha}\right)\|u\|_{2,\Omega}, \\ \|\partial_z(u - u_\tau)\|_{L^2_\tau} &\le \tau\left(c_3 + \frac{\sqrt{3} + 2\sqrt{6}\|\widehat{\boldsymbol{\beta}}\|_\infty M}{6\alpha}\right)\|u\|_{2,\Omega}, \end{aligned}$$

and

$$\|u - u_\tau\|_{H^1_\tau} \le \tau\left(c_3 + \frac{\sqrt{3} + 2\sqrt{6}\|\widehat{\boldsymbol{\beta}}\|_\infty M}{6\alpha}\right)\|u\|_{2,\Omega}.$$

Proof. We take $v_\tau = e_\tau$ in (35) so that

$$
\begin{aligned}
B_\tau(e_\tau, e_\tau) = (E_\tau, e_\tau)_{L^2_\tau} &= \sum_{k=1}^{K-1} \tau \left(E^k, e^k\right)_\omega \\
&= \alpha \sum_{k=1}^{K-1} \tau \left(\nabla \left(u\left(z_k\right) - \frac{1}{\tau} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} u \mathrm{d}z \right), \nabla e^k \right)_\omega \\
&\quad + \alpha \sum_{k=1}^{K-1} \tau \left(\mathrm{d}_z u\left(z_k\right) - \partial_z u\left(z_{k-\frac{1}{2}}\right), \mathrm{d}_z e^k \right)_\omega \\
&\quad + \sum_{k=0}^{K-1} \tau \left(\boldsymbol{\beta} \cdot \nabla \left(u\left(z_k\right) - \frac{1}{\tau} \int_{z_{k-\frac{1}{2}}}^{k+\frac{1}{2}} u \mathrm{d}z \right), e^k \right)_\omega \\
&\quad + \beta_3 \sum_{k=1}^{K-1} \tau \left(\mathrm{d}_z^* u\left(z_k\right) - \partial_z u\left(z_k\right), e^k \right)_\omega \\
&:= I + II + III + IV.
\end{aligned}
\tag{36}
$$

Let us bound the four terms I - IV in turn.

For this, using integration by parts, Young's inequality and the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
|I| =& \alpha \left| \sum_{k=1}^{K-1} \tau \left(\nabla \left(u\left(z_k\right) - \frac{1}{\tau} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} u \mathrm{d}z \right), \nabla e^k \right)_\omega \right| \\
=& \alpha \left| \sum_{k=1}^{K-1} \tau \left(\frac{1}{\tau} \int_{z_k}^{z_{k+\frac{1}{2}}} \left(z - z_{k+\frac{1}{2}} \right) \partial_z \nabla u(z) \mathrm{d}z \right. \right. \\
& \left. \left. + \frac{1}{\tau} \int_{z_{k-\frac{1}{2}}}^{z_k} \left(z - z_{k-\frac{1}{2}} \right) \partial_z \nabla u(z) \mathrm{d}z, \nabla e^k \right)_\omega \right| \\
\leq& \alpha \sum_{k=1}^{K-1} \tau \left(\frac{\varepsilon_1 \tau}{24} \left\| \left(\int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} \left(\partial_z \nabla u(z) \right)^2 \mathrm{d}z \right)^{\frac{1}{2}} \right\|_{0,\omega}^2 + \frac{\left\| \nabla e^k \right\|_{0,\omega}^2}{4\varepsilon_1} \right) \\
\leq& \frac{\alpha \varepsilon_1 \tau^2}{24} \left\| \partial_z \nabla u \right\|_{0,\Omega}^2 + \frac{\alpha}{4\varepsilon_1} \left\| \nabla e_\tau \right\|_{L^2_\tau}^2 .
\end{aligned}
$$

Next, recalling (6),

$$
\begin{aligned}
|II| =& \left| \sum_{k=1}^{K-1} \tau \left(\boldsymbol{\beta} \cdot \nabla \left(u\left(z_k\right) - \frac{1}{\tau} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} u \mathrm{d}z \right), e^k \right)_\omega \right| \\
\leq& \|\boldsymbol{\beta}\|_\infty \left| \sum_{k=1}^{K-1} \tau \left(\frac{1}{\tau} \int_{z_k}^{z_{k+\frac{1}{2}}} \left(z - z_{k+\frac{1}{2}} \right) |\partial_z \nabla u| \mathrm{d}z \right. \right. \\
& \left. \left. + \frac{1}{\tau} \int_{z_{k-\frac{1}{2}}}^{z_k} \left(z - z_{k-\frac{1}{2}} \right) |\partial_z \nabla u| \mathrm{d}z, e^k \right)_\omega \right|
\end{aligned}
$$

$$
\begin{aligned}
\leq & \|\boldsymbol{\beta}\|_{\infty} \sum_{k=1}^{K-1} \tau\left(\frac{\varepsilon_{2} \tau}{24}\left\|\int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}}\left|\partial_{z} \boldsymbol{\nabla} u\right|^{2} \mathrm{d} z\right\|_{0, \omega}^{2}+\frac{\left\|e^{k}\right\|_{0, \omega}^{2}}{4 \varepsilon_{2}}\right) \\
\leq & \frac{\|\boldsymbol{\beta}\|_{\infty} \varepsilon_{2} \tau^{2}}{24}\left\|\partial_{z} \boldsymbol{\nabla} u\right\|_{0, \Omega}^{2}+\frac{\|\boldsymbol{\beta}\|_{\infty} P^{2}}{4 \varepsilon_{2}}\left\|\boldsymbol{\nabla} e_{\tau}\right\|_{L_{\tau}^{2}}^{2} .
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
|III| \leq & \alpha\left|\sum_{k=1}^{K} \tau\left(\mathrm{d}_{z} u\left(z_{k}\right)-\partial_{z} u\left(z_{k-\frac{1}{2}}\right), \mathrm{d}_{z} e^{k}\right)_{\omega}\right| \\
= & \alpha\left|\sum_{k=1}^{K} \tau\left(\frac{1}{\tau} \int_{z_{k-\frac{1}{2}}}^{z_{k}}\left(z-z_{k}\right) \partial_{z z} u(z) \mathrm{d} z\right.\right. \\
& \left.\left.+\frac{1}{\tau} \int_{z_{k-1}}^{z_{k-\frac{1}{2}}}\left(z-z_{k-1}\right) \partial_{z z} u(z) \mathrm{d} z, \mathrm{d}_{z} e^{k}\right)_{\omega}\right| \\
\leq & \alpha \sum_{k=1}^{K} \tau\left(\frac{\varepsilon_{3} \tau}{24}\left\|\left(\int_{z_{k-1}}^{z_{k}} \partial_{z z}^{2} u(z) \mathrm{d} z\right)^{\frac{1}{2}}\right\|_{0, \omega}^{2}+\frac{\left\|\mathrm{d}_{z} e^{k}\right\|_{0, \omega}^{2}}{4 \varepsilon_{3}}\right) \\
\leq & \frac{\alpha \varepsilon_{3} \tau^{2}}{24}\left\|\partial_{z z} u\right\|_{0, \Omega}^{2}+\frac{\alpha}{4 \varepsilon_{3}}\left\|\partial_{z} e_{\tau}\right\|_{L_{\tau}^{2}}^{2} .
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
|IV| = & \left|\beta_{3}\right|\left|\sum_{k=1}^{K-1} \tau\left(\mathrm{d}_{z}^{*} u\left(z_{k}\right)-\partial_{z} u\left(z_{k}\right), e^{k}\right)_{\omega}\right| \\
= & \frac{\left|\beta_{3}\right|}{2}\left|\sum_{k=1}^{K-1} \tau\left(\frac{1}{\tau} \int_{z_{k}}^{z_{k+1}}\left(z_{k+1}-z\right) \partial_{z z} u \mathrm{d} z, e^{k}\right)_{\omega}\right. \\
& \left.+\sum_{k=1}^{K-1} \tau\left(\frac{1}{\tau} \int_{z_{k-1}}^{z_{k}}\left(z-z_{k-1}\right) \partial_{z z} u \mathrm{d} z, e^{k}\right)_{\omega}\right| \\
\leq & \left|\beta_{3}\right| \sum_{k=1}^{K} \tau\left(\frac{\varepsilon_{4} \tau}{3}\left\|\left(\int_{z_{k-1}}^{z_{k}} \partial_{z z}^{2} u(z) \mathrm{d} z\right)^{\frac{1}{2}}\right\|_{0, \omega}^{2}+\frac{\left\|e^{k}\right\|_{0, \omega}^{2}}{4 \varepsilon_{4}}\right) \\
\leq & \frac{\left|\beta_{3}\right| \varepsilon_{4} \tau^{2}}{3}\left\|\partial_{z z} u\right\|_{0, \Omega}^{2}+\frac{\left|\beta_{3}\right|\left(b_{3}-a_{3}\right)}{4 \varepsilon_{4}}\left\|\partial_{z} e_{\tau}\right\|_{L_{\tau}^{2}}^{2} .
\end{aligned}
$$

Taking $\varepsilon_1 = 1$, $\varepsilon_2 = \frac{\|\boldsymbol{\beta}\|_\infty P^2}{\alpha}$, $\varepsilon_3 = 2$, $\varepsilon_4 = \frac{|\boldsymbol{\beta}_3|(b_3-a_3)^2}{2\alpha}$, and using (26) we can obtain

$$
\begin{aligned}
\left\|\boldsymbol{\nabla} e_{\tau}\right\|_{L_{\tau}^{2}}^{2} & \leq \tau^{2}\left(\frac{\alpha^{2}+\|\boldsymbol{\beta}\|_{\infty} P^{2}}{12 \alpha^{2}}\left\|\partial_{z} \boldsymbol{\nabla} u\right\|_{0, \Omega}^{2}+\frac{\alpha^{2}+8\left|\boldsymbol{\beta}_{3}\right|^{2}\left(b_{3}-a_{3}\right)^{2}}{24 \alpha^{2}}\left\|\partial_{z z} u\right\|_{0, \Omega}^{2}\right) \\
& \leq \tau^{2}\left(\frac{\alpha^{2}+4\|\widehat{\boldsymbol{\beta}}\|_{\infty} M^{2}}{12 \alpha^{2}}\right)\|u\|_{2, \Omega}^{2},
\end{aligned}
$$

which indicates

$$\|\nabla e_\tau\|_{L^2_\tau} \le \tau \left(\frac{\sqrt{3} + 2\sqrt{3}\|\widehat{\beta}\|_\infty M}{6\alpha} \right) \|u\|_{2,\Omega}.$$

If we choose $\varepsilon_1 = \frac{1}{2}$, $\varepsilon_2 = \frac{\|\beta\|_\infty P^2}{2\alpha}$, $\varepsilon_3 = 1$, $\varepsilon_4 = \frac{|\beta_3|(b_3-a_3)^2}{\alpha}$, and use (26), we can obtain

$$\begin{aligned}\|\partial_z e_\tau\|^2_{L^2_\tau} &\le \tau^2 \left(\frac{\alpha^2 + \|\beta\|_\infty P^2}{24\alpha^2} \|\partial_z \nabla u\|^2_{0,\Omega} + \frac{\alpha^2 + 8|\beta_3|^2 (b_3 - a_3)^2}{12\alpha^2} \|\partial_{zz} u\|^2_{0,\Omega} \right) \\ &\le \tau^2 \left(\frac{\alpha^2 + 8\|\widehat{\beta}\|_\infty M^2}{12\alpha^2} \right) \|u\|^2_{2,\Omega},\end{aligned}$$

which indicates

$$\|\partial_z e_\tau\|_{L^2_\tau} \le \tau \left(\frac{\sqrt{3} + 2\sqrt{6}\|\widehat{\beta}\|_\infty M}{6\alpha} \right) \|u\|_{2,\Omega}.$$

Choosing $\varepsilon_1 = 1$, $\varepsilon_2 = \frac{\|\beta\|_\infty P^2}{\alpha}$, $\varepsilon_3 = 1$, $\varepsilon_4 = \frac{|\beta_3|(b_3-a_3)^2}{\alpha}$, and using (26), we get

$$\|e_\tau\|_{H^1_\tau} \le \tau \left(\frac{\sqrt{3} + 2\sqrt{6}\|\widehat{\beta}\|_\infty M}{6\alpha} \right) \|u\|_{2,\Omega}.$$

By using the triangle inequality and Theorem 3.1, the conclusions are drawn. □

4. Difference finite element method for the 3D convection diffusion equation

In this section, we define the fully-discrete difference finite element(DFE) method based on the $P_1 \times P_1$-conforming element for 3D convection-diffusion equations.

We define the difference finite solution for the 3D convection-diffusion equation as follows:

$$u_{h\tau}(\mathbf{x}, z) = \sum_{k=1}^{K-1} u_h^k(\mathbf{x}) \psi_k(z) \in \mathcal{X}_{h\tau}, \tag{37}$$

where $\mathcal{X}_{h\tau} = \left\{ v_{h\tau} = \sum_{k=0}^{K} v_h^k(\mathbf{x}) \psi_k(z) \middle| v_h^k \in X_h, v_h^0 = v_h^K = 0 \right\}$. We rewrite $u_{h\tau}$ defined in (37) in the form

$$u_{h\tau}(\mathbf{x}, z) = \sum_{k=1}^{K-1} \sum_{j=1}^{J} u_j^k \phi_j(\mathbf{x}) \psi_k(z). \tag{38}$$

Thus, the fully-discrete DFE solution we seek is $u_h^k \in X_h, k = 1, \cdots, K-1$, such that

(39)

$$\alpha \left(\nabla u_h^k, \nabla v_h^k \right)_\omega + \left(\beta \cdot \nabla u_h^k, v_h^k \right)_\omega - \alpha \left(\mathrm{d}_{zz} u_h^k, v_h^k \right)_\omega + \left(\beta_3 \mathrm{d}_z^* u_h^k, v_h^k \right)_\omega = \left(\overline{f}(z_k), v_h^k \right)_\omega$$
$$\forall\, v_h^k \in X_h,\ k = 1, \ldots, K-1.$$

Multiplying (39) by τ and summing for $k = 1, \ldots, K-1$, one can have an equivalent form as follows:

$$B_\tau(u_{h\tau}, v_{h\tau}) = (f_\tau, v_{h\tau})_{L^2_\tau} \quad \forall v_{h\tau} \in \mathcal{X}_{h\tau}, \tag{40}$$

where the bilinear form $B_\tau(\cdot, \cdot)$ is defined by (27).

For clarity of presentation, we recall the set of finite element basis functions in ω is

$$\phi_1(\mathbf{x}), \ldots, \phi_j(\mathbf{x}), \ldots, \phi_J(\mathbf{x}).$$

Noticing that $u_h^0 = u_h^K = 0$, (40) can be easily recast in the form of a linear system: to find the coefficients u_j^k in (38) for $j = 1, \ldots, J$ and $k = 1, \ldots, K-1$, fulfilling (39). We thus formulate this as in a linear system:

$$\mathbf{A}\mathbf{U} = \mathbf{F}, \tag{41}$$

where

$$\begin{aligned} \mathbf{U} &= \left(U^1, \ldots, U^{K-1}\right)^\top, \quad U^k = \left(u_1^k, \ldots, u_J^k\right)^\top, \\ \mathbf{F} &= \left(F^1, \ldots, F^{K-1}\right)^\top, \quad F^k = \left((f(z_k), \phi_1)_\omega, \ldots, (f(z_k), \phi_J)_\omega\right)^\top. \end{aligned}$$

In more precisely, we have

$$\mathbf{A} = \alpha \mathbf{A}_L + \mathbf{A}_{\widehat{\boldsymbol{\beta}}}, \tag{42}$$

where $\mathbf{A}_L$ and $\mathbf{A}_{\widehat{\boldsymbol{\beta}}}$ are two $(K-1) \times (K-1)$ matrices given by (see also [14])

$$\mathbf{A}_L = \begin{pmatrix} R_L + \frac{2}{\tau^2} R & -\frac{1}{\tau^2} R & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{\tau^2} R & R_L + \frac{2}{\tau^2} R & -\frac{1}{\tau^2} R & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{\tau^2} R & R_L + \frac{2}{\tau^2} R & -\frac{1}{\tau^2} R & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\tau^2} R & R_L + \frac{2}{\tau^2} R & -\frac{1}{\tau^2} R \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\tau^2} R & R_L + \frac{2}{\tau^2} R \end{pmatrix},$$

$$R_L = (a_{lm})_{J \times J}, \quad a_{lm} = (\boldsymbol{\nabla} \phi_m, \boldsymbol{\nabla} \phi_l)_\omega, R = (b_{lm})_{J \times J}, b_{lm} = (\phi_m, \phi_l)_\omega,$$

and

$$\mathbf{A}_{\widehat{\boldsymbol{\beta}}} = \begin{pmatrix} R_{\boldsymbol{\beta}} & \frac{\beta_3}{2\tau} R & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{\beta_3}{2\tau} R & R_{\boldsymbol{\beta}} & \frac{\beta_3}{2\tau} R & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{\beta_3}{2\tau} R & R_{\boldsymbol{\beta}} & \frac{\beta_3}{2\tau} R & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -\frac{\beta_3}{2\tau} R & R_{\boldsymbol{\beta}} & \frac{\beta_3}{2\tau} R \\ 0 & 0 & 0 & \cdot & \cdots & \cdot & -\frac{\beta_3}{2\tau} R & R_{\boldsymbol{\beta}} \end{pmatrix},$$

with

$$R_{\boldsymbol{\beta}} = (c_{lm})_{J \times J}, \quad c_{lm} = (\boldsymbol{\beta} \cdot \boldsymbol{\nabla} \phi_m, \phi_l)_\omega,$$

Now, we proceed to obtain error estimates of the DFE solution $u_{h\tau}$ of (39)). For this, we will assume that the data is regular enough: that is, assume that ω is convex or C^2 and $f \in L^2(\omega)$ so that the solution $u \in H_0^1(\omega)$ of (3) belongs to $H^2(\omega)$.

Lemma 4.1. *Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of* (3)*. Then, $R_h I_\tau u$ satisfies the following estimate:*

$$\|I_\tau u - R_h I_\tau u\|_{H_\tau^1} \le c_1 c_4 h \left(2 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right) \|u\|_{2,\Omega}, \tag{43}$$

where $c_4 = \max\left\{1, \alpha c_0 \left(1 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)\right\}$.

Proof. It follows from Theorem 2.1 that

$$\begin{aligned}
&\|I_\tau u - R_h I_\tau u\|_{H^1_\tau}^2 \\
&= \sum_{k=1}^{K-1} \tau \left\|\boldsymbol{\nabla}\ (u(z_k) - R_h u(z_k))\right\|_{0,\omega}^2 + \sum_{k=1}^{K-1} \tau \left\|\mathrm{d}_z\,(u(z_k) - R_h u(z_k))\right\|_{0,\omega}^2 \\
&\le c_1^2 h^2 \left(2 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)^2 \sum_{k=1}^{K-1} \tau \left\|\boldsymbol{\nabla}\, u(z_k)\right\|_{0,\omega}^2 \\
&\quad + \alpha^2 c_0^2 c_1^2 h^2 \left(1 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)^2 \left(2 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)^2 \sum_{k=1}^{K} \int_{z_{k-1}}^{z_k} \|\partial_z \boldsymbol{\nabla}\, u\|_{0,\omega}^2 \,\mathrm{d}z \\
&\le c_1^2 c_4^2 h^2 \left(2 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)^2 \|u\|_{2,\Omega}^2, \qquad c_4 = \max\left\{1, \alpha c_0 \left(1 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)\right\}.
\end{aligned}$$

This proves (43). $\square$

Lemma 4.2. *Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_{h\tau} \in \mathcal{X}_{h\tau}$ be the solutions of* (3) *and* (39)*, respecitvely. Then the following estimates hold:*

(44)

$$\|\boldsymbol{\nabla}\ (R_h I_\tau u - u_{h\tau})\|_{L^2_\tau} \le \left[\frac{\sqrt{2}}{4}\tau \left(1 + \frac{2M\|\widehat{\boldsymbol{\beta}}\|_\infty}{\alpha}\right) + \frac{\sqrt{6}}{2} c_5 h \left(1 + \frac{\sqrt{2}\beta_3 P}{\alpha}\right)\right] \|u\|_{2,\Omega},$$

(45)

$$\begin{aligned}
\|\partial_z\,(R_h I_\tau u - u_{h\tau})\|_{L^2_\tau} \le &\left[\frac{\sqrt{2}}{4}\tau \left(1 + \frac{2\sqrt{2}M\|\widehat{\boldsymbol{\beta}}\|_\infty}{\alpha}\right)\right. \\
&\left. + \sqrt{3} c_5 h \left(1 + \frac{\sqrt{2}\beta_3 P}{2\alpha}\right)\right] \|u\|_{2,\Omega},
\end{aligned}$$

and

(46)

$$\|R_h I_\tau u - u_{h\tau}\|_{H^1_\tau} \le \left[\frac{\sqrt{2}}{4}\tau \left(1 + \frac{2\sqrt{2}M\|\widehat{\boldsymbol{\beta}}\|_\infty}{\alpha}\right) + \sqrt{3} c_5 h \left(1 + \frac{\beta_3 P}{\alpha}\right)\right] \|u\|_{2,\Omega},$$

where $c_5 = \alpha c_0 c_1 \left(1 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)\left(2 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)$.

Proof. Setting $\eta_h^k = R_h u(z_k) - u_h^k$, we see that

$$\eta_{h\tau} = R_h I_\tau u - u_{h\tau} = \sum_{k=1}^{K-1} \left(R_h u(z_k) - u_h^k\right) \psi_k(z) = \sum_{k=1}^{K-1} \eta_h^k \psi_k(z).$$

Recalling E_τ is defined in (32), we put

$$E_h = \sum_{k=1}^{K-1} E_h^k \psi_k(z), \;\text{ where } E_h^k \;\text{ solves}$$

$$\begin{aligned}
\left(E_h^k, v_h^j\right)_\omega &= \alpha(\mathrm{d}_z (I - R_h) u(z_k), \mathrm{d}_z v_h^j)_\omega \\
&\quad + \beta_3 (\mathrm{d}_z^* (I - R_h) u(z_k), v_h^j)_\omega, \quad j = 1, \cdots, K-1.
\end{aligned}$$

Then, we claim that

$$B_\tau(\eta_{h\tau}, v_{h\tau}) = (E_\tau, v_{h\tau})_{L^2_\tau} + (E_h, v_{h\tau})_{L^2_\tau}, \quad \forall\, v_{h\tau} \in \mathcal{X}_{h\tau}. \tag{47}$$

Indeed, we can write

$$\begin{aligned} B_\tau(\eta_{h\tau}, v_{h\tau}) &= B_\tau(R_h I_\tau u - u_{h\tau}, v_{h\tau}) \\ &= B_\tau(R_h I_\tau u - I_\tau u, v_{h\tau}) + B_\tau(I_\tau u - u_{h\tau}, v_{h\tau}). \end{aligned} \tag{48}$$

The first bilnear form term in (48) can be put as follows:

$$\begin{aligned} &B_\tau(R_h I_\tau u - I_\tau u, v_{h\tau}) \\ &\quad = \sum_{k=1}^{K-1} \tau A\left(R_h u(z_k) - u(z_k), v_h^k\right) + \sum_{k=1}^{K-1} \tau \left(E_h^k, v_h^k\right)_\omega = (E_h, v_{h\tau})_{L^2_\tau} \end{aligned}$$

since $A\left(R_h u(z_k) - u(z_k), v_h^k\right) = 0$ due to (13). Next, recalling (40) and (34), we see from the second bilnear form term in (48) that

$$\begin{aligned} B_\tau(I_\tau u - u_{h\tau}, v_{h\tau}) &= B_\tau(I_\tau u, v_{h\tau}) - B_\tau(u_{h\tau}, v_{h\tau}) \\ &= B_\tau(I_\tau u, v_{h\tau}) - (f_\tau, v_{h\tau})_{L^2_\tau} = (E_\tau, v_{h\tau})_{L^2_\tau}. \end{aligned}$$

Therefore, (47) follows.

Taking $v_{h\tau} = \eta_{h\tau}$ in (47), we bound each term in the RHS.

First, in order to get a bound of $|(E_\tau, \eta_{h\tau})_{L_2}|$, we observe that the arguments are identical to those in the proof of Theorem 3.3, by replacing e^k by η_h^k. Thus, we proceed as follows.

$$\begin{aligned} &|(E_\tau, \eta_{h\tau})_{L_2}| \\ &\quad = \Bigg| \alpha \sum_{k=1}^{K-1} \tau \left(\boldsymbol{\nabla} \left(u(z_k) - \frac{1}{\tau} \int_{z_{k-\frac12}}^{z_{k+\frac12}} u \mathrm{d}z \right), \boldsymbol{\nabla}\, \eta_h^k \right)_\omega \\ &\qquad + \alpha \sum_{k=1}^{K-1} \tau \left(\mathrm{d}_z u\,(z_k) - \partial_z u \left(z_{k-\frac12}\right), \mathrm{d}_z \eta_h^k \right)_\omega \\ &\qquad + \sum_{k=0}^{K-1} \tau \left(\boldsymbol{\beta} \cdot \boldsymbol{\nabla} \left(u\,(z_k) - \frac{1}{\tau} \int_{z_{k-\frac12}}^{z_{k+\frac12}} u \mathrm{d}z \right), \eta_h^k \right)_\omega \\ &\qquad + \beta_3 \sum_{k=1}^{K-1} \tau \left(\mathrm{d}_z^* u\,(z_k) - \partial_z u\,(z_k), \eta_h^k \right)_\omega \Bigg| \\ &\quad =: I + II + III + IV. \end{aligned}$$

The same line of arguments as in the proof of Theorem 3.3 leads to

$$|I_1| \;\le\; \frac{\alpha \varepsilon_1 \tau^2}{24} \|\partial_z \boldsymbol{\nabla}\, u\|^2_{0,\Omega} + \frac{\alpha}{4\varepsilon_1} \|\boldsymbol{\nabla}\, \eta_{h\tau}\|^2_{L^2_\tau}, \tag{49a}$$

$$|I_2| \;\le\; \frac{\|\boldsymbol{\beta}\|_\infty \varepsilon_2 \tau^2}{24} \|\partial_z \boldsymbol{\nabla}\, u\|^2_{0,\Omega} + \frac{\|\boldsymbol{\beta}\|_\infty P^2}{4\varepsilon_2} \|\boldsymbol{\nabla}\, \eta_{h\tau}\|^2_{L^2_\tau}, \tag{49b}$$

$$|I_3| \;\le\; \frac{\alpha \varepsilon_3 \tau^2}{24} \|\partial_{zz} u\|^2_{0,\Omega} + \frac{\alpha}{4\varepsilon_3} \|\partial_z \eta_{h\tau}\|^2_{L^2_\tau}, \tag{49c}$$

$$|I_4| \;\le\; \frac{|\beta_3| \varepsilon_4 \tau^2}{3} \|\partial_{zz} u\|^2_{0,\Omega} + \frac{|\beta_3|(b_3 - a_3)^2}{4\varepsilon_4} \|\partial_z \eta_{h\tau}\|^2_{L^2_\tau}. \tag{49d}$$

Next, we proceed to bound $|(E_h, \eta_{h\tau})_{L_2}|$.

(50)

$$
\begin{aligned}
&|(E_h, \eta_{h\tau})_{L_2}| \\
&\quad = \left| \alpha \sum_{k=1}^{K-1} \tau \left(\mathrm{d}_z (I - R_h) u(z_k), \mathrm{d}_z \eta_h^k \right)_\omega + \beta_3 \sum_{k=1}^{K-1} \tau \left(\mathrm{d}_z^* (I - R_h) u(z_k), \eta_h^k \right)_\omega \right| \\
&\quad \le \alpha \sum_{k=1}^{K-1} \tau \left(\varepsilon_5 \|\mathrm{d}_z (I - R_h) u(z_k)\|_{0,\omega}^2 + \frac{\|\mathrm{d}_z \eta_h^k\|_{0,\omega}^2}{4\varepsilon_5} \right) \\
&\qquad + |\beta_3| \sum_{k=1}^{K-1} \tau \left(\varepsilon_6 \|\mathrm{d}_z^* (I - R_h) u(z_k)\|_{0,\omega}^2 + \frac{\|\eta_h^k\|_{0,\omega}^2}{4\varepsilon_6} \right) \\
&\quad \le \alpha c_5^2 h^2 \varepsilon_5 \|\partial_z \boldsymbol{\nabla}\, u\|_{0,\Omega}^2 + \frac{\alpha}{4\varepsilon_5} \|\partial_z \eta_{h\tau}\|_{L_\tau^2}^2 \\
&\qquad + |\beta_3| c_5^2 h^2 \varepsilon_6 \|\partial_z \boldsymbol{\nabla}\, u\|_{0,\Omega}^2 + \frac{|\beta_3| P^2}{4\varepsilon_6} \|\boldsymbol{\nabla}\, \eta_{h\tau}\|_{L_\tau^2}^2 \\
&\quad = c_5^2 h^2 (\alpha \varepsilon_5 + |\beta_3| \varepsilon_6) \|\partial_z \boldsymbol{\nabla}\, u\|_{0,\Omega}^2 + \frac{\alpha}{4\varepsilon_5} \|\partial_z \eta_{h\tau}\|_{L_\tau^2}^2 + \frac{|\beta_3| P^2}{4\varepsilon_6} \|\boldsymbol{\nabla}\, \eta_{h\tau}\|_{L_\tau^2}^2 ,
\end{aligned}
$$

where $c_5 = \alpha c_0 c_1 \left(1 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right) \left(2 + \frac{\|\boldsymbol{\beta}\|_\infty}{\alpha} h\right)$.

Then, with $\varepsilon_1 = \frac{3}{2}$, $\varepsilon_2 = \frac{3\|\boldsymbol{\beta}\|_\infty P^2}{2\alpha}$, $\varepsilon_3 = \frac{3}{4}$, $\varepsilon_4 = \frac{3|\beta_3|(b_3 - a_3)^2}{4\alpha}$, $\varepsilon_5 = \frac{3}{4}$, $\varepsilon_6 = \frac{3|\beta_3| P^2}{2\alpha}$, we can derive from (47), (49), and (50) that

$$
\begin{aligned}
\|\boldsymbol{\nabla}\, \eta_{h\tau}\|_{L_\tau^2}^2 \le \tau^2 &\left(\frac{\alpha^2 + \|\boldsymbol{\beta}\|_\infty^2 P^2}{8\alpha^2} \|\partial_z \boldsymbol{\nabla}\, u\|_{0,\Omega}^2 + \frac{\alpha^2 + 8\|\boldsymbol{\beta}_3\|^2 (b_3 - a_3)^2}{16\alpha^2} \|\partial_{zz} u\|_{0,\Omega}^2 \right) \\
&+ 3c_5^2 h^2 \left(\frac{\alpha^2 + 2|\boldsymbol{\beta}_3|^2 P^2}{2\alpha^2} \right) \|\partial_z \boldsymbol{\nabla}\, u\|_{0,\Omega}^2 \\
\le \tau^2 &\left(\frac{\alpha^2 + 4\|\widehat{\boldsymbol{\beta}}\|_\infty^2 M^2}{8\alpha^2} \right) \|u\|_{2,\Omega}^2 + 3c_5^2 h^2 \left(\frac{\alpha^2 + 2|\boldsymbol{\beta}_3|^2 P^2}{2\alpha^2} \right) \|u\|_{2,\Omega}^2,
\end{aligned}
$$

which shows (44).

Similarly, assigning $\varepsilon_1 = \frac{3}{4}$, $\varepsilon_2 = \frac{3\|\boldsymbol{\beta}\|_\infty P^2}{4\alpha}$, $\varepsilon_3 = \frac{3}{2}$, $\varepsilon_4 = \frac{3|\beta_3|(b_3 - a_3)^2}{2\alpha}$, $\varepsilon_5 = \frac{3}{2}$, $\varepsilon_6 = \frac{3|\beta_3| P^2}{4\alpha}$, we can deduce

$$
\begin{aligned}
\|\partial_z \eta_{h\tau}\|_{L_\tau^2}^2 \le \tau^2 &\left(\frac{\alpha^2 + \|\boldsymbol{\beta}\|_\infty^2 P^2}{16\alpha^2} \|\partial_z \boldsymbol{\nabla}\, u\|_{0,\Omega}^2 + \frac{\alpha^2 + 8\|\boldsymbol{\beta}_3\|^2 (b_3 - a_3)^2}{8\alpha^2} \|\partial_{zz} u\|_{0,\Omega}^2 \right) \\
&+ 3c_5^2 h^2 \left(\frac{2\alpha^2 + |\boldsymbol{\beta}_3|^2 P^2}{2\alpha^2} \right) \|\partial_z \boldsymbol{\nabla}\, u\|_{0,\Omega}^2 \\
\le \tau^2 &\left(\frac{\alpha^2 + 8\|\widehat{\boldsymbol{\beta}}\|_\infty^2 M^2}{8\alpha^2} \right) \|u\|_{2,\Omega}^2 + 3c_5^2 h^2 \left(\frac{2\alpha^2 + |\boldsymbol{\beta}_3|^2 P^2}{2\alpha^2} \right) \|u\|_{2,\Omega}^2,
\end{aligned}
$$

which proves (45).

Choosing $\varepsilon_1 = \frac{3}{2}$, $\varepsilon_2 = \frac{3\|\boldsymbol{\beta}\|_\infty P^2}{2\alpha}$, $\varepsilon_3 = \frac{3}{2}$, $\varepsilon_4 = \frac{3|\beta_3|(b_3-a_3)^2}{2\alpha}$, $\varepsilon_5 = \frac{3}{2}$, $\varepsilon_6 = \frac{3|\beta_3|P^2}{2\alpha}$ we can deduce that

$$\begin{aligned}\|\eta_{h\tau}\|^2_{H^1_\tau} \le{}& \tau^2 \left(\frac{\alpha^2 + \|\boldsymbol{\beta}\|^2_\infty P^2}{8\alpha^2} \|\partial_z \boldsymbol{\nabla} u\|^2_{0,\Omega} + \frac{\alpha^2 + 8\|\boldsymbol{\beta}_3\|^2 (b_3-a_3)^2}{8\alpha^2} \|\partial_{zz} u\|^2_{0,\Omega} \right) \\ &+ 3c_5^2 h^2 \left(\frac{\alpha^2 + |\boldsymbol{\beta}_3|^2 P^2}{\alpha^2} \right) \|\partial_z \boldsymbol{\nabla} u\|^2_{0,\Omega} \\ \le{}& \tau^2 \left(\frac{\alpha^2 + 8\|\widehat{\boldsymbol{\beta}}\|^2_\infty M^2}{8\alpha^2} \right) \|u\|^2_{2,\Omega} + 3c_5^2 h^2 \left(\frac{\alpha^2 + |\boldsymbol{\beta}_3|^2 P^2}{\alpha^2} \right) \|u\|^2_{2,\Omega},\end{aligned}$$

which proves (46). □

Finally, we are ready to state and prove one of our main theorems.

Theorem 4.1. *Let $u \in H^1_0(\Omega) \cap H^2(\Omega)$ and $u_{h\tau} \in \mathcal{X}_{h\tau}$ be the solutions of* (3) *and* (39)*, respecitvely. Then the following holds:*

$$\|u - u_{h\tau}\|_{1,\Omega} \le C(h+\tau)\|u\|_{2,\Omega}, \tag{51}$$

where $C > 0$ is a constant related to the diffusion coefficient α, convection field $\widehat{\boldsymbol{\beta}}$, and 3D domain Ω.

Proof. We write $u - u_{h\tau}$ into

$$u - u_{h\tau} = (u - I_\tau u) + (I_\tau u - R_h I_\tau u) + (R_h I_\tau u - u_{h\tau}).$$

A combination of the triangle inequality, Lemma 3.2, Theorem 3.3, Lemmas 4.1 and 4.2 proves the theorem. □

5. Numerical experiments

In this section, we perform numerical experiments to illustrate the obtained theoretical results. Denote error functions $E_{h\tau} = u - u_{h\tau}$, we will study the convergence behavior of $\|E_{h\tau}\|_{0,\Omega}$, $\left\|\widehat{\boldsymbol{\nabla}} E_{h\tau}\right\|_{0,\Omega}$, $\|\boldsymbol{\nabla} E_{h\tau}\|_{0,\Omega}$, $\|\partial_z E_{h\tau}\|_{0,\Omega}$ by problems with an exact solution for different types of convective fields $\widehat{\boldsymbol{\beta}}$, domain Ω and the domain mesh.

Example 1. *Let $\Omega = \omega \times (0,1)$, where $\omega = (0,1)^2$ is decomposed into $\omega_1 = \{(x,y) \in \omega \mid x \ge \frac{1}{2}, y \ge \frac{1}{2}\}$, $\omega_2 = \{(x,y) \in \omega \mid x < \frac{1}{2}, y \ge \frac{1}{2}\}$, $\omega_3 = \{(x,y) \in \omega \mid x < \frac{1}{2}, y < \frac{1}{2}\}$, and $\omega_4 = \{(x,y) \in \omega \mid x \ge \frac{1}{2}, y < \frac{1}{2}\}$.*

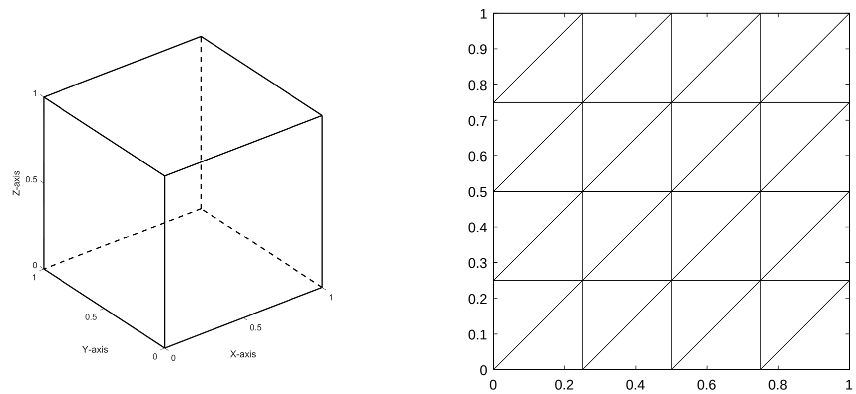
We consider the discontinuous diffusion coefficient:

$$\alpha(x,y,z) = \alpha(x,y) = 2 - \chi_{\omega_1}(x,y) + 2\chi_{\omega_3}(x,y),$$

and the convection field $\widehat{\boldsymbol{\beta}} = (\boldsymbol{\beta}, \beta_3) = \left(x+y, x-y, \frac{1}{2}\right)$. Suppose that the exact solution of (1a) *is given by*

$$\begin{aligned}u_1(x,y,z) = {}&[(4x-1)(4y-1)\chi_{\omega_1}(x,y) + 2x(4y-1)\chi_{\omega_2}(x,y) + 4xy\chi_{\omega_3}(x,y) \\ &+(4x-1)(2y)\chi_{\omega_4}(x,y)] \frac{\sin(\pi z)}{9}.\end{aligned}$$

The source function f can be produced from the exact solution u, the diffusion constant α and the convection field $\widehat{\boldsymbol{\beta}}$.



(a) (b)

FIGURE 1. Left: The computational domain Ω in Example 1. Right: Initial triangulations of size $h = 1/4$ on ω in Example 1.

Uniform meshes are considered in the (x, y)*-plane while the* z*-direction interval* $(0, 1)$ *is divided into uniform intervals of size-length* τ*. Notice that the uniform mesh is obtained by dividing each square of size h into two triangles by drawing a diagonal line from the left-lower corner to the right-upper corner. We show the initial triangulations for* (x, y)*-plane in Figure 1(b). We use the conforming* P_1 *elements for triangular meshes in the* (x, y) *direction with uniform meshes in the* z *direction. Numerical results are given in Table 1 and Figure 2 from which we observe that the asymptotic convergence orders of the DFE* (40) *are optimal in* H_1 *norm and* L_2 *norm.*

TABLE 1. Numerical results of u_1 for the DFE method on uniform meshes.

Mesh size (h, τ)	$\|E_{h\tau}\|_{0,\Omega}$	*Rate*	$\|\boldsymbol{\nabla} E_{h\tau}\|_{0,\Omega}$	*Rate*	$\|\partial_z E_{h\tau}\|_{0,\Omega}$	*Rate*	$\left\|\widehat{\boldsymbol{\nabla}} E_{h\tau}\right\|_{0,\Omega}$	*Rate*
$(1/4, 1/4)$	$6.4944e-3$	—	$1.1126e-1$	—	$1.3044e-1$	—	$1.7144e-1$	—
$(1/8, 1/8)$	$1.5745e-3$	2.04	$5.6422e-2$	0.98	$6.5274e-2$	1.00	$8.6280e-02$	1.00
$(1/16, 1/16)$	$3.8833e-4$	2.02	$2.8315e-2$	0.99	$3.2643e-2$	1.00	$4.3212e-2$	1.00
$(1/32, 1/32)$	$9.6558e-5$	2.01	$1.4171e-2$	1.00	$1.6322e-02$	1.00	$2.1615e-02$	1.00

Example 2. *In the next example, we consider an L-shaped domain* ω *with the exact solution* $u_2(\mathbf{x}, z) = \phi(r, \theta)\psi(z)$ *in cylindrical coordinates with singularity at the origin in* ω *given as*

$$u_2(r, \theta, z) = r^{\frac{2}{3}} \sin(\frac{2}{3}\theta) \exp(-z).$$

The source function f *can be acquired by setting the diffusion coefficient* $\alpha = 1$ *and the convection field* $\widehat{\boldsymbol{\beta}} = (\boldsymbol{\beta}, \beta_3) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$*. We consider the computational domain* $\Omega = \omega \times (-1, 1)$*, where* $\omega = (-1, 1)^2 \setminus [0, 1] \times [-1, 0]$ *is an L-shape domain.*

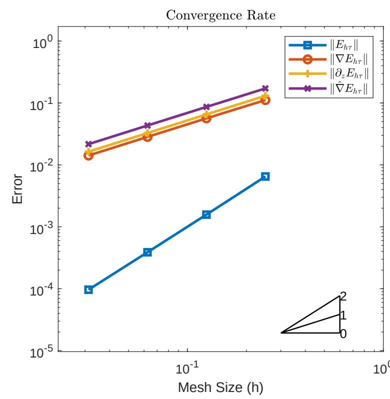


FIGURE 2. Convergence rates in Example 1.

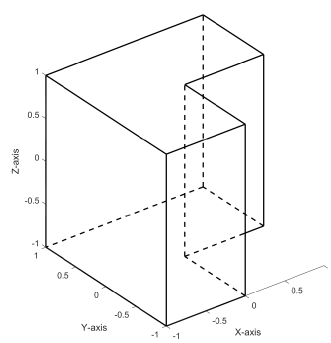
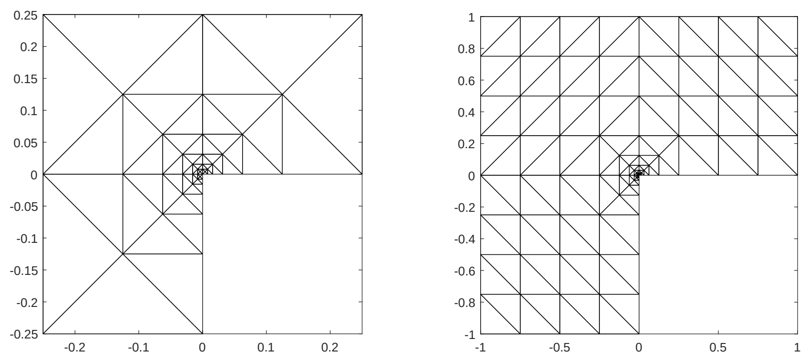


FIGURE 3. The computational domain Ω in Example 2.

(See Figure 3). The z-direction interval $(0, 1)$ is divided into uniform intervals of size-length τ. We proceed to work on graded meshes in the $(x, y)-$plane. We begin with a uniform triangulation by rectangles $\mathcal{T}_h$ and mesh size $h = \frac{1}{n}$. Mesh points in $\overline{\omega}$ are given as $(x_j, y_k) = h(j, k), j, k = 0, \cdots, 2n$. They are three elements of size $h \times h$ from $\mathcal{T}_h$ that share their vertices with $(0, 0)$:

$$R_{n,n-1} = (x_{n-1}, x_n) \times (y_{n-2}, y_{n-1}) := G_b^{(0)},$$
$$R_{nn} = (x_{n-1}, x_n) \times (y_{n-1}, y_n) := G_c^{(0)},$$
$$R_{n+1,n} = (x_n, x_{n+1}) \times (y_{n-1}, y_n) := G_r^{(0)}.$$

Denote the above three rectangles by $G_b^{(0)}$, $G_c^{(0)}$, $G_r^{(0)}$ and their centers $g_b^{(0)}$, $g_c^{(0)}$, $g_r^{(0)}$, respectively. The superscript (s) denotes the s-th level of mesh grading. Drawing two diagonal lines, we subdivide each of the rectangles $G_\iota^{(0)}$, $\iota = b, c, t$ into four triangles $T_{\iota,0}^{(1)}$, $T_{\iota,0}^{(2)}$, $T_{\iota,0}^{(3)}$, $T_{\iota,0}^{(4)}$ that $T_{\iota,0}^{(3)}$ and $T_{\iota,0}^{(4)}$ can form a square $G_\iota^{(1)}$ of size $h/2$. The cubes $G_\iota^{(1)}$ are supposed to have common vertex $(0, 0)$ and edges are parallel to the x- and y-axes. This procedure is a recursive procedure. S is represented as the grading step number.Other element from $\mathcal{T}_h$ besides $R_{n-1,n}$, R_{nn}, $R_{n,n+1}$ are



(a) (b)

FIGURE 4. Left: The graded mesh on the corner singular elements with the grading step number $s = 5$. Right: Initial triangulations of size $h = 1/4$ on ω in Example 2.

divided two triangles by drawing a diagonal line. We show the graded mesh on the corner singular elements and the initial mesh on the Ω *in Figure 4.*

There report the numerical results for the $P_1 \times P_1$ *elements in Table 2 and Figure 5, from which we observe that the asymptotic convergence orders of the DFE* (40) *are optimal in* H_1 *norm and* L_2 *norm.*

TABLE 2. Numerical results of u_2 for the DFE method on graded meshes with the grading step number $s = 5$.

Mesh size (h, τ)	$\|E_{h\tau}\|_{0,\Omega}$	*Rate*	$\|\nabla E_{h\tau}\|_{0,\Omega}$	*Rate*	$\|\partial_z E_{h\tau}\|_{0,\Omega}$	*Rate*	$\left\|\widehat{\nabla} E_{h\tau}\right\|_{0,\Omega}$	*Rate*
$(1/4, 1/4)$	$1.2974e-2$	—	$2.0440e-1$	—	$1.4294e-1$	—	$2.4943e-1$	—
$(1/8, 1/8)$	$3.6432e-3$	1.83	$1.3765e-1$	0.57	$7.1802e-2$	0.99	$1.5525e-1$	0.68
$(1/16, 1/16)$	$1.2506e-3$	1.54	$9.0270e-2$	0.61	$3.5957e-2$	1.00	$9.7168e-2$	0.68
$(1/32, 1/32)$	$5.0609e-4$	1.31	$5.8268e-2$	0.63	$1.7986e-2$	1.00	$6.0980e-2$	0.67

Example 3. *We set* $\Omega = \omega \times (0, 10)$, *where* ω *is a ring composed of two concentric circles centered at* $(0, 0)$, *with the outer circle having a radius of 0.5 and the inner circle having a radius of 0.05. The 3D domain* Ω *and initial triangulations on* ω *are shown in Figure 6. We consider the convection field* $\widehat{\boldsymbol{\beta}} = (\boldsymbol{\beta}, \beta_3) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$. *Let the exact solution of* (1a) *be given by*

$$u_3(x, y, z) = (0.5 - x^2 - y^2)^2(0.1 - x^2 - y^2)^2 z^2 (10 - z)^2 \sin(xyz)\exp(xyz).$$

Numerical results are shown for the $P_1 \times P_1$ *elements in Table 3 and Figure 7 where* $\tau = 10h$, *from which we observe that the convergence orders of the DFE are optimal in* H_1 *norm and* L_2 *norm.*

6. Conclusions

In this paper, we have proposed the Difference Finite Element Method using the continuous piecewise $P_1 \times P_1$ elements to approximate solution of the Dirichlet

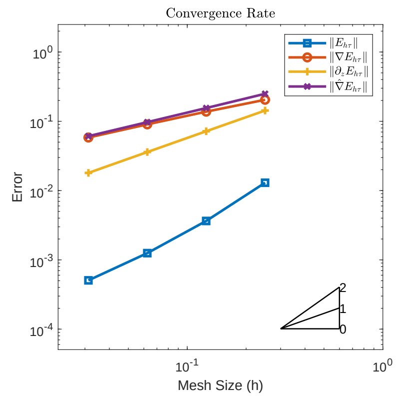
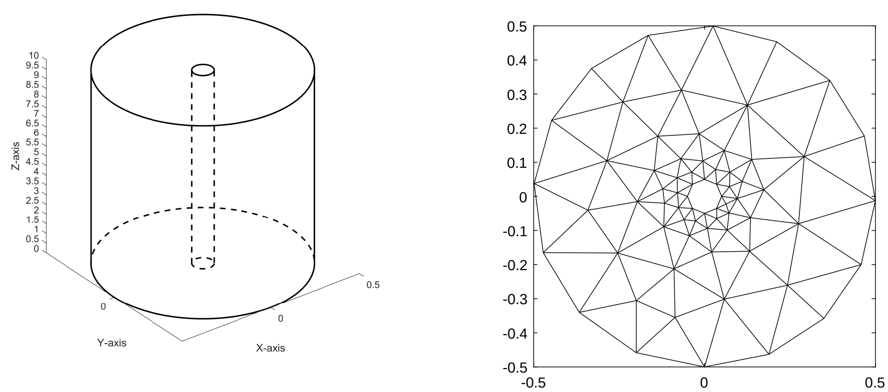


FIGURE 5. Convergence rates in Example 2.



(a) (b)

FIGURE 6. Left: The computational domain Ω in Example 3. Bottom: Initial triangulations of size $h = 1/4$ on ω in Example 3.

TABLE 3. Numerical results of u_3 for the DFE method with $\tau = 10h$.

Mesh size (h, τ)	$\|E_{h\tau}\|_{0,\Omega}$	*Rate*	$\|\boldsymbol{\nabla} E_{h\tau}\|_{0,\Omega}$	*Rate*	$\|\partial_z E_{h\tau}\|_{0,\Omega}$	*Rate*	$\left\|\widehat{\boldsymbol{\nabla}} E_{h\tau}\right\|_{0,\Omega}$	*Rate*
$(1/4, 5/2)$	$1.6298e-1$	—	$3.9460e+0$	—	$1.1780e-1$	—	$3.9478e+0$	—
$(1/8, 5/4)$	$4.1698e-2$	1.97	$1.8614e+0$	1.08	$7.2313e-2$	0.70	$1.8628e+0$	1.08
$(1/16, 5/8)$	$1.0857e-2$	1.94	$9.1723e-1$	1.02	$3.8795e-2$	0.90	$9.1805e-1$	1.02
$(1/32, 5/16)$	$2.7689e-3$	1.97	$4.5978e-1$	1.00	$1.9752e-2$	0.97	$4.6020e-1$	1.00

boundary value problem for the 3D convection-diffusion equation. Here, we provide the existence and uniqueness of the DFE solution $u_{h\tau}$. The optimal convergence rate of the DFE solution $u_{h\tau}$ with respect to the exact solution u for the H_1-norm is deduced. Finally, we have provided some numerical examples to verify the accuracy and flexibility of the proposed method. Perhaps, we could also use finite

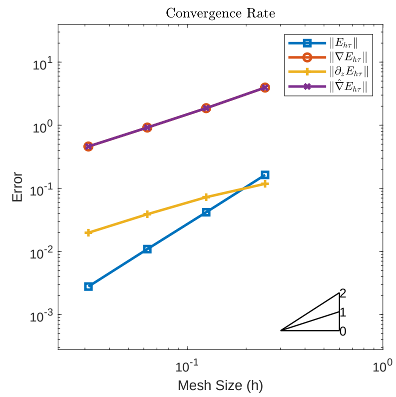


FIGURE 7. Convergence rates in Example 3.

element discretization in the z-direction and finite difference discretization in the (x, y)-plane if this proves to be more effectively applicable to other models. The research is underway to develop computationally cheaper and more stable numerical methods for 3D convection-dominated problems.

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