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RICHARDSON EXTRAPOLATION OF THE CRANK-NICOLSON SCHEME FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this work, we consider Richardson extrapolation of the Crank-Nicolson (CN) scheme for backward stochastic differential equations (BSDEs). First, applying the Adomian decomposition to the nonlinear generator of BSDEs, we introduce a new system of BSDEs. Then we theoretically prove that the solution of the CN scheme for BSDEs admits an asymptotic expansion with its coefficients the solutions of the new system of BSDEs. Based on the expansion, we propose Richardson extrapolation algorithms for solving BSDEs. Finally, some numerical tests are carried out to verify our theoretical conclusions and to show the stability, efficiency and high accuracy of the algorithms.

Key words. Backward stochastic differential equations, Crank-Nicolson scheme, Adomian decomposition, Richardson extrapolation, asymptotic error expansion.

1. Introduction

This paper is concerned with the numerical solution of the following BSDE defined on a filtered complete probability space $(\Omega,\ \mathcal{F},\ \mathbb{F},\ \mathbb{P})$ with the natural filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ generated by a standard d_1-dimensional Brownian motion $W_t = (W_t^1, W_t^2, \cdots, W_t^{d_1})^\top$, $0 \leq t \leq T$.

$$(1)\qquad Y_t =\varphi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s,$$

where T is a deterministic terminal time instant; $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^q$ and $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^q \times \mathbb{R}^{q\times d_1} \longrightarrow \mathbb{R}^q$ are the terminal condition and the generator of BSDE (1), respectively. Note that the stochastic integral with respect to W_t is of Itô's type, and X_t is a diffusion process. In this paper, we only consider the case where

$$(2)\qquad X_t = X_0 + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}W_s,\ 0 \leq t \leq T,$$

where the functions $b : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d_1}$ are called the drift and the diffusion coefficients of the SDE (2). A pair of processes $(Y_t,\ Z_t)$ is called an L^2-adapted solution of (1) if it is $\mathcal{F}_t$-adapted, square integrable, and satisfies BSDE (1).

In 1990, the existence and uniqueness of the solution of BSDEs were proved by Pardoux and Peng [28]. Since then, lots of efforts have been devoted to the study of BSDEs due to their applications in various important fields such as mathematical finance, stochastic optimal control, risk measure, game theory, and so on (see, e.g., [12, 32, 26, 29] and references therein).

As BSDEs seldom admit explicitly closed-form solutions, numerical methods have played an important role in applications. In recent years, great efforts have been made for designing efficient numerical schemes for BSDEs and forward

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backward stochastic differential equations (FBSDEs). There are two main types of numerical schemes: the first one is based on numerical solution of a parabolic PDE which is related to a FBSDE [11, 25], while the second type of schemes focus on discretizing FBSDEs directly [3, 5, 10, 18, 24, 33, 40]. From the temporal discretization point of view, popular strategies include Euler-type methods [15, 16, 38], θ-schemes [34, 43], Runge-Kutta schemes [8], multistep schemes [7, 14, 41, 44, 45], and strong stability preserving multistep (SSPM) schemes [13], to name a few. For fully coupled FBSDEs, there exist only few numerical studies and satisfactory results [27, 41]. We mention the work in [41], where a class of multistep type schemes are proposed, which turns out to be effective in obtaining highly accurate solutions of FBSDEs, and the work in [35], where the classical deferred correction (DC) method is adopted to design highly accurate numerical methods for fully coupled FBSDEs.

In this paper, we will approximate the solution of BSDE (1) based on the Richardson extrapolation (RiE) method. It is well known that Richardson extrapolation method, which was established by Richardson [31], is an efficient procedure for increasing the accuracy of approximations of many problems in numerical analysis. For example, the applications of the RiE to ordinary differential equations (ODEs) based on one-step schemes, e.g., Runge-Kutta methods are described in [6, 17]. In addition, this method has been well demonstrated in its applications to finite element and mixed finite element methods for elliptic partial differential equations [4], Sobolev- and viscoelasticity-type equations [22], partial integro-differential equations [23], Fredholm and Volterra integral equations of the second kind [20], Volterra integro-differential equations [39], and to collocation methods in [21], etc. As for the applications of the RiE to BSDEs, we mention the work in [9], where an explicit error expansion for the solution of BSDEs is obtained by using the cubature on Wiener spaces method.

In this work, we will design highly accurate Richardson extrapolation algorithms with the solutions of the Crank-Nicolson scheme for BSDE (1). To this end, we first introduce a new system of BSDEs by applying the Adomian decomposition to the nonlinear generator of BSDEs. Then we theoretically prove that the solution of the Crank-Nicolson scheme for BSDEs admits an asymptotic expansion with its coefficients being the solutions of the new system of BSDEs. Finally, based on the expansion, we propose the Richardson extrapolation algorithms of the Crank-Nicolson scheme (RiE-CN, for short) for solving BSDEs. The RiE-CN algorithms are very easy in use. We can obtain accurate solutions with high order rate of convergence only by combining linearly the numerical solutions of the CN scheme with different time step sizes. Moreover, our numerical tests verify our theoretical conclusions, and show that the RiE-CN algorithms are stable, very efficient and high accurate.

The rest of the paper is organized as follows. In Section 2, we recall the nonlinear Feynman-Kac formula, the generator of a diffusion process, the Adomian decomposition and the Richardson extrapolation method in brief. We present the asymptotic error expansion of the solution of the Crank-Nicolson scheme for BSDEs in Section 3. The construction of the RiE-CN algorithms for BSDEs is presented in Section 4. And in Section 5, numerical tests are carried out to support the theoretical results. Finally, some concluding remarks are given in Section 6.

2. Preliminaries

In this Section, we will recall the non-linear Feynman-Kac formula, the generator of a diffusion process, the Adomian decomposition and the Richardson extrapolation method in brief.

2.1. The nonlinear Feynman-Kac formula. Let $u \in C^{1,2}([0,T] \times \mathbb{R}^d; \mathbb{R}^q)$ be the solution of the parabolic partial differential equation (PDE)

$$L^0 u(t,x) + f(t,x,u(t,x), \nabla_x u(t,x)\sigma(t,x)) = 0,\ (t,x) \in [0,T) \times \mathbb{R}^d \tag{3}$$

with the terminal condition $u(T,x) = \varphi(x)$, where L^0 is a second order differential operator defined by

$$L^0 := \frac{\partial}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{l=1}^{d_1}(\sigma_{il}\sigma_{jl})(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial}{\partial x_i}. \tag{4}$$

Here C^{k_1,k_2} refers to the set of functions $g(t,x)$ with continuous partial derivatives up to k_1 with respect to t, and up to k_2 with respect to x. And we denote by $C_b^{k_1,k_2}$ the space that consists of all functions $(t,x) \mapsto g(t,x)$ with bounded continuous partial derivatives up to the orders k_1 and k_2 with respect to $t \in [0,T]$ and $x \in \mathbb{R}^d$, respectively.

In 1991, Peng [30] proved that under certain regularity conditions, the solution u of the PDE (3) can be expressed as

$$u(t,X_t) = Y_t,\ \nabla_x u(t,X_t)\sigma(t,X_t) = Z_t,\ t \in [0,T). \tag{5}$$

The first formula in (5) is known as the nonlinear Feynman-Kac formula.

2.2. The diffusion process generator.

Definition 1. *Let X_t be a diffusion process in $\mathbb{R}^d$ satisfying* (2). *Then the generator D_t^x of X_t on $g : [0,T] \times \mathbb{R}^d$ is defined by*

$$D_t^x g(t,x) = \lim_{s \downarrow t} \frac{\mathbb{E}_t^x[g(s,X_s)] - g(t,x)}{s-t},\ x \in \mathbb{R}^d \tag{6}$$

if the limit exists, where $\mathbb{E}_t^x[\cdot]$ is the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t,\ X_t = x]$ for $(t, x) \in [0,T] \times \mathbb{R}^d$.

Note that $D_t^x g(t,x) = L^0 g(t,x)$ when $g \in C^{1,2}([0,T] \times \mathbb{R}^d)$. By Definition 1, Itô's formula and the tower rule of conditional expectations, we have the following Lemma.

Lemma 2 ([41])**.** *Let $t \in [0,s]$ be a fixed time. If*

$$g \in C_b^{1,2}([0,T] \times \mathbb{R}^d),\ \text{and } \mathbb{E}_t^x[L^0 g(s,X_s)] < +\infty,$$

then for $s \in [t,T)$ we have the identity

$$\frac{d\mathbb{E}_t^x[g(s,X_s)]}{ds} = \mathbb{E}_t^x[L^0 g(s,X_s)].$$

Proof. By Definition 1, we have

$$L^0 g(s,X_s) = \lim_{r \downarrow s} \frac{\mathbb{E}_s^{X_s}[g(r,X_r)] - g(s,X_s)}{r-s}. \tag{7}$$

Taking the conditional expectation $\mathbb{E}_t^x[\cdot]$ on both sides of (7), we have

$$\mathbb{E}_t^x[L^0 g(s,X_s)] = \mathbb{E}_t^x\left[\lim_{r \downarrow s} \frac{\mathbb{E}_s^{X_s}[g(r,X_r)] - g(s,X_s)}{r-s}\right]. \tag{8}$$

Note that we can exchange the order of the limit and the conditional expectation in (8) on account of the condition $g \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$. Then we have

$$
\begin{aligned}
\mathbb{E}_t^x[L^0 g(s,X_s)] =& \mathbb{E}_t^x\left[\lim_{r\downarrow s} \frac{\mathbb{E}_s^{X_s}[g(r,X_r)] - g(s,X_s)}{r-s}\right] \\
=& \lim_{r\downarrow s} \frac{\mathbb{E}_t^x[\mathbb{E}_s^{X_s}[g(r,X_r)]] - \mathbb{E}_t^x[g(s,X_s)]}{r-s} \\
=& \lim_{r\downarrow s} \frac{\mathbb{E}_t^x[g(r,X_r)] - \mathbb{E}_t^x[g(s,X_s)]}{r-s} \\
=& \frac{d\mathbb{E}_t^x[g(s,X_s)]}{ds}.
\end{aligned} \tag{9}
$$

The proof ends. □

As a direct corollary of Lemma 2, we have

Corollary 3. *If* $g \in C_b^{k,2k}([0,T] \times \mathbb{R}^d)$, and $\mathbb{E}_t^x[(L^0)^{(k)} g(s,X_s)] < +\infty$, *then for* $t \in [0,s]$ *we have*

$$\frac{d^k \mathbb{E}_t^x[g(s,X_s)]}{ds^k} = \mathbb{E}_t^x[(L^0)^{(k)} g(s,X_s)],$$

where $(L^0)^{(k)} = \underbrace{L^0 \circ \cdots \circ L^0}_{k \text{ times}}$.

2.3. Adomian decomposition. Let $G : \mathscr{X} \to \mathscr{Y}$ be a nonlinear operator, where $\mathscr{X}$ and $\mathscr{Y}$ are two Banach spaces, and $\mathbf{u} \in \mathscr{X}$ have the series form $\mathbf{u} = \sum_{j=0}^{\infty} \mathbf{u}_j$. Then $G\mathbf{u}$ can be decomposed into an infinite series of the form

$$G\mathbf{u} = \sum_{j=0}^{\infty} A_j^G, \tag{10}$$

where A_j^G are the so-called Adomian polynomials of $\mathbf{u}_0$, $\mathbf{u}_1$, $\cdots$, $\mathbf{u}_j$ and are calculated by

$$A_j^G = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j} G(\sum_{i=0}^{\infty} \lambda^i \mathbf{u}_i)\right]_{\lambda=0}, \quad j = 0,1,2,\cdots. \tag{11}$$

Note that the polynomials A_j^G are generated for the nonlinearity so that each A_j^G depends only on $\mathbf{u}_0, \mathbf{u}_1, \cdots, \mathbf{u}_j$ for $j \geq 0$. We call (10) the Adomian decomposition of $G\mathbf{u}$. The Adomian decomposition was proposed by Adomian [1, 2] initially with the aims to solve frontier nonlinear problems in physics, biology and chemical reactions, etc. To show the use of the Adomian decomposition in solving nonlinear problems, we choose the nonlinear equation as

$$G\mathbf{u} = L\mathbf{u} + F\mathbf{u} = 0, \tag{12}$$

where $L\mathbf{u}$ is the linear term, $F\mathbf{u}$ is the nonlinear term, and $\mathbf{u} = (u,v)$.

Assume the inverse L^{-1} of the linear operator L exist. Taking L^{-1} in both sides of (12) gives

$$\mathbf{u} = -L^{-1} F\mathbf{u}. \tag{13}$$

Assume $\mathbf{u}(t) = \sum_{j=0}^{\infty} \mathbf{u}_j(t) = \left(\sum_{j=0}^{\infty} u_j(t), \sum_{j=0}^{\infty} v_j(t)\right)$. Then by applying the Adomian decomposition to $F\mathbf{u} = F(t,\mathbf{u}(t))$, we have

$$\sum_{j=0}^{\infty} \mathbf{u}_j = -L^{-1} \sum_{j=0}^{\infty} A_j^F, \tag{14}$$

where A_j^F are calculated by

$$A_j^F(t) = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j}F\left(t, \sum_{i=0}^{\infty}\lambda^i u_i(t), \sum_{i=0}^{\infty}\lambda^i v_i(t)\right)\right]_{\lambda=0}, \quad j = 0, 1, 2, \cdots.$$

Here we list the first few Adomian polynomials $A_j^F(t)$, $j = 0, 1, 2, 3$ which are

$$\begin{aligned} A_0^F(t) =& F_{0,0}, \\ A_1^F(t) =& u_1(t)F_{1,0} + v_1(t)F_{0,1}, \\ A_2^F(t) =& u_2(t)F_{1,0} + v_2(t)F_{0,1} + (u_1^2(t)/2!)F_{2,0} \\ & + u_1(t)v_1(t)F_{1,1} + (v_1^2(t)/2!)F_{0,2}, \\ A_3^F(t) =& u_3(t)F_{1,0} + v_3(t)F_{0,1} + u_1(t)u_2(t)F_{2,0} \\ & + [u_1(t)v_2(t) + u_2(t)v_1(t)]\, F_{1,1} + v_1(t)v_2(t)F_{0,2} \\ & + (u_1^3(t)/3!)F_{3,0} + (u_1^2(t)/2!)v_1(t)F_{2,1} \\ & + u_1(t)(v_1^2(t)/2!)F_{1,2} + (v_1^3(t)/3!)F_{0,3}, \end{aligned} \tag{15}$$

where $F_{\mu,\nu} = \frac{\partial^{\mu+\nu}}{\partial u^\mu \partial v^\nu}F(t, u_0(t), v_0(t))$. It is worthy of noting that in (15), $A_0^F(t) = F(t, u_0(t), v_0(t))$, and for $j \geq 1$, A_j^F is linear with respect to u_j and v_j.

Given $\mathbf{u}_0$, we solve the $\mathbf{u}_j (j = 1, 2, \cdots)$ by

$$\mathbf{u}_j = -L^{-1}A_{j-1}^F. \tag{16}$$

We call the procedure (14), (15) and (16) the Adomian decomposition method for solving the nonlinear problem (12).

2.4. Richardson extrapolation. Consider a problem with exact solution $y(t)$, where $t \in [0, T]$, and T is a positive real number. Let $\tilde{y}(t; \Delta t)$ be a numerical solution of $y(t)$ on a uniform grid $\pi_N := \{t_n | t_n = n\Delta t, \Delta t = \frac{T}{N}, n = 0, 1, \cdots, N\}$, where Δt is the step size, N is a positive integer. Assume that the exact solution $y(t)$ is smooth enough on the domain $[0, T]$ such that $\tilde{y}(t; \Delta t)$ admits the asymptotic expansion on π_N

$$\tilde{y}(t; \Delta t) - y(t) = \sum_{j=1}^{K-1} e_j(t)(\Delta t)^{a_j} + E_K(t)(\Delta t)^{a_K}, \tag{17}$$

where the $e_j(t)$ are independent of Δt with $e_j(t_0) = 0$, and $E_K(t)$ is bounded, and the sequence $\{a_j\}_{j=1}^K$ is monotonically increasing.

Now we choose a sequence of positive integers

$$1 = N_0 < N_1 < N_2 < \cdots, \tag{18}$$

and define the corresponding uniform grids $\pi_{N,i} (i = 0, 1, \cdots, K-1)$ by

$$\pi_{N,i} = \{t_n | t_n = n\Delta t_i, \Delta t_i = \frac{T}{N \cdot N_i}, n = 0, 1, \cdots, N \cdot N_i\}. \tag{19}$$

Note that $\pi_{N,0} = \pi_N$, and all the $\pi_{N,i}, i = 0, 1, \cdots, K-1$ have the common grid points in $\pi_{N,0}$. Then for any $t_n \in \pi_{N,0}$ (Sometimes we also say $n \in \pi_{N,0}$ which means n is a nonnegative integer such that $t_n \in \pi_{N,0}$), and $1 \leq p \leq m \leq K-1$, by (17), we have

$$\tilde{y}(t_n; \Delta t_i) - y(t_n) = \sum_{j=1}^{p} e_j(t_n)(\Delta t_i)^{a_j} + \mathcal{O}\left((\Delta t_i)^{a_{p+1}}\right). \tag{20}$$

By multiplying $c_i \in \mathbb{R}$ on both sides of (20) and adding the derived equations up from $i = m-p$ to m, we obtain

$$\begin{aligned}
&\sum_{i=m-p}^{m} c_i \tilde{y}(t_n;\Delta t_i) - \left(\sum_{i=m-p}^{m} c_i\right) y(t_n) \\
=&\sum_{i=m-p}^{m}\sum_{j=1}^{p} c_i e_j(t_n)(\Delta t_i)^{a_j} + \mathcal{O}\left((\Delta t_{m-p})^{a_{p+1}} \sum_{i=m-p}^{m} c_i\right) \\
=&\sum_{j=1}^{p}\left(\sum_{i=m-p}^{m} \frac{c_i}{N_i^{a_j}}\right) e_j(t_n)(\Delta t)^{a_j} + \mathcal{O}\left((\Delta t)^{a_{p+1}} \sum_{i=m-p}^{m} c_i\right).
\end{aligned} \tag{21}$$

Since $N_i \neq N_j$ for $i \neq j$, the system of equations (22)

$$\begin{pmatrix} 1 & \cdots & 1 \\ N_{m-p}^{-a_1} & \cdots & N_m^{-a_1} \\ \vdots & \ddots & \vdots \\ N_{m-p}^{-a_p} & \cdots & N_m^{-a_p} \end{pmatrix} \begin{pmatrix} c_{m-p} \\ c_{m-p+1} \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{22}$$

has a unique solution $\boldsymbol{c} = (c_{m-p}, c_{m-p+1}, \cdots, c_m)^\top$. Then from (21), we have

$$\sum_{i=m-p}^{m} c_i \tilde{y}(t_n;\Delta t_i) - y(t_n) = \mathcal{O}\left((\Delta t)^{a_{p+1}}\right). \tag{23}$$

Let $T_{i,0}^n = \tilde{y}(t_n;\Delta t_i)$, and define $T_{m,p}^n = \sum_{i=m-p}^{m} c_i T_{i,0}^n$, $1 \leq p \leq m \leq K-1$. All $T_{m,p}^n$, $0 \leq p \leq m \leq K-1$ can be arranged in the form

$$\begin{array}{ccccc}
T_{0,0}^n & & & & \\
T_{1,0}^n & T_{1,1}^n & & & \\
T_{2,0}^n & T_{2,1}^n & T_{2,2}^n & & \\
\vdots & \vdots & \vdots & \ddots & \\
T_{K-1,0}^n & T_{K-1,1}^n & T_{K-1,2}^n & \cdots & T_{K-1,K-1}^n
\end{array} \tag{24}$$

The procedure of obtaining $T_{m,p}^n = \sum_{i=m-p}^{m} c_i T_{i,0}^n$, $1 \leq p \leq m \leq K-1$ from $T_{i,0}^n$ is called the Richardson extrapolation. And we call $T_{m,p}^n$, $1 \leq p \leq m \leq K-1$ the extrapolation solutions of $\tilde{y}(t_n;\Delta t_i)$. It is worthy of mentioning that all the values $T_{\cdot,p}^n$ located in the pth column in (24) are the approximations to the exact solution $y(t_n)$ with error $\mathcal{O}\left((\Delta t)^{a_{p+1}}\right)$. In particular, the entry $T_{K-1,K-1}^n$ is an approximation to $y(t_n)$ with error $\mathcal{O}\left((\Delta t)^{a_K}\right)$.

If $a_j = k \cdot j$ in (17), where k is a positive integer, we can recursively realize the Richardson extrapolation by the following Aitken-Neville algorithm.

$$\begin{aligned}
T_{m,0}^n =& \tilde{y}(t_n;\Delta t_m), \\
T_{m,p}^n =& T_{m,p-1}^n + \frac{T_{m,p-1}^n - T_{m-1,p-1}^n}{\left(\frac{N_m}{N_{m-p}}\right)^k - 1}.
\end{aligned} \tag{25}$$

Note that different N_i, $i = 0, 1, \cdots$ in (18) lead to different step-number sequences. Here we list two of them that are frequently used as follows.

- Romberg sequence: $N_i = 2^i, i = 0, 1, \cdots$.

$$1, 2, 4, 8, 16, 32, 64, \cdots.$$

- Bulirsch sequence: $N_i = \begin{cases} 1, \ i = 0, \\ 2^{(i+1)/2}, \ i \text{ is odd}, \\ 1.5 \cdot 2^{i/2}, \ i \text{ is even}. \end{cases}$

$$1, 2, 3, 4, 6, 8, 12, \cdots.$$

The above two sequences have the same first two elements 1 and 2. And for $i \geq 2$, the N_i in the Romberg sequence is larger than the one in the Bulirsch sequence.

3. Asymptotic expansion of the Crank-Nicolson Scheme for BSDEs

We outline this Section as follows. In Subsection 3.1, we give a brief review of the Crank-Nicolson scheme for BSDEs. And then the asymptotic expansion of this scheme is carefully derived in Subsection 3.2, which is the foundation to investigation of the Richardson extrapolation approximations. Without loss of generality, we only consider the case of one-dimensional BSDEs (i.e., $d_1 = d = q = 1$). However we remark that all results obtained in the sequel also hold for multidimensional BSDEs.

3.1. Review of the Crank-Nicolson Scheme. To begin with, we introduce a regular time partition on the time interval $[0, T]$ as

$$\pi_N := \{t_n : t_n = n\Delta t, \ n = 0, 1, \cdots, N, \ \Delta t = \frac{T}{N}\}, \tag{26}$$

where N is a positive integer. Then we introduce some notations. By $\Delta W_{r,s}$ the increment $W_s - W_r$ of the Brownian motion W_t for $s \geq r$. For simplicity, we represent $W_{t_{n+1}} - W_{t_n}$ by ΔW_{n+1} for $0 \leq n \leq N-1$. Note that the increment ΔW_{n+1} admits the Gaussian distribution with mean zero and variance Δt.

It follows from (1) that

$$Y_{t_n} = Y_{t_{n+1}} + \int_{t_n}^{t_{n+1}} f_s \, \mathrm{d}s - \int_{t_n}^{t_{n+1}} Z_s \, \mathrm{d}W_s, \tag{27}$$

where $f_s = f(s, X_s, Y_s, Z_s)$.

For fixed $x \in \mathbb{R}$, taking the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ on (27), we obtain

$$Y_{t_n} = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f_s] \, \mathrm{d}s. \tag{28}$$

We use the following CN scheme to approximate the integral in (28):

$$\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f_s] \, \mathrm{d}s = \frac{1}{2}\Delta t f_{t_n} + \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[f_{t_{n+1}}] + R_y^n, \tag{29}$$

where

$$R_y^n = \int_{t_n}^{t_{n+1}} \left(\mathbb{E}_{t_n}^x[f_s] - \frac{1}{2} f_{t_n} - \frac{1}{2}\mathbb{E}_{t_n}^x[f_{t_{n+1}}] \right) \mathrm{d}s. \tag{30}$$

Inserting (29) into (28) leads to the following reference equation

$$Y_{t_n} = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \frac{1}{2}\Delta t f_{t_n} + \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[f_{t_{n+1}}] + R_y^n. \tag{31}$$

By multiplying ΔW_{n+1} on both sides of (27), taking conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ and then using the isometry property of Itô's integral, we obtain

$$0 = \mathbb{E}_{t_n}^x[Y_{t_{n+1}} \Delta W_{n+1}] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f_s \Delta W_{t_n,s}] \, \mathrm{d}s - \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[Z_s] \, \mathrm{d}s. \tag{32}$$

We rewrite the two standard integrals on the right-hand side of (32) in the following forms.

$$\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f_s \Delta W_{t_n,s}]\,\mathrm{d}s = \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[f_{t_{n+1}}\Delta W_{n+1}] + R_{z1}^n, \tag{33}$$

$$-\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[Z_s]\,\mathrm{d}s = -\frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[Z_{t_{n+1}}] - \frac{1}{2}\Delta t Z_{t_n} + R_{z2}^n, \tag{34}$$

where

$$R_{z1}^n = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f_s \Delta W_{t_n,s}]\,\mathrm{d}s - \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[f_{t_{n+1}}\Delta W_{n+1}],$$

$$R_{z2}^n = -\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[Z_s]\,\mathrm{d}s + \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[Z_{t_{n+1}}] + \frac{1}{2}\Delta t Z_{t_n}.$$

From (32), (33) and (34), we obtain

$$\begin{aligned} \frac{1}{2}\Delta t Z_{t_n} =& \mathbb{E}_{t_n}^x[Y_{t_{n+1}}\Delta W_{n+1}] + \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[f_{t_{n+1}}\Delta W_{n+1}] \\ & - \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[Z_{t_{n+1}}] + R_z^n, \end{aligned} \tag{35}$$

where $R_z^n = R_{z1}^n + R_{z2}^n$.

For the temporal semi-discretizations, we use $(Y^n,\ Z^n)$ to represent the approximate value of the solution $(Y_t,\ Z_t)$ of BSDE (1) at the time level $t = t_n$, $n = N, N-1, \cdots, 0$. Based on the two reference equations (31) and (35), we obtain the following Crank-Nicolson scheme for solving BSDEs.

Scheme 4 (Crank-Nicolson scheme)**.** *Given Y^N and Z^N, for $n = N-1, \cdots, 0$, solve random variables Y^n and Z^n by*

$$\begin{aligned} Y^n =& \mathbb{E}_{t_n}^x[Y^{n+1}] + \frac{1}{2}\Delta t f(t_n, x, Y^n, Z^n) \\ & + \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[f(t_{n+1}, X_{t_{n+1}}, Y^{n+1}, Z^{n+1})], \\ \frac{1}{2}\Delta t Z^n =& \mathbb{E}_{t_n}^x[Y^{n+1}\Delta W_{n+1}] - \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[Z^{n+1}] \\ & + \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[f(t_{n+1}, X_{t_{n+1}}, Y^{n+1}, Z^{n+1})\Delta W_{n+1}]. \end{aligned} \tag{36}$$

We call $\{(Y^n,\ Z^n)\}_{n=0}^{N-1}$ with the terminal conditions Y^N and Z^N the CN solution of BSDE (1)*.*

The above Scheme 4 is a special case of the generalized θ-scheme proposed in [43] and its error estimates were presented in [42]. It was proved in [42] that Scheme 4 possesses convergence rate of 2 for sufficiently small time step Δt under certain regularity condistions on f and φ. In this paper, we pay attention to improve the accuracy of the CN solutions of BSDEs by the Richardson extrapolation method. To this end, we shall give the asymptotic error expansions of the CN solutions which are the theoretical basis for the discussions of Richardson extrapolation method.

3.2. Asymptotic expansions of Scheme 4. The purpose of this Subsection is to deduce the asymptotic expansion of the CN Scheme 4. To this end, we first derive the asymptotic expansions of the truncation errors R_y^n and R_z^n of the CN scheme in Subsection 3.2.1. Then in Subsection 3.2.2, we define two processes $Y^{n,[K]}$ and $Z^{n,[K]}$ with given processes $e_t^{y,[j]}$ and $e_t^{z,[j]}$, $1 \leq j \leq K$, and introduce two truncation errors $R_y^{n,[K]}$ and $R_z^{n,[K]}$, which have the expansions (51) and (52), respectively. When the $e_t^{y,[j]}$ and $e_t^{z,[j]}$ are defined by the BSDE system (70), the $R_y^{n,[K]}$ and $R_z^{n,[K]}$ have the estimates given in Theorem 14. Finally by using the CN scheme and Theorem 14, we obtain the asymptotic expansion of the CN Scheme in Theorem 15.

In our analysis below, we shall need the following Assumption.

Assumption 5. *The functions φ and f in* (1) *are bounded and smooth enough with bounded derivatives.*

Remark 6. *Assumption 5 guarantees the smoothness of solution of BSDE* (1)*. It is just for the simplicity in the derivation of the asymptotic expansion of the CN scheme for BSDEs.*

3.2.1. Asymptotic expansions of R_y^n and R_z^n. For the sake of simplicity, we define the functions

$$\begin{aligned} U(t) =&\mathbb{E}_{t_n}^x[Y_t], \qquad F(t) =\mathbb{E}_{t_n}^x[f_t], \qquad V(t) = \mathbb{E}_{t_n}^x[Z_t], \\ \bar{U}(t) =&\mathbb{E}_{t_n}^x[Y_t W_{t_n,t}], \quad \bar{F}(t) =\mathbb{E}_{t_n}^x[f_t W_{t_n,t}]. \end{aligned} \tag{37}$$

Note that U, V, F, $\bar{U}$ and $\bar{F}$ depend on t, t_n and x.

Under Assumption 5, the Feynman-Kac formula (5) implies that U, V, F, $\bar{U}$ and $\bar{F}$ are all deterministic functions satisfying

$$U^{'}(t) = -F(t),\ \bar{U}(t_n) = 0,\ \bar{F}(t_n) = 0,\ \bar{U}^{'}(t_n) = V(t_n),\ V(t) = \bar{U}^{'}(t) + \bar{F}(t), \tag{38}$$

and by taking the jth derivative with respect to t on both sides of the first and the fourth equations in (37), and taking the limit $t \to t_n^+$, one obtains

$$U^{(j)}(t_n) = \left.\frac{d^j \mathbb{E}_{t_n}^x[Y_t]}{dt^j}\right|_{t\to t_n^+}, \tag{39}$$

$$\bar{U}^{(j)}(t_n) = \left.\frac{d^j \mathbb{E}_{t_n}^x[Y_t \Delta W_{t_n,t}]}{dt^j}\right|_{t\to t_n^+}. \tag{40}$$

On R_y^n and R_z^n in (31) and (35), respectively, we have the following Lemma.

Lemma 7. *Under Assumption 5, the local truncation errors R_y^n and R_z^n defined in* (31) *and* (35)*, respectively, have the asymptotic expansions*

$$\begin{cases} R_y^n = \displaystyle\sum_{j=3}^{2K+2} \gamma_{t_n,j}(\Delta t)^j + \mathcal{O}\left((\Delta t)^{2K+3}\right), \\ R_z^n = \displaystyle\sum_{j=3}^{2K+2} \zeta_{t_n,j}(\Delta t)^j + \mathcal{O}\left((\Delta t)^{2K+3}\right), \end{cases} \tag{41}$$

where $\gamma_{t_n,j} = \frac{(j-2)U^{(j)}(t_n)}{2\cdot j!}$ and $\zeta_{t_n,j} = \frac{(j-2)\bar{U}^{(j)}(t_n)}{2\cdot j!}$.

Proof. By (31), (35) and (37), we obtain

$$R_y^n =U(t_n) - U(t_{n+1}) - \frac{1}{2}\Delta t F(t_n) - \frac{1}{2}\Delta t F(t_{n+1}), \tag{42}$$

$$R_z^n = \frac{1}{2}\Delta t V(t_n) - \bar{U}(t_{n+1}) + \frac{1}{2}\Delta t V(t_{n+1}) - \frac{1}{2}\Delta t \bar{F}(t_{n+1}). \tag{43}$$

Then taking Taylor's expansion on U, V, $\bar{U}$, F and $\bar{F}$ at $t = t_n$ and using (38), we deduce

$$\begin{aligned} R_y^n = &- \sum_{j=1}^{2K+2} \frac{U^{(j)}(t_n)}{j!}(\Delta t)^j - \frac{1}{2}\Delta t\left(2F(t_n) + \sum_{j=1}^{2K+1}\frac{F^{(j)}(t_n)}{j!}(\Delta t)^j\right) \\ &+ \mathcal{O}(\Delta t)^{2K+3} \\ =& \sum_{j=3}^{2K+2} \frac{(j-2)U^{(j)}(t_n)}{2\cdot j!}(\Delta t)^j + \mathcal{O}(\Delta t)^{2K+3} \end{aligned} \tag{44}$$

and

$$\begin{aligned} R_z^n =& \frac{1}{2}\Delta t\left(2V(t_n) + \sum_{j=1}^{2K+1}\frac{V^{(j)}(t_n)}{j!}(\Delta t)^j\right) - \sum_{j=1}^{2K+2}\frac{\bar{U}^{(j)}(t_n)}{j!}(\Delta t)^j \\ &- \frac{1}{2}\Delta t \sum_{j=1}^{2K+1}\frac{\bar{F}^{(j)}(t_n)}{j!}(\Delta t)^j + \mathcal{O}(\Delta t)^{2K+3} \\ =& \sum_{j=3}^{2K+2}\frac{(j-2)\bar{U}^{(j)}(t_n)}{2\cdot j!}(\Delta t)^j + \mathcal{O}(\Delta t)^{2K+3}. \end{aligned} \tag{45}$$

The proof ends. □

3.2.2. Asymptotic expansion of the Crank-Nicolson Scheme 4. Define $Y^{n,[K]}$ and $Z^{n,[K]}$ as

$$Y^{n,[K]} = Y^n - \sum_{j=1}^{K} e_{t_n}^{y,[j]}(\Delta t)^{2j},\ Z^{n,[K]} = Z^n - \sum_{j=1}^{K} e_{t_n}^{z,[j]}(\Delta t)^{2j}, \tag{46}$$

where $e_t^{y,[j]}$ and $e_t^{z,[j]}$, $1 \le j \le K$ are undetermined processes. By the Crank-Nicolson Scheme 4, we have the two identities

$$\begin{aligned} Y^{n,[K]} =& \mathbb{E}_{t_n}^x[Y^{n+1,[K]}] + \sum_{j=1}^{K}\left(E^{y,[j]}(t_{n+1}) - E^{y,[j]}(t_n)\right)(\Delta t)^{2j} \\ &+ \frac{1}{2}\Delta t f^{[K]}\left(t_n, x, Y^{n,[K]}, Z^{n,[K]}\right) \\ &+ \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x\left[f^{[K]}\left(t_{n+1}, X_{t_{n+1}}, Y^{n+1,[K]}, Z^{n+1,[K]}\right)\right], \\ \frac{1}{2}\Delta t Z^{n,[K]} =& \mathbb{E}_{t_n}^x[Y^{n+1,[K]}\Delta W_{n+1}] - \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[Z^{n+1,[K]}] \\ &+ \sum_{j=1}^{K}\left(\bar{E}^{y,[j]}(t_{n+1}) - \frac{1}{2}\Delta t E^{z,[j]}(t_n) - \frac{1}{2}\Delta t E^{z,[j]}(t_{n+1})\right)(\Delta t)^{2j} \\ &+ \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x\left[f^{[K]}\left(t_{n+1}, X_{t_{n+1}}, Y^{n+1,[K]}, Z^{n+1,[K]}\right)\Delta W_{n+1}\right], \end{aligned} \tag{47}$$

where

$$E^{y,[j]}(t) = \mathbb{E}_{t_n}^x[e_t^{y,[j]}],\ E^{z,[j]}(t) = \mathbb{E}_{t_n}^x[e_t^{z,[j]}],\ \bar{E}^{y,[j]}(t) = \mathbb{E}_{t_n}^x[e_t^{y,[j]}\Delta W_{t_n,t}] \tag{48}$$

and

$$f^{[K]}(t,x,y,z) = f\big(t, x, y + \sum_{j=1}^{K} E^{y,[j]}(t)(\Delta t)^{2j}, z + \sum_{j=1}^{K} E^{z,[j]}(t)(\Delta t)^{2j}\big). \tag{49}$$

Now we define local truncation errors $R_y^{n,[K]}$ and $R_z^{n,[K]}$ as

$$
\begin{aligned}
R_y^{n,[K]} =& Y_{t_n} - \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] - \sum_{j=1}^{K}\left(E^{y,[j]}(t_{n+1}) - E^{y,[j]}(t_n)\right)(\Delta t)^{2j} \\
& - \frac{1}{2}\Delta t f^{[K]}\left(t_n, x, Y_{t_n}, Z_{t_n}\right) \\
& - \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x\left[f^{[K]}\left(t_{n+1}, X_{t_{n+1}}, Y_{t_{n+1}}, Z_{t_{n+1}}\right)\right], \\
R_z^{n,[K]} =& \frac{1}{2}\Delta t Z_{t_n} - \mathbb{E}_{t_n}^x[Y_{t_{n+1}}\Delta W_{n+1}] + \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[Z_{t_{n+1}}] \\
& - \sum_{j=1}^{K}\left(\bar{E}^{y,[j]}(t_{n+1}) - \frac{1}{2}\Delta t E^{z,[j]}(t_n) - \frac{1}{2}\Delta t E^{z,[j]}(t_{n+1})\right)(\Delta t)^{2j} \\
& - \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x\left[f^{[K]}\left(t_{n+1}, X_{t_{n+1}}, Y_{t_{n+1}}, Z_{t_{n+1}}\right)\Delta W_{n+1}\right],
\end{aligned} \tag{50}
$$

where $(Y_t,\, Z_t)$ is the solution of BSDE (1). About $R_y^{n,[K]}$ and $R_z^{n,[K]}$, we have the following Lemma.

Lemma 8. *Under Assumption 5, we have*

$$
R_y^{n,[K]} = \sum_{j=3}^{2K+2}\mathbb{A}_y^j(\Delta t)^j + \mathcal{O}\left((\Delta t)^{2K+3}\right), \tag{51}
$$

$$
R_z^{n,[K]} = \sum_{j=3}^{2K+2}\mathbb{A}_z^j(\Delta t)^j + \mathcal{O}\left((\Delta t)^{2K+3}\right), \tag{52}
$$

where $\mathbb{A}_y^j$ and $\mathbb{A}_z^j$ are defined by

$$
\mathbb{A}_y^j = \begin{cases} \mathbb{A}_{y,o}^j, & j \text{ is odd}, \\ \mathbb{A}_{y,e}^j, & j \text{ is even}, \end{cases} \quad \mathbb{A}_z^j = \begin{cases} \mathbb{A}_{z,o}^j, & j \text{ is odd}, \\ \mathbb{A}_{z,e}^j, & j \text{ is even} \end{cases}
$$

with

$$
\begin{aligned}
\mathbb{A}_{y,o}^j =& \frac{j-2}{2\cdot j!}\left.\frac{d^j\mathbb{E}_{t_n}^x[Y_t]}{dt^j}\right|_{t\to t_n^+} - \sum_{l=2}^{j-3}\mathbb{I}_{\{l \text{ is even}\}}\frac{1}{2\cdot l!}\mathbb{B}_{\frac{j-l-1}{2}}^{(l)}(t_n) \\
& - \mathbb{B}_{\frac{j-1}{2}}(t_n) - \sum_{l=1}^{j-2}\mathbb{I}_{\{l \text{ is odd}\}}\left.\frac{1}{l!}\frac{d^l\mathbb{E}_{t_n}^x[e_t^{y,[\frac{j-l}{2}]}]}{dt^l}\right|_{t\to t_n^+},
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
\mathbb{A}_{z,o}^j =& \frac{j-2}{2\cdot j!}\left.\frac{d^j\mathbb{E}_{t_n}^x[Y_t\Delta W_{t_n,t}]}{dt^j}\right|_{t\to t_n^+} \\
& - \sum_{l=2}^{j-3}\mathbb{I}_{\{l \text{ is even}\}}\frac{1}{2\cdot l!}\left(\bar{\mathbb{B}}_{\frac{j-l-1}{2}}^{(l)}(t_n) - \left.\frac{d^l\mathbb{E}_{t_n}^x[e_t^{z,[\frac{j-l-1}{2}]}]}{dt^l}\right|_{t\to t_n^+}\right) \\
& + e_{t_n}^{z,[\frac{j-1}{2}]} - \sum_{l=1}^{j-2}\mathbb{I}_{\{l \text{ is odd}\}}\left.\frac{1}{l!}\frac{d^l\mathbb{E}_{t_n}^x[e_t^{y,[\frac{j-l}{2}]}\Delta W_{t_n,t}]}{dt^l}\right|_{t\to t_n^+},
\end{aligned} \tag{54}
$$

$$\begin{aligned}\mathbb{A}_{y,e}^{j} =&\frac{j-2}{2\cdot j!}\left.\frac{d^j\mathbb{E}_{t_n}^x[Y_t]}{dt^j}\right|_{t\to t_n^+} - \sum_{l=1}^{j-3}\mathbb{I}_{\{l\text{ is odd}\}}\frac{1}{2\cdot l!}\mathbb{B}_{\frac{j-l-1}{2}}^{(l)}(t_n)\\ &-\sum_{l=2}^{j-2}\mathbb{I}_{\{l\text{ is even}\}}\left.\frac{1}{l!}\frac{d^l\mathbb{E}_{t_n}^x[e_t^{y,[\frac{j-l}{2}]}]}{dt^l}\right|_{t\to t_n^+}\end{aligned}\tag{55}$$

and

$$\begin{aligned}\mathbb{A}_{z,e}^{j} =&\frac{j-2}{2\cdot j!}\left.\frac{d^j\mathbb{E}_{t_n}^x[Y_t\Delta W_{t_n,t}]}{dt^j}\right|_{t\to t_n^+}\\ &-\sum_{l=1}^{j-3}\mathbb{I}_{\{l\text{ is odd}\}}\frac{1}{2\cdot l!}\left(\bar{\mathbb{B}}_{\frac{j-l-1}{2}}^{(l)}(t_n) - \left.\frac{d^l\mathbb{E}_{t_n}^x[e_t^{z,[\frac{j-l-1}{2}]}]}{dt^l}\right|_{t\to t_n^+}\right)\\ &-\sum_{l=2}^{j-2}\mathbb{I}_{\{l\text{ is even}\}}\frac{1}{l!}\left.\frac{d^l\mathbb{E}_{t_n}^x[e_t^{y,[\frac{j-l}{2}]}\Delta W_{t_n,t}]}{dt^l}\right|_{t\to t_n^+},\end{aligned}\tag{56}$$

where $\mathbb{B}_j^{(l)}(t_n) = \left.\frac{d^l\mathbb{E}_{t_n}^x[B_j(t)]}{dt^l}\right|_{t\to t_n^+}$, $\bar{\mathbb{B}}_j^{(l)}(t_n) = \left.\frac{d^l\mathbb{E}_{t_n}^x[B_j(t)\Delta W_{t_n,t}]}{dt^l}\right|_{t\to t_n^+}$. *In particular,* $\mathbb{B}_j(t) = \mathbb{E}_{t_n}^x[B_j(t)]$, $\bar{\mathbb{B}}_j(t) = \mathbb{E}_{t_n}^x[B_j(t)\Delta W_{t_n,t}]$ *with*

$$B_j(t) =\frac{1}{j!}\left[\frac{d^j}{d\gamma^j}f\left(t, X_t, Y_t+\sum_{i=1}^{K}\gamma^i e_t^{y,[i]}(\Delta t)^{2i}, Z_t+\sum_{i=1}^{K}\gamma^i e_t^{z,[i]}(\Delta t)^{2i}\right)\right]_{\gamma=0}.\tag{57}$$

Proof. By (37) and (50), we have

$$\begin{aligned}R_y^{n,[K]} =&U(t_n)-U(t_{n+1})-\frac{1}{2}\Delta tF^{[K]}(t_n)-\frac{1}{2}\Delta tF^{[K]}(t_{n+1})\\ &-\sum_{j=1}^{K}\left(E^{y,[j]}(t_{n+1})-E^{y,[j]}(t_n)\right)(\Delta t)^{2j},\end{aligned}\tag{58}$$

$$\begin{aligned}R_z^{n,[K]} =&\frac{1}{2}\Delta tV(t_n)+\frac{1}{2}\Delta tV(t_{n+1})-\bar{U}(t_{n+1})-\frac{1}{2}\Delta t\bar{F}^{[K]}(t_{n+1})\\ &-\sum_{j=1}^{K}\left(\bar{E}^{y,[j]}(t_{n+1})-\frac{1}{2}\Delta tE^{z,[j]}(t_n)-\frac{1}{2}\Delta tE^{z,[j]}(t_{n+1})\right)(\Delta t)^{2j},\end{aligned}\tag{59}$$

where $F^{[K]}(t) = \mathbb{E}_{t_n}^x\left[f^{[K]}(t, X_t, Y_t, Z_t)\right]$ and $\bar{F}^{[K]}(t) = \mathbb{E}_{t_n}^x\left[f^{[K]}(t, X_t, Y_t, Z_t)\Delta W_{t_n,t}\right]$. By using the Adomian decomposition to $f^{[K]}$ defined by (49) in (58) and (59), we deduce

$$f^{[K]}(t_n, x, Y_{t_n}, Z_{t_n}) = \sum_{j=0}^{K}B_j(t_n)(\Delta t)^{2j}+\mathcal{O}\left((\Delta t)^{2(K+1)}\right),\tag{60}$$

$$f^{[K]}(t_{n+1}, X_{t_{n+1}}, Y_{t_{n+1}}, Z_{t_{n+1}}) = \sum_{j=0}^{K}B_j(t_{n+1})(\Delta t)^{2j}+\mathcal{O}\left((\Delta t)^{2(K+1)}\right),\tag{61}$$

where the function $B_j(t)$ is defined by (57). Inserting (60) and (61) into (58), and then by (31) and (37), we obtain

$$\begin{aligned}R_y^{n,[K]} =&R_y^n-\frac{1}{2}\Delta t\sum_{j=1}^{K}\mathbb{B}_j(t_n)(\Delta t)^{2j}-\frac{1}{2}\Delta t\sum_{j=1}^{K}\mathbb{B}_j(t_{n+1})(\Delta t)^{2j}\\ &-\sum_{j=1}^{K}\left(E^{y,[j]}(t_{n+1})-E^{y,[j]}(t_n)\right)(\Delta t)^{2j}+\mathcal{O}\left((\Delta t)^{2K+3}\right).\end{aligned}\tag{62}$$

Substituting (61) into (59), and then by (35) and (37), we deduce

$$
\begin{aligned}
R_z^{n,[K]} =& R_z^n - \frac{1}{2}\Delta t\sum_{j=1}^{K}\bar{\mathbb{B}}_j(t_{n+1})(\Delta t)^{2j} \\
&- \sum_{j=1}^{K}\left(\bar{E}^{y,[j]}(t_{n+1}) - \frac{1}{2}\Delta t E^{z,[j]}(t_n) - \frac{1}{2}\Delta t E^{z,[j]}(t_{n+1})\right)(\Delta t)^{2j} \\
&+ \mathcal{O}\left((\Delta t)^{2K+3}\right).
\end{aligned} \tag{63}
$$

By Taylor's expansions of $\mathbb{B}_j(t_{n+1})$ and $\bar{\mathbb{B}}_j(t_{n+1})$ at $t = t_n$, respectively, we have

$$
\mathbb{B}_j(t_{n+1}) = \sum_{l=0}^{2K-2j+1}\frac{1}{l!}\mathbb{B}_j^{(l)}(t_n)(\Delta t)^l + \mathcal{O}\left((\Delta t)^{2K-2j+2}\right). \tag{64}
$$

$$
\bar{\mathbb{B}}_j(t_{n+1}) = \sum_{l=1}^{2K-2j+1}\frac{1}{l!}\bar{\mathbb{B}}_j^{(l)}(t_n)(\Delta t)^l + \mathcal{O}\left((\Delta t)^{2K-2j+2}\right). \tag{65}
$$

By the definitions of $E^{y,[j]}$, $\bar{E}^{y,[j]}$ and $E^{z,[j]}$ in (48), and Taylor's expansions again, we obtain

$$
\begin{aligned}
& E^{y,[j]}(t_{n+1}) - E^{y,[j]}(t_n) \\
=& \sum_{l=1}^{2K-2j+2}\frac{1}{l!}(E^{y,[j]})^{(l)}(t_n)(\Delta t)^l + \mathcal{O}\left((\Delta t)^{2K-2j+3}\right) \\
=& \sum_{l=1}^{2K-2j+2}\frac{1}{l!}\left.\frac{d^l\mathbb{E}_{t_n}^x[e_t^{y,[j]}]}{dt^l}\right|_{t\to t_n^+}(\Delta t)^l + \mathcal{O}\left((\Delta t)^{2K-2j+3}\right),
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
& \bar{E}^{y,[j]}(t_{n+1}) - \frac{1}{2}\Delta t E^{z,[j]}(t_n) - \frac{1}{2}\Delta t E^{z,[j]}(t_{n+1}) \\
=& \sum_{l=1}^{2K-2j+2}\left(\left.\frac{d^l\mathbb{E}_{t_n}^x[e_t^{y,[j]}\Delta W_{t_n,t}]}{dt^l}\right|_{t\to t_n^+} - \frac{\Delta t}{2\cdot l!}\left.\frac{d^l\mathbb{E}_{t_n}^x[e_t^{z,[j]}]}{dt^l}\right|_{t\to t_n^+}\right)(\Delta t)^l \\
& - \Delta t e_{t_n}^{z,[j]} + \mathcal{O}\left((\Delta t)^{2K-2j+3}\right).
\end{aligned} \tag{67}
$$

Substituting (64) and (66) into (62), and by (44), we deduce

$$
\begin{aligned}
R_y^{n,[K]} =& R_y^n - \frac{1}{2}\Delta t\sum_{j=1}^{K}\mathbb{B}_j(t_n)(\Delta t)^{2j} \\
&- \frac{1}{2}\Delta t\sum_{j=1}^{K}\left(\sum_{l=0}^{2K-2j+1}\frac{1}{l!}\mathbb{B}_j^{(l)}(t_n)(\Delta t)^l\right)(\Delta t)^{2j} \\
&- \sum_{j=1}^{K}\left(\sum_{l=1}^{2K-2j+2}\frac{1}{l!}\left.\frac{d^l\mathbb{E}_{t_n}^x[e_t^{y,[j]}]}{dt^l}\right|_{t\to t_n^+}(\Delta t)^l\right)(\Delta t)^{2j} + \mathcal{O}\left((\Delta t)^{2K+3}\right) \\
=& \sum_{j=3}^{2K+2}\mathbb{A}_y^j(\Delta t)^j + \mathcal{O}\left((\Delta t)^{2K+3}\right).
\end{aligned} \tag{68}
$$

Inserting (65) and (67) into (63), and by (45), we obtain

$$
\begin{aligned}
R_z^{n,[K]} =& R_z^n - \frac{1}{2}\Delta t \sum_{j=1}^{K}\left(\sum_{l=1}^{2K-2j+1}\frac{1}{l!}\bar{\mathbb{B}}_j^{(l)}(t_n)(\Delta t)^l\right)(\Delta t)^{2j} \\
& - \sum_{j=1}^{K}\left(\sum_{l=1}^{2K-2j+2}\left(\left.\frac{d^l\mathbb{E}_{t_n}^x[e_t^{y,[j]}\Delta W_{t_n,t}]}{dt^l}\right|_{t\to t_n^+}\right.\right. \\
& \left.\left. - \frac{\Delta t}{2\cdot l!}\left.\frac{d^l\mathbb{E}_{t_n}^x[e_t^{z,[j]}]}{dt^l}\right|_{t\to t_n^+}\right)(\Delta t)^l - \Delta t e_{t_n}^{z,[j]}\right)(\Delta t)^{2j} + \mathcal{O}\left((\Delta t)^{2K+3}\right) \\
=& \sum_{j=3}^{2K+2}\mathbb{A}_z^j(\Delta t)^j + \mathcal{O}\left((\Delta t)^{2K+3}\right).
\end{aligned}
\tag{69}
$$

The proof ends. □

Now, we introduce $\mathcal{F}_t$-adapted stochastic processes $(e_t^{y,[\frac{j-1}{2}]}, e_t^{z,[\frac{j-1}{2}]})$, $j \in \mathbb{I}_K := \{2i+1 | i = 1, 2, \cdots, K\}$, $t \in [0,T]$, which are the solutions of the following system of BSDEs (70). These processes will be used in our asymptotic expansion of the solution of the Crank-Nicolson scheme for BSDE (1).

$$
\begin{aligned}
e_t^{y,[\frac{j-1}{2}]} =& \int_t^T \left(\lambda_s^{y,[\frac{j-1}{2}]} + \lambda_s^y e_s^{y,[\frac{j-1}{2}]} + \lambda_s^z e_s^{z,[\frac{j-1}{2}]}\right) \mathrm{d}s \\
& - \int_t^T \left(e_s^{z,[\frac{j-1}{2}]} + \lambda_s^{z,[\frac{j-1}{2}]}\right) \mathrm{d}W_s,
\end{aligned}
\tag{70}
$$

where λ_s^y, λ_s^z, $\lambda_s^{y,[\frac{j-1}{2}]}$ and $\lambda_s^{z,[\frac{j-1}{2}]}$ are defined by

$$
\begin{aligned}
\lambda_s^y =& \frac{\partial f}{\partial y}(s, X_s, Y_s, Z_s),\ \lambda_s^z = \frac{\partial f}{\partial z}(s, X_s, Y_s, Z_s), \\
\lambda_s^{y,[\frac{j-1}{2}]} =& -\frac{(j-2)Y_s^{\langle j\rangle}}{2\cdot j!} + B_{\frac{j-1}{2}}(s) - \lambda_s^y e_s^{y,[\frac{j-1}{2}]} - \lambda_s^z e_s^{z,[\frac{j-1}{2}]} \\
& + \sum_{l=2}^{j-3}\mathbb{I}_{\{l \text{ is even}\}}\frac{1}{2\cdot l!}B_{\frac{j-l-1}{2}}^{\langle l\rangle}(s) + \sum_{l=1}^{j-2}\mathbb{I}_{\{l \text{ is odd}\}}\frac{1}{l!}(e_s^{y,[\frac{j-l}{2}]})^{\langle l\rangle}, \\
\lambda_s^{z,[\frac{j-1}{2}]} =& \frac{(j-2)\bar{Y}_s^{\langle j\rangle}}{2\cdot j!} - \sum_{l=2}^{j-3}\mathbb{I}_{\{l \text{ is even}\}}\frac{1}{2\cdot l!}\left(\bar{B}_{\frac{j-l-1}{2}}^{\langle l\rangle}(s) - (e_s^{z,[\frac{j-l-1}{2}]})^{\langle l\rangle}\right) \\
& - \sum_{l=1}^{j-2}\mathbb{I}_{\{l \text{ is odd}\}}\frac{1}{l!}(\bar{e}_s^{y,[\frac{j-l}{2}]})^{\langle l\rangle}
\end{aligned}
\tag{71}
$$

with

$$
\begin{aligned}
Y_s^{\langle j\rangle} =& (L^0)^{(j)}u(s, X_s),\ \bar{Y}_s^{\langle j\rangle} = (L^0)^{(j)}\tilde{u}(s, \Delta W_{t,s}), \\
(e_s^{y,[\frac{j-l}{2}]})^{\langle l\rangle} =& (L^0)^{(l)}u^{[\frac{j-l}{2}]}(s, X_s),\ \bar{B}_{\frac{j-l-1}{2}}(s) = B_{\frac{j-l-1}{2}}(s)\Delta W_{t,s} \\
(e_s^{z,[\frac{j-l-1}{2}]})^{\langle l\rangle} =& (L^0)^{(l)}\left(\nabla_x u^{[\frac{j-l-1}{2}]}(s, X_s) - \lambda_s^{z,[\frac{j-l-1}{2}]}\right), \\
(\bar{e}_s^{y,[\frac{j-l}{2}]})^{\langle l\rangle} =& (L^0)^{(l)}\tilde{u}^{[\frac{j-l}{2}]}(s, \Delta W_{t,s}).
\end{aligned}
\tag{72}
$$

Here $\tilde{u}(s, \Delta W_{t,s}) = u(s, X_s)\Delta W_{t,s}$, $\tilde{u}^{[\frac{j-l}{2}]}(s, \Delta W_{t,s}) = u^{[\frac{j-l}{2}]}(s, X_s)\Delta W_{t,s}$, L^0 is defined in (4), and $(L^0)^{(k)}$ is defined in Corollary 3, where the $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ is the solution of the PDE

$$
L^0 u(t,x) + f(t, u(t,x), \nabla_x u(t,x)) = 0,\ (t,x) \in [0,T) \times \mathbb{R}
$$

with the terminal condition $u(T,x)=\varphi(x)$, and $u^{[\frac{j-l}{2}]}:[0,T]\times\mathbb{R}\to\mathbb{R}$, are the solutions of the PDEs

$$\begin{aligned}&L^0u^{[\frac{j-l}{2}]}(t,x)+\lambda_t^{y,[\frac{j-l}{2}]}-\lambda_t^z\lambda_t^{z,[\frac{j-l}{2}]}\\&+\lambda_t^yu^{[\frac{j-l}{2}]}(t,x)+\lambda_t^z\nabla_xu^{[\frac{j-l}{2}]}(t,x)=0,\ (t,x)\in[0,T)\times\mathbb{R}\end{aligned}$$

with the terminal conditions $u^{[\frac{j-l}{2}]}(T,x)=0$, $1\le l\le j-2$, $j\in\mathbb{I}_K$.

Remark 9. *Let $\tilde{e}_s^{z,[\frac{j-1}{2}]}=e_s^{z,[\frac{j-1}{2}]}+\lambda_s^{z,[\frac{j-1}{2}]}$, then for all $t\in[0,T]$, (70) can be written as*

$$\begin{aligned}e_t^{y,[\frac{j-1}{2}]}=&\int_t^T\left(\lambda_s^{y,[\frac{j-1}{2}]}-\lambda_s^z\lambda_s^{z,[\frac{j-1}{2}]}+\lambda_s^ye_s^{y,[\frac{j-1}{2}]}+\lambda_s^z\tilde{e}_s^{z,[\frac{j-1}{2}]}\right)\mathrm{d}s\\&-\int_t^T\tilde{e}_s^{z,[\frac{j-1}{2}]}\,\mathrm{d}W_s,\ j\in\mathbb{I}_K.\end{aligned}\tag{73}$$

Note that the BSDEs (73) are linear with unknown $(e_t^{y,[\frac{j-1}{2}]},\tilde{e}_t^{z,[\frac{j-1}{2}]})$ and the unique solvability of (73) can be guaranteed by Assumption 5 which implies that the BSDEs (70) have the unique solutions $(e_t^{y,[\frac{j-1}{2}]},e_t^{z,[\frac{j-1}{2}]})$, $j\in\mathbb{I}_K$.

Taking the conditional expectation $\mathbb{E}_t^x[\cdot]$ on $Y_s^{\langle j\rangle}$, $\bar{Y}_s^{\langle j\rangle}$, $(e_s^{y,[\frac{j-l}{2}]})^{\langle l\rangle}$, $(e_s^{z,[\frac{j-l-1}{2}]})^{\langle l\rangle}$ and $(\bar{e}_s^{y,[\frac{j-l}{2}]})^{\langle l\rangle}$ defined in (72), then for $t\le s\le T$, by Corollary 3, we have the identities

$$\begin{aligned}&\mathbb{E}_t^x[Y_s^{\langle j\rangle}]=\frac{d^j\mathbb{E}_t^x[Y_s]}{ds^j}, &&\mathbb{E}_t^x[\bar{Y}_s^{\langle j\rangle}]=\frac{d^j\mathbb{E}_t^x[Y_s\Delta W_{t,s}]}{ds^j},\\&\mathbb{E}_t^x[(e_s^{y,[\frac{j-l}{2}]})^{\langle l\rangle}]=\frac{d^l\mathbb{E}_t^x[e_s^{y,[\frac{j-l}{2}]}]}{ds^l}, &&\mathbb{E}_t^x[(\bar{e}_s^{y,[\frac{j-l}{2}]})^{\langle l\rangle}]=\frac{d^l\mathbb{E}_t^x[e_s^{y,[\frac{j-l}{2}]}\Delta W_{t,s}]}{ds^l},\\&\mathbb{E}_t^x[(e_s^{z,[\frac{j-l-1}{2}]})^{\langle l\rangle}]=\frac{d^l\mathbb{E}_t^x[e_s^{z,[\frac{j-l-1}{2}]}]}{ds^l}.\end{aligned}\tag{74}$$

We claim that all the coefficients $\mathbb{A}_y^j$ and $\mathbb{A}_z^j$ in (51) and (52) are equal to zeros if the process $e_t^{y,[j]}$ and $e_t^{z,[j]}$, $1\le j\le K$ in (46) are the solutions of the system of BSDEs (70). We shall show this conclusion in Lemmas 10 and 13 below.

Lemma 10. *Under Assumption 5, let $(e_t^{y,[\frac{j-1}{2}]},e_t^{z,[\frac{j-1}{2}]})$, $j\in\mathbb{I}_K$, be the solutions of BSDEs (70). Then all the $\mathbb{A}_y^j$ in (51) and $\mathbb{A}_z^j$ in (52) are zeros for $j\in\mathbb{I}_K$.*

Proof. For $t\in[t_n,T]$ and any $j\in\mathbb{I}_K$, by (73), we obtain

$$\begin{aligned}e_{t_n}^{y,[\frac{j-1}{2}]}=e_t^{y,[\frac{j-1}{2}]}&+\int_{t_n}^t\left(\lambda_s^{y,[\frac{j-1}{2}]}-\lambda_s^z\lambda_s^{z,[\frac{j-1}{2}]}+\lambda_s^ye_s^{y,[\frac{j-1}{2}]}+\lambda_s^z\tilde{e}_s^{z,[\frac{j-1}{2}]}\right)\mathrm{d}s\\&-\int_{t_n}^t\tilde{e}_s^{z,[\frac{j-1}{2}]}\,\mathrm{d}W_s.\end{aligned}\tag{75}$$

For fixed $x\in\mathbb{R}$, taking the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ on (75), we obtain

$$\begin{aligned}e_{t_n}^{y,[\frac{j-1}{2}]}=&\mathbb{E}_{t_n}^x[e_t^{y,[\frac{j-1}{2}]}]\\&+\int_{t_n}^t\mathbb{E}_{t_n}^x\left[\lambda_s^{y,[\frac{j-1}{2}]}-\lambda_s^z\lambda_s^{z,[\frac{j-1}{2}]}+\lambda_s^ye_s^{y,[\frac{j-1}{2}]}+\lambda_s^z\tilde{e}_s^{z,[\frac{j-1}{2}]}\right]\mathrm{d}s.\end{aligned}\tag{76}$$

By taking the derivative with respect to t on both sides of (76), and taking the limit $t \to t_n^+$, one obtains

(77)
$$\begin{aligned}\frac{d\mathbb{E}^x_{t_n}[e_t^{y,[\frac{j-1}{2}]}]}{dt}\bigg|_{t\to t_n^+} &= \mathbb{E}^x_{t_n}\left[-\lambda_t^{y,[\frac{j-1}{2}]} + \lambda_t^z\lambda_t^{z,[\frac{j-1}{2}]} - \lambda_t^y e_t^{y,[\frac{j-1}{2}]} - \lambda_t^z \tilde{e}_t^{z,[\frac{j-1}{2}]}\right]\bigg|_{t\to t_n^+},\\ &= \mathbb{E}^x_{t_n}\left[-\lambda_t^{y,[\frac{j-1}{2}]} - \lambda_t^y e_t^{y,[\frac{j-1}{2}]} - \lambda_t^z e_t^{z,[\frac{j-1}{2}]}\right]\bigg|_{t\to t_n^+}.\end{aligned}$$

Then, by (71) and (74), we deduce

$$\begin{aligned}\mathbb{A}_y^j =& \frac{j-2}{2\cdot j!}\,\frac{d^j\mathbb{E}^x_{t_n}[Y_t]}{dt^j}\bigg|_{t\to t_n^+} - \sum_{l=2}^{j-3}\mathbb{I}_{\{l \text{ is even}\}}\frac{1}{2\cdot l!}\mathbb{B}^{(l)}_{\frac{j-l-1}{2}}(t_n)\\ &- B_{\frac{j-1}{2}}(t_n) - \sum_{l=1}^{j-2}\mathbb{I}_{\{l \text{ is odd}\}}\,\frac{1}{l!}\frac{d^l\mathbb{E}^x_{t_n}[e_t^{y,[\frac{j-l}{2}]}]}{dt^l}\bigg|_{t\to t_n^+} = 0.\end{aligned}\tag{78}$$

By multiplying $\Delta W_{t_n,t}$ on both sides of (75), taking conditional expectation $\mathbb{E}^x_{t_n}[\cdot]$ and then using Itô's isometry property, we have

$$\begin{aligned}&-\mathbb{E}^x_{t_n}[e_t^{y,[\frac{j-1}{2}]}\Delta W_{t_n,t}]\\ =&\int_{t_n}^t \mathbb{E}^x_{t_n}[(\lambda_s^{y,[\frac{j-1}{2}]} - \lambda_s^z\lambda_s^{z,[\frac{j-1}{2}]} + \lambda_s^y e_s^{y,[\frac{j-1}{2}]} + \lambda_s^z\tilde{e}_s^{z,[\frac{j-1}{2}]})\Delta W_{t_n,s}]\,\mathrm{d}s\\ &-\int_{t_n}^t \mathbb{E}^x_{t_n}[\tilde{e}_s^{z,[\frac{j-1}{2}]}]\,\mathrm{d}s.\end{aligned}\tag{79}$$

Similarly, by (71) and (74), taking the derivative with respect to t on both sides of (79), and taking the limit $t \to t_n^+$, we deduce

$$\begin{aligned}\mathbb{A}_z^j =& \frac{j-2}{2\cdot j!}\,\frac{d^j\mathbb{E}^x_{t_n}[Y_t\Delta W_{t_n,t}]}{dt^j}\bigg|_{t\to t_n^+}\\ &- \sum_{l=2}^{j-3}\mathbb{I}_{\{l \text{ is even}\}}\frac{1}{2\cdot l!}\left(\bar{\mathbb{B}}^{(l)}_{\frac{j-l-1}{2}}(t_n) - \frac{d^l\mathbb{E}^x_{t_n}[e_t^{z,[\frac{j-l-1}{2}]}]}{dt^l}\bigg|_{t\to t_n^+}\right) + e_{t_n}^{z,[\frac{j-1}{2}]}\\ &- \sum_{l=1}^{j-2}\mathbb{I}_{\{l \text{ is odd}\}}\,\frac{1}{l!}\frac{d^l\mathbb{E}^x_{t_n}[e_t^{y,[\frac{j-l}{2}]}\Delta W_{t_n,t}]}{dt^l}\bigg|_{t\to t_n^+} = 0,\ j\in\mathbb{I}_K.\end{aligned}\tag{80}$$

The proof ends. □

And further, we will show that all the coefficients $\mathbb{A}_y^j$ and $\mathbb{A}_z^j$, $j \in \bar{\mathbb{I}}_K := \{2i+2|i = 1, 2, \cdots, K\}$ in (51) and (51), respectively, are also equal to zeros if the process $e_t^{y,[j]}$ and $e_t^{z,[j]}$, $1 \le j \le K$ in (46) are the solutions of (70). To this end, we make the following Assumption.

TABLE 1. The solution of the system of equation (81).

	$K=1$	$K=2$	$K=3$	$K=4$	$K=5$
α_3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
α_5		$(-\frac{1}{4!},\frac{1}{2})$	$(-\frac{1}{4!},\frac{1}{2})$	$(-\frac{1}{4!},\frac{1}{2})$	$(-\frac{1}{4!},\frac{1}{2})$
α_7			$(\frac{3}{6!},-\frac{1}{4!},\frac{1}{2})$	$(\frac{3}{6!},-\frac{1}{4!},\frac{1}{2})$	$(\frac{3}{6!},-\frac{1}{4!},\frac{1}{2})$
α_9				$(-\frac{17}{8!},\frac{3}{6!},-\frac{1}{4!},\frac{1}{2})$	$(-\frac{17}{8!},\frac{3}{6!},-\frac{1}{4!},\frac{1}{2})$
α_{11}					$(\frac{155}{10!},-\frac{17}{8!},\frac{3}{6!},-\frac{1}{4!},\frac{1}{2})$

Assumption 11. *Let K be a fixed positive integer. For any fixed $j \in \mathbb{I}_K$, define $\mathcal{I}_j = \{2\ell+1 | \ell = 1,2,\cdots,\frac{j-1}{2}\}$. Assume that for $j \in \mathbb{I}_K$ the equations*

$$
\left\{
\begin{aligned}
\sum_{i\in\mathcal{I}_j} \alpha_{j,i}\frac{i-2}{2\cdot i!} &= \frac{j-1}{2\cdot(j+1)!},\\
\sum_{i\in\mathcal{I}_j} \alpha_{j,i}\left(\frac{\mathbb{I}_{\{i\ge k+3\}}}{2\cdot(i-k-1)!}\right) + \alpha_{j,k+1} &= \frac{1}{2\cdot(j-k)!},\ k=2,4,\cdots,j-3,\\
\sum_{i\in\mathcal{I}_j} \alpha_{j,i}\frac{\mathbb{I}_{\{i\ge k+2\}}}{(i-k-1)!} &= \frac{1}{(j-k)!},\ k=1,3,\cdots,j-2,\\
\alpha_{j,j} &= \frac{1}{2}
\end{aligned}
\right. \tag{81}
$$

has a unique solution $\alpha_j = (\alpha_{j,3},\alpha_{j,5},\cdots,\alpha_{j,j})^\top$.

Remark 12. *For generic positive integer K, we are not able to prove the solvability of the system (81) now, but for each K, $1 \le K \le 5$, the system (81) has unique solutions α_j. We list $\alpha_j, j\in\mathbb{I}_K, 1\le K\le 5$ in Table 1.*

Lemma 13. *Under Assumption 5 and Assumption 11, let $(e_t^{y,[\frac{j-1}{2}]}, e_t^{z,[\frac{j-1}{2}]})$, $j \in \mathbb{I}_K$, be the solutions of BSDEs (70). Then all the $\mathbb{A}_y^j$ in (51) and $\mathbb{A}_z^j$ in (52), $j\in\bar{\mathbb{I}}_K$, are equal to zeros.*

Proof. Note that the set $\{j-1|j\in\bar{\mathbb{I}}_K\}$ is identical to the set $\mathbb{I}_K$. Now we give the proof in two steps.

• **Step 1: The proof of $\mathbb{A}_y^j = 0, j\in\bar{\mathbb{I}}_K$.** Given any $j\in\bar{\mathbb{I}}_K$, for $i\in\mathcal{I}_{j-1}$, similar to (76), we have

$$
\begin{aligned}
e_{t_n}^{y,[\frac{i-1}{2}]} =& \mathbb{E}_{t_n}^x[e_t^{y,[\frac{i-1}{2}]}]\\
&+\int_{t_n}^t \mathbb{E}_{t_n}^x\left[\lambda_s^{y,[\frac{i-1}{2}]} - \lambda_s^z\lambda_s^{z,[\frac{i-1}{2}]} + \lambda_s^y e_s^{y,[\frac{i-1}{2}]} + \lambda_s^z \tilde{e}_s^{z,[\frac{i-1}{2}]}\right] \mathrm{d}s.
\end{aligned} \tag{82}
$$

By taking the $(j-i+1)$th derivative with respect to t on both sides of (82), and taking the limit $t\to t_n^+$, we deduce

$$
\begin{aligned}
0 =& \frac{i-2}{2\cdot i!}\left.\frac{d^j\mathbb{E}_{t_n}^x[Y_t]}{dt^j}\right|_{t\to t_n^+} - \sum_{l=2}^{i-3}\mathbb{I}_{\{l\text{ is even}\}}\frac{1}{2\cdot l!}\mathbb{B}_{\frac{i-l-1}{2}}^{(j-i+l)}(t_n)\\
&-\mathbb{B}_{\frac{i-1}{2}}^{(j-i)}(t_n) - \sum_{l=1}^{i-2}\mathbb{I}_{\{l\text{ is odd}\}}\frac{1}{l!}\left.\frac{d^{j-i+l}\mathbb{E}_{t_n}^x[e_t^{y,[\frac{i-l}{2}]}]}{dt^{j-i+l}}\right|_{t\to t_n^+}
\end{aligned} \tag{83}
$$

$$=\frac{i-2}{2\cdot i!}\left.\frac{d^j\mathbb{E}_{t_n}^x[Y_t]}{dt^j}\right|_{t\to t_n^+}-\sum_{l=3}^{i-2}\mathbb{I}_{\{l\text{ is odd}\}}\frac{1}{2\cdot(l-1)!}\mathbb{B}_{\frac{i-l}{2}}^{(j-i+l-1)}(t_n)$$
$$-\mathbb{B}_{\frac{i-1}{2}}^{(j-i)}(t_n)-\sum_{l=2}^{i-1}\mathbb{I}_{\{l\text{ is even}\}}\frac{1}{(l-1)!}\left.\frac{d^{j-i+l-1}\mathbb{E}_{t_n}^x[e_t^{y,[\frac{i-l+1}{2}]}]}{dt^{j-i+l-1}}\right|_{t\to t_n^+}.$$

By multiplying $\alpha_{j-1,i}\in\mathbb{R}$ on both sides of (83) and adding the derived equations up for all $i\in\mathcal{I}_{j-1}$, we obtain

$$0=\sum_{i\in\mathcal{I}_{j-1}}\alpha_{j-1,i}\frac{i-2}{2\cdot i!}\left.\frac{d^j\mathbb{E}_{t_n}^x[Y_t]}{dt^j}\right|_{t\to t_n^+}-\sum_{i\in\mathcal{I}_{j-1}}\alpha_{j-1,i}\mathbb{B}_{\frac{i-1}{2}}^{(j-i)}(t_n)$$
$$-\sum_{i\in\mathcal{I}_{j-1}}\alpha_{j-1,i}\sum_{l=3}^{i-2}\mathbb{I}_{\{l\text{ is odd}\}}\frac{1}{2\cdot(l-1)!}\mathbb{B}_{\frac{i-l}{2}}^{(j-i+l-1)}(t_n)$$
$$-\sum_{i\in\mathcal{I}_{j-1}}\alpha_{j-1,i}\sum_{l=2}^{i-1}\mathbb{I}_{\{l\text{ is even}\}}\frac{1}{(l-1)!}\left.\frac{d^{j-i+l-1}\mathbb{E}_{t_n}^x[e_t^{y,[\frac{i-l+1}{2}]}]}{dt^{j-i+l-1}}\right|_{t\to t_n^+}$$

(84)
$$=\left(\sum_{i\in\mathcal{I}_{j-1}}\alpha_{j-1,i}\frac{i-2}{2\cdot i!}\right)\left.\frac{d^j\mathbb{E}_{t_n}^x[Y_t]}{dt^j}\right|_{t\to t_n^+}-\alpha_{j-1,j-1}\mathbb{B}_{\frac{j-2}{2}}^{(1)}(t_n)$$
$$-\sum_{k=2}^{j-4}\mathbb{I}_{\{k\text{ is even}\}}\left(\sum_{i\in\mathcal{I}_{j-1}}\alpha_{j-1,i}\left(\frac{\mathbb{I}_{\{i\geq k+3\}}}{2\cdot(i-k-1)!}\right)+\alpha_{j-1,k+1}\right)$$
$$\mathbb{B}_{\frac{k}{2}}^{(j-k-1)}(t_n)-\sum_{k=1}^{j-3}\mathbb{I}_{\{k\text{ is odd}\}}\left(\sum_{i\in\mathcal{I}_{j-1}}\alpha_{j-1,i}\left(\frac{\mathbb{I}_{\{i\geq k+2\}}}{(i-k-1)!}\right)\right)$$
$$\left.\frac{d^{j-k-1}\mathbb{E}_{t_n}^x[e_t^{y,[\frac{k+1}{2}]}]}{dt^{j-k-1}}\right|_{t\to t_n^+}.$$

Then letting α_{j-1}, $j-1\in\mathbb{I}_K$ be the solutions of the equations (81) in Assumption 11, we deduce

(85)
$$0=\frac{j-2}{2\cdot j!}\left.\frac{d^j\mathbb{E}_{t_n}^x[Y_t]}{dt^j}\right|_{t\to t_n^+}$$
$$-\sum_{k=2}^{j-4}\mathbb{I}_{\{k\text{ is even}\}}\frac{1}{2\cdot(j-k-1)!}\mathbb{B}_{\frac{k}{2}}^{(j-k-1)}(t_n)-\frac{1}{2}\mathbb{B}_{\frac{j-2}{2}}^{(1)}(t_n)$$
$$-\sum_{k=1}^{j-3}\mathbb{I}_{\{k\text{ is odd}\}}\frac{1}{(j-k-1)!}\left.\frac{d^{j-k-1}\mathbb{E}_{t_n}^x[e_t^{y,[\frac{k+1}{2}]}]}{dt^{j-k-1}}\right|_{t\to t_n^+},$$

or equivalently

(86)
$$0=\frac{j-2}{2\cdot j!}\left.\frac{d^j\mathbb{E}_{t_n}^x[Y_t]}{dt^j}\right|_{t\to t_n^+}-\sum_{l=1}^{j-3}\mathbb{I}_{\{l\text{ is odd}\}}\frac{1}{2\cdot l!}\mathbb{B}_{\frac{j-l-1}{2}}^{(l)}(t_n)$$
$$-\sum_{l=2}^{j-2}\mathbb{I}_{\{l\text{ is even}\}}\frac{1}{l!}\left.\frac{d^l\mathbb{E}_{t_n}^x[e_t^{y,[\frac{j-l}{2}]}]}{dt^l}\right|_{t\to t_n^+},\ j\in\bar{\mathbb{I}}_K.$$

By the definition of $\mathbb{A}_y^j$, we deduce $\mathbb{A}_y^j=0$ for $j\in\bar{\mathbb{I}}_K$.

• **Step 2: The proof of $\mathbb{A}_z^j = 0, j \in \bar{\mathbb{I}}_K$.** Given any $j \in \bar{\mathbb{I}}_K$, for $i \in \mathcal{I}_{j-1}$, from (75), we have

$$
\begin{aligned}
e_{t_n}^{y,[\frac{i-1}{2}]} = e_t^{y,[\frac{i-1}{2}]} &+ \int_{t_n}^t \left(\lambda_s^{y,[\frac{i-1}{2}]} - \lambda_s^z \lambda_s^{z,[\frac{i-1}{2}]} + \lambda_s^y e_s^{y,[\frac{i-1}{2}]} + \lambda_s^z \tilde{e}_s^{z,[\frac{i-1}{2}]}\right) \mathrm{d}s \\
&- \int_{t_n}^t \tilde{e}_s^{z,[\frac{i-1}{2}]} \,\mathrm{d}W_s.
\end{aligned} \tag{87}
$$

By multiplying $\Delta W_{t_n,t}$ on both sides of the equation (87), taking conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ and then using the isometry property of Itô's integral, we have

$$
\begin{aligned}
&- \mathbb{E}_{t_n}^x[e_t^{y,[\frac{i-1}{2}]} \Delta W_{t_n,t}] \\
=& \int_{t_n}^t \mathbb{E}_{t_n}^x[(\lambda_s^{y,[\frac{i-1}{2}]} - \lambda_s^z \lambda_s^{z,[\frac{i-1}{2}]} + \lambda_s^y e_s^{y,[\frac{i-1}{2}]} + \lambda_s^z \tilde{e}_s^{z,[\frac{i-1}{2}]}) \Delta W_{t_n,s}] \,\mathrm{d}s \\
&- \int_{t_n}^t \mathbb{E}_{t_n}^x[\tilde{e}_s^{z,[\frac{i-1}{2}]}] \,\mathrm{d}s.
\end{aligned} \tag{88}
$$

Taking the $(j-i+1)$th derivative with respect to t on both sides of (88), and then taking the limit $t \to t_n^+$, we deduce

$$
\begin{aligned}
0 =& \frac{i-2}{2\cdot i!} \left.\frac{d^j \mathbb{E}_{t_n}^x[Y_t \Delta W_{t_n,t}]}{dt^j}\right|_{t\to t_n^+} - \sum_{l=2}^{i-3} \mathbb{I}_{\{l \text{ is even}\}} \frac{1}{2\cdot l!} \Bigg(\bar{\mathbb{B}}_{\frac{i-l-1}{2}}^{(j-i+l)}(t_n) \\
&- \left.\frac{d^{j-i+l}\mathbb{E}_{t_n}^x[e_t^{z,[\frac{i-l-1}{2}]}]}{dt^{j-i+l}}\right|_{t\to t_n^+}\Bigg) + \left.\frac{d^{j-i}\mathbb{E}_{t_n}^x[e_t^{z,[\frac{i-1}{2}]}]}{dt^{j-i}}\right|_{t\to t_n^+} + \bar{\mathbb{B}}_{\frac{i-1}{2}}^{(j-i)}(t_n) \\
&- \sum_{l=1}^{i-2} \mathbb{I}_{\{l \text{ is odd}\}} \frac{1}{l!} \left.\frac{d^{j-i+l}\mathbb{E}_{t_n}^x[e_t^{y,[\frac{i-l}{2}]}\Delta W_{t_n,t}]}{dt^{j-i+l}}\right|_{t\to t_n^+} \\
=& \frac{i-2}{2\cdot i!} \left.\frac{d^j \mathbb{E}_{t_n}^x[Y_t \Delta W_{t_n,t}]}{dt^j}\right|_{t\to t_n^+} - \sum_{l=3}^{i-2} \mathbb{I}_{\{l \text{ is odd}\}} \frac{1}{2\cdot (l-1)!} \Bigg(\bar{\mathbb{B}}_{\frac{i-l}{2}}^{(j-i+l-1)}(t_n) \\
&- \left.\frac{d^{j-i+l-1}\mathbb{E}_{t_n}^x[e_t^{z,[\frac{i-l}{2}]}]}{dt^{j-i+l-1}}\right|_{t\to t_n^+}\Bigg) + \left.\frac{d^{j-i}\mathbb{E}_{t_n}^x[e_t^{z,[\frac{i-1}{2}]}]}{dt^{j-i}}\right|_{t\to t_n^+} + \bar{\mathbb{B}}_{\frac{i-1}{2}}^{(j-i)}(t_n) \\
&- \sum_{l=2}^{i-1} \mathbb{I}_{\{l \text{ is even}\}} \frac{1}{(l-1)!} \left.\frac{d^{j-i+l-1}\mathbb{E}_{t_n}^x[e_t^{y,[\frac{i-l+1}{2}]}\Delta W_{t_n,t}]}{dt^{j-i+l-1}}\right|_{t\to t_n^+}.
\end{aligned} \tag{89}
$$

By multiplying $\beta_{j-1,i} \in \mathbb{R}$ on both sides of (89) and adding the derived equations up for all $i \in \mathcal{I}_{j-1}$, we obtain

$$
\begin{aligned}
0 =& \sum_{i\in\mathcal{I}_{j-1}} \beta_{j-1,i} \frac{i-2}{2\cdot i!} \left.\frac{d^j \mathbb{E}_{t_n}^x[Y_t \Delta W_{t_n,t}]}{dt^j}\right|_{t\to t_n^+} + \sum_{i\in\mathcal{I}_{j-1}} \beta_{j-1,i} \bar{\mathbb{B}}_{\frac{i-1}{2}}^{(j-i)}(t_n) \\
&- \sum_{i\in\mathcal{I}_{j-1}} \beta_{j-1,i} \sum_{l=3}^{i-2} \mathbb{I}_{\{l \text{ is odd}\}} \frac{1}{2\cdot (l-1)!} \Bigg(\bar{\mathbb{B}}_{\frac{i-l}{2}}^{(j-i+l-1)}(t_n) \\
&- \left.\frac{d^{j-i+l-1}\mathbb{E}_{t_n}^x[e_t^{z,[\frac{i-l}{2}]}]}{dt^{j-i+l-1}}\right|_{t\to t_n^+}\Bigg) + \sum_{i\in\mathcal{I}_{j-1}} \beta_{j-1,i} \left.\frac{d^{j-i}\mathbb{E}_{t_n}^x[e_t^{z,[\frac{i-1}{2}]}]}{dt^{j-i}}\right|_{t\to t_n^+}
\end{aligned} \tag{90}
$$

$$- \sum_{i\in\mathcal{I}_{j-1}} \beta_{j-1,i} \sum_{l=2}^{i-1} \mathbb{I}_{\{l \text{ is even}\}} \frac{1}{(l-1)!} \frac{d^{j-i+l-1}\mathbb{E}_{t_n}^x[e_t^{y,[\frac{i-l+1}{2}]}\Delta W_{t_n,t}]}{dt^{j-i+l-1}}\Bigg|_{t\to t_n^+}.$$

Then by some elementary calculation, (90) becomes

$$\begin{aligned}
0 = &\sum_{i\in\mathcal{I}_{j-1}} \beta_{j-1,i}\frac{i-2}{2\cdot i!} \frac{d^j\mathbb{E}_{t_n}^x[Y_t\Delta W_{t_n,t}]}{dt^j}\Bigg|_{t\to t_n^+} + \beta_{j-1,j-1} \frac{d\mathbb{E}_{t_n}^x[e_t^{z,[\frac{j-2}{2}]}]}{dt}\Bigg|_{t\to t_n^+} \\
&- \sum_{k=2}^{j-4} \mathbb{I}_{\{k \text{ is even}\}} \left(\sum_{i\in\mathcal{I}_{j-1}} \beta_{j-1,i}\frac{\mathbb{I}_{\{i\geq k+3\}}}{2\cdot(i-k-1)!} + \beta_{j-1,k+1}\right) \bar{\mathbb{B}}_{\frac{k}{2}}^{(j-k-1)}(t_n) \\
&+ \sum_{k=2}^{j-4} \mathbb{I}_{\{k \text{ is even}\}} \left(\sum_{i\in\mathcal{I}_{j-1}} \beta_{j-1,i}\frac{\mathbb{I}_{\{i\geq k+3\}}}{2\cdot(i-k-1)!} + \beta_{j-1,k+1}\right) \\
&\frac{d^{j-k-1}\mathbb{E}_{t_n}^x[e_t^{z,[\frac{k}{2}]}]}{dt^{j-k-1}}\Bigg|_{t\to t_n^+} - \sum_{k=1}^{j-3} \mathbb{I}_{\{k \text{ is odd}\}} \sum_{i\in\mathcal{I}_{j-1}} \beta_{j-1,i} \\
&\frac{\mathbb{I}_{\{i\geq k+2\}}}{(i-k-1)!} \frac{d^{j-k-1}\mathbb{E}_{t_n}^x[e_t^{y,[\frac{k+1}{2}]}\Delta W_{t_n,t}]}{dt^{j-k-1}}\Bigg|_{t\to t_n^+} + \beta_{j-1,j-1}\bar{\mathbb{B}}_{\frac{j-2}{2}}^{(1)}(t_n).
\end{aligned} \tag{91}$$

Similar to Step 1, letting β_{j-1}, $j-1\in\mathbb{I}_K$ be the solutions of the equations (81), we deduce

$$\begin{aligned}
0 = &\frac{j-2}{2\cdot j!} \frac{d^j\mathbb{E}_{t_n}^x[Y_t\Delta W_{t_n,t}]}{dt^j}\Bigg|_{t\to t_n^+} + \frac{1}{2}\left(\frac{d\mathbb{E}_{t_n}^x[e_t^{z,[\frac{j-2}{2}]}]}{dt}\Bigg|_{t\to t_n^+} - \bar{\mathbb{B}}_{\frac{j-2}{2}}^{(1)}(t_n)\right) \\
&- \sum_{k=2}^{j-4} \mathbb{I}_{\{k \text{ is even}\}} \frac{1}{2\cdot(j-k-1)!} \left(\bar{\mathbb{B}}_{\frac{k}{2}}^{(j-k-1)}(t_n) - \frac{d^{j-k-1}\mathbb{E}_{t_n}^x[e_t^{z,[\frac{k}{2}]}]}{dt^{j-k-1}}\Bigg|_{t\to t_n^+}\right) \\
&- \sum_{k=1}^{j-3} \mathbb{I}_{\{k \text{ is odd}\}} \frac{1}{(j-k-1)!} \frac{d^{j-k-1}\mathbb{E}_{t_n}^x[e_t^{y,[\frac{k+1}{2}]}\Delta W_{t_n,t}]}{dt^{j-k-1}}\Bigg|_{t\to t_n^+},
\end{aligned} \tag{92}$$

or equivalently

$$\begin{aligned}
0 = &\frac{j-2}{2\cdot j!} \frac{d^j\mathbb{E}_{t_n}^x[Y_t\Delta W_{t_n,t}]}{dt^j}\Bigg|_{t\to t_n^+} \\
&- \sum_{l=1}^{j-3} \mathbb{I}_{\{l \text{ is odd}\}} \frac{1}{2\cdot l!} \left(\bar{\mathbb{B}}_{\frac{j-l-1}{2}}^{(l)}(t_n) - \frac{d^l\mathbb{E}_{t_n}^x[e_t^{z,[\frac{j-l-1}{2}]}]}{dt^l}\Bigg|_{t\to t_n^+}\right) \\
&- \sum_{l=2}^{j-2} \mathbb{I}_{\{l \text{ is even}\}} \frac{1}{l!} \frac{d^l\mathbb{E}_{t_n}^x[e_t^{y,[\frac{j-l}{2}]}\Delta W_{t_n,t}]}{dt^l}\Bigg|_{t\to t_n^+}, \ j\in\bar{\mathbb{I}}_K,
\end{aligned} \tag{93}$$

from which we deduce $\mathbb{A}_z^j = 0$ for $j\in\bar{\mathbb{I}}_K$ by the definition of $\mathbb{A}_z^j$. □

Combining Lemmas 8, 10 and 13, we have the following Theorem.

Theorem 14. *Under Assumption 5 and Assumption 11, let* $(e_t^{y,[\frac{j-1}{2}]}, e_t^{z,[\frac{j-1}{2}]})$, $j\in\mathbb{I}_K$, *be the solutions of BSDEs* (70). *Then*

$$R_y^{n,[K]} = \mathcal{O}\left((\Delta t)^{2K+3}\right) \text{ and } R_z^{n,[K]} = \mathcal{O}\left((\Delta t)^{2K+3}\right).$$

Now we state our asymptotic expansion results for the Crank-Nicolson Scheme 4 in the following Theorem.

Theorem 15. *Under Assumption 5 and Assumption 11, and if* $\mathbb{E}[|Y^N - Y_{t_N}|^2] = \mathcal{O}\left((\Delta t)^{4K+4}\right)$, $\mathbb{E}[|Z^N - Z_{t_N}|^2] = \mathcal{O}\left((\Delta t)^{4K+4}\right)$, *the numerical solutions* Y^n *and* Z^n *of the Crank-Nicolson Scheme 4 have the expansions*

$$Y^n = Y_{t_n} + \sum_{j=1}^{K} e_{t_n}^{y,[j]}(\Delta t)^{2j} + \eta_{t_n}^{y,[K]},\ Z^n = Z_{t_n} + \sum_{j=1}^{K} e_{t_n}^{z,[j]}(\Delta t)^{2j} + \eta_{t_n}^{z,[K]} \tag{94}$$

with the estimate

$$\mathbb{E}[|\eta_{t_n}^{y,[K]}|^2] + \Delta t \sum_{i=n}^{N-1} \mathbb{E}[|\eta_{t_i}^{z,[K]}|^2] \leq C(\Delta t)^{4K+4}, \tag{95}$$

where $(e_t^{y,[j]}, e_t^{z,[j]})$ *are the solutions of the BSDEs* (70)*, and* C *is a positive constant depending only on* T*,* f*, and* φ*.*

Proof. We define $\eta_{t_n}^{y,[K]}$ and $\eta_{t_n}^{z,[K]}$ by

$$\eta_{t_n}^{y,[K]} = Y^{n,[K]} - Y_{t_n} \text{ and } \eta_{t_n}^{z,[K]} = Z^{n,[K]} - Z_{t_n}, \tag{96}$$

where $Y^{n,[K]}$ and $Z^{n,[K]}$ are defined by (46). Then we have

$$Y^n = Y_{t_n} + \sum_{j=1}^{K} e_{t_n}^{y,[j]}(\Delta t)^{2j} + \eta_{t_n}^{y,[K]},\ Z^n = Z_{t_n} + \sum_{j=1}^{K} e_{t_n}^{z,[j]}(\Delta t)^{2j} + \eta_{t_n}^{z,[K]}. \tag{97}$$

By (47) and (50), we deduce

$$\begin{aligned}
\eta_{t_n}^{y,[K]} =& \mathbb{E}_{t_n}^x[\eta_{t_{n+1}}^{y,[K+1]}] \\
&+ \frac{1}{2}\Delta t \left(f^{[K]}\left(t_n, x, Y^{n,[K]}, Z^{n,[K]}\right) - f^{[K]}\left(t_n, x, Y_{t_n}, Z_{t_n}\right)\right) \\
&+ \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x\left[f^{[K]}\left(t_{n+1}, X_{t_{n+1}}, Y^{n+1,[K]}, Z^{n+1,[K]}\right)\right. \\
&\left. - f^{[K]}\left(t_{n+1}, X_{t_{n+1}}, Y_{t_{n+1}}, Z_{t_{n+1}}\right)\right] - R_y^{n,[K]}
\end{aligned} \tag{98}$$

and

$$\begin{aligned}
\frac{1}{2}\Delta t \eta_{t_n}^{z,[K]} =& \mathbb{E}_{t_n}^x[\eta_{t_{n+1}}^{y,[K+1]}\Delta W_{n+1}] - \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x[\eta_{t_{n+1}}^{z,[K+1]}] \\
&+ \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x\left[f^{[K]}\left(t_{n+1}, X_{t_{n+1}}, Y^{n+1,[K]}, Z^{n+1,[K]}\right)\Delta W_{n+1}\right] \\
&- \frac{1}{2}\Delta t \mathbb{E}_{t_n}^x\left[f^{[K]}\left(t_{n+1}, X_{t_{n+1}}, Y_{t_{n+1}}, Z_{t_{n+1}}\right)\Delta W_{n+1}\right] - R_z^{n,[K]}.
\end{aligned} \tag{99}$$

Based on Theorem 14 and the above two equations, following the proof of the error estimates of the Crank-Nicolson Scheme in [42], we can prove the estimate (95). □

4. Extrapolation algorithms of the Crank-Nicolson Scheme for BSDEs

In this Section, based on the asymptotic expansions (94) in Theorem 15, we will apply the Richardson extrapolation to the solutions of the Crank-Nicolson Scheme 4 to obtain much accurate approximations to the solution of BSDE (1). To this end, we shall construct our Richardson extrapolation algorithms for BSDEs.

For any $t_n \in \pi_N$, let $(\mathcal{Y}_{i,0}^n, \mathcal{Z}_{i,0}^n)$ be the numerical approximations of the exact solution (Y_{t_n}, Z_{t_n}) of BSDE (1) by (4) with time step sizes Δt_i, $i = 0, 1, \cdots, K-1$. Then we define the extrapolation solutions of $(\mathcal{Y}_{i,0}^n, \mathcal{Z}_{i,0}^n)$ by $\mathcal{Y}_{m,p}^n = \sum_{i=m-p}^{m} c_i \mathcal{Y}_{i,0}^n$ and $\mathcal{Z}_{m,p}^n = \sum_{i=m-p}^{m} c_i \mathcal{Z}_{i,0}^n$, $1 \leq p \leq m \leq K-1$. Here π_N, Δt_i and c_i are defined in Subsection 2.4.

All the extrapolation solutions $\mathcal{Y}^n_{m,p}$ and $\mathcal{Z}^n_{m,p}$, $1 \le p \le m \le K-1$ can be obtained by the Aitken-Neville algorithm in Subsection 2.4 with $k = 2$.

$$
\begin{aligned}
\mathcal{Y}^n_{m,p} =& \mathcal{Y}^n_{m,p-1} + \frac{\mathcal{Y}^n_{m,p-1} - \mathcal{Y}^n_{m-1,p-1}}{\left(\frac{N_m}{N_{m-p}}\right)^2 - 1}, \\
\mathcal{Z}^n_{m,p} =& \mathcal{Z}^n_{m,p-1} + \frac{\mathcal{Z}^n_{m,p-1} - \mathcal{Z}^n_{m-1,p-1}}{\left(\frac{N_m}{N_{m-p}}\right)^2 - 1}.
\end{aligned} \tag{100}
$$

We summarize our Richardson extrapolation algorithms for solving BSDE (1) in the following Algorithm.

Algorithm 4.1 Richardson extrapolation of the solution of the Crank-Nicolson Scheme for BSDEs

Input: $n_0 \in \pi_{N,0}$, K, $\{N_m\}_{m=0}^{K-1}$, X^{n_0}, $Y^{N_{K-1}}$;
for $m = 0, 1, \cdots, K-1$ **do**
  Let $N = N * N_m$; Solve $\{(Y^n, Z^n)\}_{n=n_0}^{N-1}$ by Scheme 4 on $\pi_{N,m}$; Let $\mathcal{Y}^{n_0}_{m,0} = Y^{n_0}$, $\mathcal{Z}^{n_0}_{m,0} = Z^{n_0}$;
end for
for $m = 1, 2, \cdots, K-1$ **do**
  for $p = 1, 2, \cdots, m$ **do**
    $\mathcal{Y}^{n_0}_{m,p} = \mathcal{Y}^{n_0}_{m,p-1} + \frac{\mathcal{Y}^{n_0}_{m,p-1} - \mathcal{Y}^{n_0}_{m-1,p-1}}{\left(\frac{N_m}{N_{m-p}}\right)^2 - 1}$;
    $\mathcal{Z}^{n_0}_{m,p} = \mathcal{Z}^{n_0}_{m,p-1} + \frac{\mathcal{Z}^{n_0}_{m,p-1} - \mathcal{Z}^{n_0}_{m-1,p-1}}{\left(\frac{N_m}{N_{m-p}}\right)^2 - 1}$;
  end for
end for
return $\mathcal{Y}^{n_0}_{K-1,K-1}$, $\mathcal{Z}^{n_0}_{K-1,K-1}$.

Remark 16. *Algorithm 4.1 has the following features including that*

(1) *Algorithm 4.1 returns the CN solution when $K = 1$;*
(2) *$(\mathcal{Y}^0_{m,p}, \mathcal{Z}^0_{m,p})$ is an approximation to the exact solution (Y_{t_0}, Z_{t_0}) of BSDE (1) with error $\mathcal{O}\left((\Delta t)^{2p+2}\right)$;*
(3) *the $N_m, m = 0, 1, \cdots, K-1$ are the first K elements of any step-number sequence for Richardson extrapolation, and different $\{N_m\}_{m=0}^{K-1}$ lead to different extrapolation algorithms;*
(4) *compared with other high order multistep methods* [41, 45]*, the RiE-CN algorithms are self-starting ones. So they can be used to give the initializations of numerical solutions of other multistep schemes.*

For Algorithm 4.1, we have the following conclusion.

Theorem 17. *Under Assumption 5 and Assumption 11, and if $\mathbb{E}[|Y^N - Y_{t_N}|^2] = \mathcal{O}\left((\Delta t)^{4K+4}\right)$, $\mathbb{E}[|Z^N - Z_{t_N}|^2] = \mathcal{O}\left((\Delta t)^{4K+4}\right)$, the numerical solutions $\mathcal{Y}^{n_0}_{K-1,K-1}$ and $\mathcal{Z}^{n_0}_{K-1,K-1}$ of Algorithm 4.1 have the estimates*

$$
\mathbb{E}[|\mathcal{Y}^{n_0}_{K-1,K-1} - Y_0|^2] \le C(\Delta t)^{4K+4},\ \mathbb{E}[|\mathcal{Z}^{n_0}_{K-1,K-1} - Z_0|^2] \le C(\Delta t)^{4K+4}, \tag{101}
$$

where (Y_0, Z_0) refers to the exact solution of BSDE (1) at $t = 0$.

Based on the asymptotic expansion (94) in Theorem 15, the estimates (101) can be obtained by the convergence result of the Aitken-Neville algorithm [19].

TABLE 2. Errors and convergence rates of the RiE-CN algorithm using Romberg sequence for (102) with $T = 1.0$.

		$N=8$	$N=10$	$N=12$	$N=14$	$N=16$	C.R.
$K=1$	$\lvert Y_0-\mathcal{Y}^0_{0,0}\rvert$	1.692E-02	1.080E-02	7.494E-03	5.502E-03	4.210E-03	2.007
	$\lvert Z_0-\mathcal{Z}^0_{0,0}\rvert$	5.590E-03	3.597E-03	2.505E-03	1.843E-03	1.413E-03	1.984
$K=2$	$\lvert Y_0-\mathcal{Y}^0_{1,1}\rvert$	2.749E-05	1.119E-05	5.380E-06	2.899E-06	1.697E-06	4.017
	$\lvert Z_0-\mathcal{Z}^0_{1,1}\rvert$	2.058E-05	8.467E-06	4.091E-06	2.211E-06	1.297E-06	3.988
$K=3$	$\lvert Y_0-\mathcal{Y}^0_{2,2}\rvert$	2.188E-08	5.395E-09	1.797E-09	7.118E-10	3.069E-10	6.131
	$\lvert Z_0-\mathcal{Z}^0_{2,2}\rvert$	1.154E-08	2.945E-09	1.001E-09	4.000E-10	1.833E-10	5.973
$K=4$	$\lvert Y_0-\mathcal{Y}^0_{3,3}\rvert$	1.953E-12	3.355E-13	7.350E-14	2.043E-14	8.660E-15	7.932
	$\lvert Z_0-\mathcal{Z}^0_{3,3}\rvert$	9.182E-12	1.429E-12	2.138E-13	4.041E-14	5.551E-15	10.563

TABLE 3. Errors and convergence rates of the RiE-CN algorithm using Bulirsch sequence for (102) with $T = 1.0$.

		$N=8$	$N=10$	$N=12$	$N=14$	$N=16$	C.R.
$K=1$	$\lvert Y_0-\mathcal{Y}^0_{0,0}\rvert$	1.692E-02	1.080E-02	7.494E-03	5.502E-03	4.210E-03	2.007
	$\lvert Z_0-\mathcal{Z}^0_{0,0}\rvert$	5.590E-03	3.597E-03	2.505E-03	1.843E-03	1.413E-03	1.984
$K=2$	$\lvert Y_0-\mathcal{Y}^0_{1,1}\rvert$	2.749E-05	1.119E-05	5.380E-06	2.899E-06	1.697E-06	4.017
	$\lvert Z_0-\mathcal{Z}^0_{1,1}\rvert$	2.058E-05	8.467E-06	4.091E-06	2.211E-06	1.297E-06	3.988
$K=3$	$\lvert Y_0-\mathcal{Y}^0_{2,2}\rvert$	3.904E-08	9.595E-09	3.195E-09	1.266E-09	5.677E-10	6.093
	$\lvert Z_0-\mathcal{Z}^0_{2,2}\rvert$	2.052E-08	5.228E-09	1.778E-09	7.107E-10	3.200E-10	5.992
$K=4$	$\lvert Y_0-\mathcal{Y}^0_{3,3}\rvert$	1.387E-11	2.326E-12	5.571E-13	1.488E-13	3.908E-14	8.377
	$\lvert Z_0-\mathcal{Z}^0_{3,3}\rvert$	7.022E-11	1.109E-11	1.636E-12	2.890E-13	8.410E-14	9.890

5. Numerical tests

In this Section, we will provide several numerical tests to verify our theoretical conclusions and show the effectiveness, efficiency and high-order convergence rate of the proposed RiE-CN algorithms. For simplicity, the $2K$-order extrapolation algorithm is denoted by RiE-CN(K). The conditional mathematical expectations $\mathbb{E}^x_{t_n}[\cdot]$ in Scheme 4 are evaluated by the Sinc quadrature rule. For more details about the Sinc quadrature rule for $\mathbb{E}^x_{t_n}[\cdot]$, readers may refer to [37] and [36].

The tested BSDE from [44] is the equation (1) with

$$\begin{aligned} \varphi(x) &= \exp(T^2)\ln(\sin x+3), \\ f(t,x,y,z) &= \frac{1}{2}\left[\exp(t^2) - 4ty - 3\exp\left(t^2 - y\exp(-t^2)\right) + z^2\exp(-t^2)\right]. \end{aligned} \tag{102}$$

The analytic solution is $Y_t = \exp(t^2)\ln(\sin X_t + 3)$, $Z_t = \exp(t^2)\frac{\cos X_t}{\sin X_t+3}$.

To show the accuracy and the efficiency of the RiE-CN algorithms, we will report the errors $\left|Y_0 - \mathcal{Y}^0_{K-1,K-1}\right|$ and $\left|Z_0 - \mathcal{Z}^0_{K-1,K-1}\right|$ between the numerical solution $(\mathcal{Y}^n_{K-1,K-1},\ \mathcal{Z}^n_{K-1,K-1})$ of the RiE-CN algorithms at $n = 0$ and the exact solution $(Y_t,\ Z_t)$ at $t = 0$, and the associated running times (R.T.). In all the tests, if not specified, we take $X_0 = 0.5$ and $T = 1.0$. The time convergence rates (C.R.) are obtained by linear square fitting. All the numerical tests are implemented in Python 3.9.16 on a laptop with Intel Core i5-12500H 12-Core Processor (2.5GHz), and 16 GB DDR5 RAM (4800MHz).

5.1. Accuracy tests. In this Subsection, we shall verify the convergence rate with respect to Δt and the high accuracy of the RiE-CN algorithms.

We adopt Algorithm 4.1 with $K = 1, 2, 3, 4$ to solve the BSDE (102). Specifically, we calculate the numerical solutions of the BSDE (102) with various time step sizes by the RiE-CN algorithms with the Romberg sequence and Bulirsch sequence and list the absolute errors and the convergence rates in Tables 2 and 3. And we use the

TABLE 4. Errors and convergence rates of the RiE-CN algorithm using Romberg sequence for (102) with $T = 2.0$.

		$N = 16$	$N = 20$	$N = 24$	$N = 28$	$N = 32$	C.R.
$K = 1$	$\|Y_0 - \mathcal{Y}_{0,0}^0\|$	1.367E-01	1.240E-01	8.682E-02	6.030E-02	4.382E-02	1.697
	$\|Z_0 - \mathcal{Z}_{0,0}^0\|$	6.381E-01	3.730E-01	2.323E-01	1.559E-01	1.123E-01	2.522
$K = 2$	$\|Y_0 - \mathcal{Y}_{1,1}^0\|$	1.625E-02	6.535E-03	3.126E-03	1.680E-03	9.823E-04	4.047
	$\|Z_0 - \mathcal{Z}_{1,1}^0\|$	4.602E-03	2.131E-03	1.099E-03	6.182E-04	3.722E-04	3.630
$K = 3$	$\|Y_0 - \mathcal{Y}_{2,2}^0\|$	3.528E-05	7.642E-06	2.247E-06	8.137E-07	3.421E-07	6.690
	$\|Z_0 - \mathcal{Z}_{2,2}^0\|$	9.030E-05	2.585E-05	9.103E-06	3.725E-06	1.707E-06	5.727
$K = 4$	$\|Y_0 - \mathcal{Y}_{3,3}^0\|$	2.125E-07	3.756E-08	8.986E-09	2.663E-09	9.254E-10	7.844
	$\|Z_0 - \mathcal{Z}_{3,3}^0\|$	3.005E-07	5.570E-08	1.371E-08	4.141E-09	1.457E-09	7.690

TABLE 5. Errors and convergence rates of the RiE-CN algorithm using Bulirsch sequence for (102) with $T = 2.0$.

		$N = 16$	$N = 20$	$N = 24$	$N = 28$	$N = 32$	C.R.
$K = 1$	$\|Y_0 - \mathcal{Y}_{0,0}^0\|$	1.367E-01	1.240E-01	8.682E-02	6.030E-02	4.382E-02	1.697
	$\|Z_0 - \mathcal{Z}_{0,0}^0\|$	6.381E-01	3.730E-01	2.323E-01	1.559E-01	1.123E-01	2.522
$K = 2$	$\|Y_0 - \mathcal{Y}_{1,1}^0\|$	1.625E-02	6.535E-03	3.126E-03	1.680E-03	9.823E-04	4.047
	$\|Z_0 - \mathcal{Z}_{1,1}^0\|$	4.602E-03	2.131E-03	1.099E-03	6.182E-04	3.722E-04	3.630
$K = 3$	$\|Y_0 - \mathcal{Y}_{2,2}^0\|$	6.387E-05	1.379E-05	4.044E-06	1.461E-06	6.133E-07	6.704
	$\|Z_0 - \mathcal{Z}_{2,2}^0\|$	1.589E-04	4.565E-05	1.611E-05	6.599E-06	3.026E-06	5.715
$K = 4$	$\|Y_0 - \mathcal{Y}_{3,3}^0\|$	1.494E-06	2.652E-07	6.360E-08	1.887E-08	6.563E-09	7.832
	$\|Z_0 - \mathcal{Z}_{3,3}^0\|$	2.092E-06	3.903E-07	9.652E-08	2.922E-08	1.030E-08	7.667

same time step sizes to solve the BSDE (102) to $T = 2.0$ and list the experiment results in Tables 4 and 5.

Tables 2-5 show that

(1) RiE-CN(K) are stable and enjoy the $2K$-order time convergence rates for $1 \leq K \leq 4$ for both Romberg and Bulirsch sequences. Such results are consistent with our theoretical results.

(2) for the same time step size $\Delta t = \frac{T}{N}$, the RiE-CN(K), $K = 1, 2$, with Romberg and Bulirsch step-number sequences are the same algorithm, the RiE-CN(K), $K = 3, 4$, with Romberg sequence are more accurate than the ones with Bulirsch sequence. Such results are consistent with the discussions of the Richardson extrapolation algorithm described in Subsection 2.4.

5.2. Efficiency Tests. In this Subsection, we are concerned about the efficiency of the RiE-CN algorithms.

We first compare the RiE-CN(2) with the Crank-Nicolson Scheme. And then we compare the RiE-CN(K), where the Bulirsch step-number sequence is used, with the multistep schemes proposed in [41], and use DM(K) to denote the K-step Kth-order one. All the numerical results are listed in Tables 6-8. In all the tables, Y_K^0 and Z_K^0 is the numerical solution at $n = 0$ by the DM(K) scheme.

To compare the RiE-CN(2) with the Crank-Nicolson Scheme, we calculate the numerical solutions of the BSDE (102) with various time step sizes by the Crank-Nicolson Scheme and the RiE-CN(2), respectively, and list the absolute errors and the runing times in Table 6. The errors and running times in Table 6 show that to achieve the same or smaller errors, the RiE-CN(2) which enjoys theoretical time convergence rate 4 costs less time than the Crank-Nicolson Scheme, which means that the RiE-CN(2) is more efficient than the Crank-Nicolson Scheme.

To compare the efficiency of the RiE-CN(K) algorithms with the DM(K) scheme, we numerically solve the BSDE (102) with various time step sizes by the DM(K)

Table 6. Errors and running times of the Crank-Nicolson Scheme and the RiE-CN(2).

CN($K=1$)		$N=112$	$N=120$	$N=128$	$N=136$	$N=144$
	$\lvert Y_0-\mathcal{Y}_{0,0}^0\rvert$	8.363E-05	7.523E-05	6.612E-05	5.857E-05	5.224E-05
	$\lvert Z_0-\mathcal{Z}_{0,0}^0\rvert$	4.194E-05	3.653E-05	3.211E-05	2.844E-05	2.537E-05
	R.T.(s)	1.477	1.710	1.944	2.200	2.473
RiE-CN(2)		$N=8$	$N=10$	$N=12$	$N=14$	$N=16$
	$\lvert Y_0-\mathcal{Y}_{1,1}^0\rvert$	2.271E-05	9.227E-06	4.432E-06	2.387E-06	1.397E-06
	$\lvert Z_0-\mathcal{Z}_{1,1}^0\rvert$	1.963E-05	8.078E-06	3.901E-06	2.107E-06	1.235E-06
	R.T.(s)	0.0198	0.0326	0.0336	0.0498	0.0698

Table 7. Errors and running times the DM(4) scheme and the RiE-CN(2).

DM(4)		$N=50$	$N=55$	$N=60$	$N=65$	$N=70$
	$\lvert Y_0-Y_4^0\rvert$	4.007E-06	2.816E-06	2.036E-06	1.508E-06	1.141E-06
	$\lvert Z_0-Z_4^0\rvert$	2.581E-07	1.791E-07	1.282E-07	9.414E-08	7.067E-08
	R.T.(s)	0.782	0.895	1.024	1.193	1.378
RiE-CN(2)		$N=26$	$N=28$	$N=30$	$N=32$	$N=34$
	$\lvert Y_0-\mathcal{Y}_{1,1}^0\rvert$	1.988E-07	1.485E-07	1.126E-07	8.700E-08	6.826E-08
	$\lvert Z_0-\mathcal{Z}_{1,1}^0\rvert$	1.772E-07	1.318E-07	9.998E-08	7.723E-08	6.060E-08
	R.T.(s)	0.329	0.383	0.451	0.504	0.598

Table 8. Errors and running times the DM(6) scheme and the RiE-CN(3).

DM(6)		$N=65$	$N=70$	$N=75$	$N=80$	$N=85$
	$\lvert Y_0-Y_6^0\rvert$	3.331E-09	2.252E-09	1.546E-09	1.082E-09	7.728E-10
	$\lvert Z_0-Z_6^0\rvert$	9.845E-11	5.987E-11	4.026E-11	2.827E-11	2.025E-11
	R.T.(s)	1.804	2.104	2.385	2.724	3.158
RiE-CN(3)		$N=16$	$N=18$	$N=20$	$N=22$	$N=24$
	$\lvert Y_0-\mathcal{Y}_{2,2}^0\rvert$	6.044E-10	2.980E-10	1.583E-10	8.930E-11	5.299E-11
	$\lvert Z_0-\mathcal{Z}_{2,2}^0\rvert$	4.014E-11	1.660E-11	8.010E-12	4.277E-12	2.454E-12
	R.T.(s)	0.327	0.449	0.542	0.687	0.845

scheme and the RiE-CN(K), and list the absolute errors and the running times in Tables 7 and 8. The errors and the running times in Tables 7 and 8 show that to achieve the same or smaller errors the RiE-CN(K) cost less time than the DM($2K$) schemes for the same rates of convergence 4 and 6. So the RiE-CN algorithms with the Bulirsch sequence are more efficient than DM(K) schemes.

All the above numerical tests show that

(1) the RiE-CN(K) algorithms enjoy $2K$-order convergence in time discretization for solving BSDEs for $1 \le K \le 4$ which is consistent with our theoretical conclusions;
(2) the RiE-CN(K) algorithms are stable and very efficient.

6. Conclusions

In this work, by the theory of backward stochastic differential equations and the Adomian decomposition, we theoretically proved that the solution of the Crank-Nicolson scheme for solving BSDEs admits an asymptotic expansion with its coefficients the solutions of the new system of BSDEs we introduced. Then based on the expansion, we proposed Richardson extrapolation algorithms for solving B-SDEs which are very easy in use. Some numerical tests were carried out. The numerical results of the tests verified our theoretical conclusions, and showed that the algorithms are stable, very efficient and high accurate.

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