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# AN A PRIORI ERROR ANALYSIS OF A PROBLEM INVOLVING MIXTURES OF CONTINUA WITH GRADIENT ENRICHMENT

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**Abstract.** In this work, we study a strain gradient problem involving mixtures. The variational formulation is written as a first-order in time coupled system of parabolic variational equations. An existence and uniqueness result is recalled. Then, we introduce a fully discrete approximation by using the finite element method and the implicit Euler scheme. A discrete stability property and a priori error estimates are proved. Finally, some one- and two-dimensional numerical simulations are performed.

**Key words.** Mixtures, strain gradient, finite elements, discrete energy decay, a priori error estimates, numerical simulations.

## 1. Introduction

We refer as a mixture of materials to the combination of two or more solids and/or fluids. It is very usual to find mixture of materials in our daily life as far we can use them in the chemical industry or in steel manufacturing (among others). It is common to consider them in the creation of composites which combine several materials with different chemical or physical properties. The main aim is to obtain a new issue which satisfies new specific properties.

To describe these materials it was considered the continuum theory of mixtures. It has become an important field of work for physics, engineers and mathematicians. A mathematical perspective of this theory suggests a relevant family of problems concerning systems of partial differential equations and/or integro-differential equations. It is worth recalling that the current formulation of this theory can be found in the contributions of Bowen and Wise [8], Eringen and Ingram [11, 22], Green and Naghdi [14, 15] and Truesdell and Toupin [29]. Books and classical references on this theory are the works of Atkin and Craine [5, 4], Bedford and Drumheller [6] and Bowen [7]. This theory is totally accepted in the scientific community and it has been extended to consider viscous effects on the different constituents and/or the whole mixture. Some studies concerning these materials can be found in [24, 25, 16, 17, 19, 21, 20], but they are only a few examples in the huge quantity of contributions in this theory.

It is of interest (from a mathematical point of view) to clarify the qualitative and quantitative properties of the solutions to the systems of the differential equations describing mixture of materials. From a qualitative perspective, it is relevant to clarify the existence, uniqueness, continuous dependence and the asymptotic behavior of the solutions. We can cite several contributions [1, 3, 2, 12, 13, 26, 27, 28, 18] in this line.

In this paper, we center our attention in the strain gradient theory of mixtures proposed by Ieşan [18] and we consider several dissipative mechanisms on the conservative structure. It is worth recalling that, from a mathematical perspective, the

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strain gradient theories provide a fourth order spatial derivative in the system of equations and it will be of high interest to clarify the consequences of these components there. We will consider a cylinder of constant cross-section and we will study the behavior of the anti-plane shear deformations. The qualitative study of this problem can be found in [23]. In this new paper, we want to make a numerical contribution to the same problem.

In the next section we describe the mathematical model, we state the basic assumptions to obtain the well-posedness of the problem and we recall an existence and uniqueness result as well as an energy decay property. In Section 3 a fully discrete approximation is introduced by using the finite element method and the implicit Euler scheme. A discrete stability property is proved and a priori error estimates are obtained. Finally, in Section 4 some numerical simulations, involving examples in one and two dimensions, are presented to demonstrate the accuracy of the approximation, the decay of the discrete energy and the behavior of the solution.

## 2. The basic equations and the variational formulation

In this work, we consider a mixture of two interacting materials. Our domain will consist of one cylinder $R$ of constant cross-section, $R = B \times [0, L]$, where $B$ is a bounded two-dimensional region whose boundary, $\partial B$, is a curve assumed smooth enough to allow the application of the divergence theorem.

Following [23], we consider the isotropic and homogeneous case for anti-plane shear deformations. It means that we impose the following conditions on the displacements of the two interacting continua $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{w} = (w_1, w_2, w_3)$:

$$u_1 = u_2 = w_1 = w_2 = 0, \quad u_3 = u(x_1, x_2), \quad w_3 = w(x_1, x_2),$$

where $u$ and $w$ are two-dimensional functions which define the displacements of the two constituents in the domain $B$.

The general problem modeling the evolution of the mixtures with some dissipation mechanisms is written as follows:

$$\left.\begin{array}{rl} \rho_1 \ddot{u} = & \mu_1 \Delta u + \mu \Delta w - \gamma_1 \Delta^2 u - \gamma \Delta^2 w - a(u - w) - \gamma^* \Delta^2 \dot{u} \\ & -a^*(\dot{u} - \dot{w}) + \mu^* \Delta \dot{u}, \\ \rho_2 \ddot{w} = & \mu \Delta u + \mu_2 \Delta w - \gamma \Delta^2 u - \gamma_2 \Delta^2 w + a(u - w) + a^*(\dot{u} - \dot{w}), \end{array}\right\} \tag{1}$$

where $\Delta$ is the two-dimensional Laplacian operator.

In the previous system of equations, we have assumed three possible dissipation mechanisms which correspond to the hyperviscosity (if $\gamma^* \neq 0$ and $a^* = \mu^* = 0$), the weak viscosity (if $a^* \neq 0$ and $\gamma^* = \mu^* = 0$), and the viscosity (if $\mu^* \neq 0$ and $a^* = \gamma^* = 0$). Anyway, we assume that $a^*$, $\gamma^*$, $\mu^* \geq 0$.

As usual in this context, $\rho_1$ and $\rho_2$ are assumed to be positive because they represent mass densities. Moreover, in order to guarantee the elastic stability of the materials we will suppose that

$$\mu_1 \mu_2 > \mu^2, \quad \gamma_1 \gamma_2 > \gamma^2, \quad a,\ \mu_1,\ \gamma_1 > 0. \tag{2}$$

We assume also that $\gamma \neq 0$ to ensure the coupling between the materials.

To have a well-posed problem we need to introduce initial and boundary conditions. As initial conditions we consider:

$$\begin{array}{ll} u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \dot{u}(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}) & \text{for a.e. } \boldsymbol{x} \in B, \\ w(\boldsymbol{x}, 0) = w_0(\boldsymbol{x}), \quad \dot{w}(\boldsymbol{x}, 0) = e_0(\boldsymbol{x}) & \text{for a.e. } \boldsymbol{x} \in B, \end{array} \tag{3}$$

and we set the following boundary conditions:

$$u = \Delta u = w = \Delta w = 0 \text{ on } \partial B. \tag{4}$$

In [23], the authors gave some results concerning the existence, uniqueness and the time decay of the solutions to this system. We recall the main result in the following theorem.

**Theorem 1.** *Under the assumptions (2), it follows that problem (1), (3) and (4) admits one solution with the regularity:*

$$u, w \in C^1([0,T]; H_0^1(B) \cap H^2(B)) \cap C^2([0,T]; L^2(B)).$$

*Moreover, when $\mu^* \neq 0$ or $\gamma^* \neq 0$ (that is, the hyperviscosity or viscosity cases), if we also assume that $\gamma m_n^2 + \mu m_n \neq a$ for all $m_n$ the eigenvalues of the problem:*

$$\begin{aligned} &\Delta^2\Phi - m_n^2\Phi = 0 \quad in \quad B, \\ &\Phi = \Delta\Phi = 0 \quad on \quad \partial B, \end{aligned}$$

*then the corresponding solution to problem (1), (3) and (4) decays exponentially.*

*When $a^* \neq 0$ and $\mu^* = \gamma^* = 0$ (that is, the weak viscosity case), if we assume again that $\gamma m_n^2 + \mu m_n \neq a$ for all $m_n$ and also that $(\gamma+\gamma_1)\rho_2 - (\gamma+\gamma_2)\rho_1 \neq 0$, then the solution to problem (1), (3) and (4) decays exponentially. However, if $(\gamma+\gamma_1)\rho_2 - (\gamma+\gamma_2)\rho_1 = 0$, then the solution decays slowly.*

In the rest of this section, we will derive the weak form of the problem defined by system (1), initial conditions (3) and boundary conditions (4). So, let us denote by $Y$, $H$ and $V$ the variational spaces $L^2(B)$, $[L^2(B)]^2$ and $H_0^2(B)$, respectively, and let $(\cdot,\cdot)_Y$ and $\|\cdot\|_Y$ be the scalar product and the norm in $Y$ (resp. by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ the scalar product and the norm in $H$). As usual, we must replace boundary conditions (4) by the new ones:

$$u = w = 0, \quad \nabla u = \nabla w = \mathbf{0} \text{ on } \partial B. \tag{5}$$

It is worth noting that Theorem 1 could be also proved with these new boundary conditions (5) (see [23] for further details).

Multiplying system (1) by adequate test functions and applying Green's formula, by using the new boundary conditions (5) we have the weak form written in terms of the velocity of the first constituent $v = \dot{u}$ and the velocity of the second constituent $e = \dot{w}$.

**Problem VP**. *Find the velocity of the first constituent $v : [0,T] \to V$ and the velocity of the second constituent $e : [0,T] \to V$ such that $v(0) = v_0$, $e(0) = e_0$, and, for a.e. $t \in (0,T)$ and for all $r, z \in V$,*

$$\begin{aligned} &\rho_1(\dot{v}(t), r)_Y + \mu_1(\nabla u(t), \nabla r)_H + \mu(\nabla w(t), \nabla r)_H + \gamma_1(\Delta u(t), \Delta r)_Y \\ &\quad + \gamma(\Delta w(t), \Delta r)_Y + a(u(t) - w(t), r)_Y + \mu^*(\nabla v(t), \nabla r)_H \\ &\quad + \gamma^*(\Delta v(t), \Delta r)_Y + a^*(v(t) - e(t), r)_Y = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} &\rho_2(\dot{e}(t), z)_Y + \mu(\nabla u(t), \nabla z)_H + \mu_2(\nabla w(t), \nabla z)_H + \gamma(\Delta u(t), \Delta z)_Y \\ &\quad + \gamma_2(\Delta w(t), \Delta z)_Y + a(w(t) - u(t), z)_Y + a^*(e(t) - v(t), z)_Y = 0. \end{aligned} \tag{7}$$

In the above variational equations, the displacements of the first and second constituents are then recovered from the relations:

$$u(t) = \int_0^t v(s)\, ds + u_0, \quad w(t) = \int_0^t e(s)\, ds + w_0. \tag{8}$$

Finally, we recall that the energy of the system is given by

$$E(t) = \frac{1}{2}\Big(\rho_1\|v(t)\|_Y^2 + \rho_2\|e(t)\|_Y^2 + \mu_1\|\nabla u(t)\|_H^2 + \mu_2\|\nabla w(t)\|_H^2 + \gamma_1\|\Delta u(t)\|_Y^2 + \gamma_2\|\Delta w(t)\|_Y^2\Big). \tag{9}$$

## 3. An a priori error analysis of a fully discrete problem

Here, we numerically analyze a fully discrete approximation of the variational problem (6)-(8). As usual, this is done in two steps. First, we obtain the spatial approximation and so, let us construct the finite element space $V^h$ as follows:

$$V^h = \{r^h \in C^1(\overline{B}) \cap V\ ;\ r^h_{Tr} \in P_3(Tr) \quad \forall Tr \in \mathcal{T}^h\}. \tag{10}$$

In this definition, we have assumed that the domain $\overline{B}$ is polyhedral and we have denoted by $\mathcal{T}^h$ a regular triangulation (in the sense of [10]), where the space of cubic polynomials in $Tr$ is represented by $P_3(Tr)$ and parameter $h > 0$ is the spatial discretization size. As we can see, we have used $C^1$ and piecewise cubic functions for the approximation of the variational space $V$.

Secondly, we discretize the time derivatives. Then, let us define a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, where $k = T/N$ is the time step size. For a continuous function $f$, let $f_n = f(t_n)$ and, for a sequence $\{z_n\}_{n=0}^N$, let $\delta z_n = (z_n - z_{n-1})/k$ be its divided differences.

Applying the classical implicit Euler scheme, the fully discrete approximation of Problem VP is the following.

**Problem VP$^{hk}$.** *Find the discrete velocity of the first constituent $v^{hk} = \{v_n^{hk}\}_{n=0}^N \subset V^h$ and the discrete velocity of the second constituent $e^{hk} = \{e_n^{hk}\}_{n=0}^N \subset V^h$ such that $v_0^{hk} = v_0^h$, $e_0^{hk} = e_0^h$, and, for $n = 1, \ldots N$, and for all $r^h, z^h \in V^h$,*

$$\begin{aligned}&\rho_1(\delta v_n^{hk}, r^h)_Y + \mu_1(\nabla u_n^{hk}, \nabla r^h)_H + \mu(\nabla w_n^{hk}, \nabla r^h)_H + \gamma_1(\Delta u_n^{hk}, \Delta r^h)_Y\\ &\quad +\gamma(\Delta w_n^{hk}, \Delta r^h)_Y + a(u_n^{hk} - w_n^{hk}, r^h)_Y + \mu^*(\nabla v_n^{hk}, \nabla r^h)_H\\ &\quad +\gamma^*(\Delta v_n^{hk}, \Delta r^h)_Y + a^*(v_n^{hk} - e_n^{hk}, r^h)_Y = 0,\end{aligned} \tag{11}$$

$$\begin{aligned}&\rho_2(\delta e_n^{hk}, z^h)_Y + \mu(\nabla u_n^{hk}, \nabla z^h)_H + \mu_2(\nabla w_n^{hk}, \nabla z^h)_H + \gamma(\Delta u_n^{hk}, \Delta z^h)_Y\\ &\quad +\gamma_2(\Delta w_n^{hk}, \Delta z^h)_Y + a(w_n^{hk} - u_n^{hk}, z^h)_Y + a^*(e_n^{hk} - v_n^{hk}, z^h)_Y = 0.\end{aligned} \tag{12}$$

In the previous discrete variational equations, the discrete displacements of the first and second constituents are then recovered from the relations:

$$u_n^{hk} = k\sum_{j=1}^n v_j^{hk} + u_0^h, \quad w_n^{hk} = k\sum_{j=1}^n e_j^{hk} + w_0^h. \tag{13}$$

If we denote by $\mathcal{P}^h$ the interpolation operator over the finite element space $V^h$ (see, again, [10]), in the fully discrete problem $VP^{hk}$ the discrete initial conditions $u_0^h$, $v_0^h$, $w_0^h$ and $e_0^h$ are approximations of the initial conditions $u_0$, $v_0$, $w_0$ and $e_0$ defined as

$$u_0^h = \mathcal{P}^h u_0, \quad v_0^h = \mathcal{P}^h v_0, \quad w_0^h = \mathcal{P}^h w_0, \quad e_0^h = \mathcal{P}^h e_0.$$

It is worth noting that, using Lax-Milgram lemma and assumptions (2), it is straightforward to show the existence of a unique discrete solution to Problem VP$^{hk}$.

Now, we provide an a priori error analysis of the fully discrete problem $VP^{hk}$. First, we will prove below a discrete stability property.

**Lemma 2.** *If we assume that the conditions of Theorem 1 hold, then the sequences* $\{u^{hk}, v^{hk}, w^{hk}, e^{hk}\}$, *defined in Problem* $VP^{hk}$, *satisfy the stability estimate:*

$$\|v_n^{hk}\|_Y^2 + \|u_n^{hk}\|_Y^2 + \|\nabla u_n^{hk}\|_H^2 + \|\Delta u_n^{hk}\|_Y^2 + \|e_n^{hk}\|_Y^2 + \|w_n^{hk}\|_Y^2 + \|\nabla w_n^{hk}\|_H^2 + \|\Delta w_n^{hk}\|_Y^2 \le C,$$

*for a given positive constant* $C$ *independent of the parameters* $h$ *and* $k$.

*Proof.* Here, we remove the superscripts over all the variables for the sake of clarity in the writing of the calculations presented below.

If we take $r^h = v_n^{hk}$ as a test function in (11) it follows that

$$\begin{aligned}&\rho_1(\delta v_n, v_n)_Y + \mu_1(\nabla u_n, \nabla v_n)_H + \mu(\nabla w_n, \nabla v_n)_H + \gamma_1(\Delta u_n, \Delta v_n)_Y\\ &\quad +\gamma(\Delta w_n, \Delta v_n)_Y + a(u_n - w_n, v_n)_Y + \mu^*(\nabla v_n, \nabla v_n)_H\\ &\quad +\gamma^*(\Delta v_n, \Delta v_n) + a^*(v_n - e_n, v_n) = 0,\end{aligned}$$

and so, keeping in mind that

$$\begin{aligned}&\rho_1(\delta v_n, v_n)_Y \ge \frac{\rho_1}{2k}\left\{\|v_n\|_Y^2 - \|v_{n-1}\|_Y^2\right\},\\ &a(u_n, v_n)_Y \ge \frac{a}{2k}\left\{\|u_n\|_Y^2 - \|u_{n-1}\|_Y^2\right\},\\ &\mu_1(\nabla u_n, \nabla v_n)_H = \frac{\mu_1}{2k}\left\{\|\nabla u_n\|_H^2 - \|\nabla u_{n-1}\|_H^2 + \|\nabla(u_n - u_{n-1})\|_H^2\right\},\\ &\gamma_1(\Delta u_n, \Delta v_n)_Y = \frac{\gamma_1}{2k}\left\{\|\Delta u_n\|_Y^2 - \|\Delta u_{n-1}\|_Y^2 + \|\Delta(u_n - u_{n-1})\|_Y^2\right\},\end{aligned}$$

using several times Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality

$$ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2 \quad \forall a, b \in \mathbb{R},\ \epsilon > 0, \tag{14}$$

we find that

$$\begin{aligned}&\frac{\rho_1}{2k}\left\{\|v_n\|_Y^2 - \|v_{n-1}\|_Y^2\right\} + \mu(\nabla w_n, \nabla v_n)_H + \gamma(\Delta w_n, \Delta v_n)_Y\\ &+ \frac{\mu_1}{2k}\left\{\|\nabla u_n\|_H^2 - \|\nabla u_{n-1}\|_H^2 + \|\nabla(u_n - u_{n-1})\|_H^2\right\}\\ &+ \frac{\gamma_1}{2k}\left\{\|\Delta u_n\|_Y^2 - \|\Delta u_{n-1}\|_Y^2 + \|\Delta(u_n - u_{n-1})\|_Y^2\right\}\\ &+ \frac{a}{2k}\left\{\|u_n\|_Y^2 - \|u_{n-1}\|_Y^2\right\} \le C\Big(\|v_n\|_Y^2 + \|w_n\|_Y^2 + \|u_n\|_Y^2 + \|e_n\|_Y^2\Big).\end{aligned}$$

Proceeding in a similar form, we obtain the following estimates for the discrete velocity of the second constituent:

$$\begin{aligned}&\frac{\rho_2}{2k}\left\{\|e_n\|_Y^2 - \|e_{n-1}\|_Y^2\right\} + \mu(\nabla u_n, \nabla e_n)_H + \gamma(\Delta u_n, \Delta e_n)_Y\\ &+ \frac{\mu_2}{2k}\left\{\|\nabla w_n\|_H^2 - \|\nabla w_{n-1}\|_H^2 + \|\nabla(w_n - w_{n-1})\|_H^2\right\}\\ &+ \frac{\gamma_2}{2k}\left\{\|\Delta w_n\|_Y^2 - \|\Delta w_{n-1}\|_Y^2 + \|\Delta(w_n - w_{n-1})\|_Y^2\right\}\\ &+ \frac{a}{2k}\left\{\|w_n\|_Y^2 - \|w_{n-1}\|_Y^2\right\} \le C\Big(\|e_n\|_Y^2 + \|w_n\|_Y^2 + \|u_n\|_Y^2 + \|v_n\|_Y^2\Big).\end{aligned}$$

Observing that

$$\begin{aligned}\mu(\nabla w_n,\nabla v_n)_H+\mu(\nabla u_n,\nabla e_n)_H =&\frac{\mu}{k}\Big\{(\nabla u_n,\nabla w_n)_H-(\nabla u_{n-1},\nabla w_{n-1})_H\\ &+(\nabla(u_n-u_{n-1}),\nabla(w_n-w_{n-1}))_H\Big\},\\ \gamma(\Delta w_n,\Delta v_n)_Y+\gamma(\Delta u_n,\Delta e_n)_Y =&\frac{\gamma}{k}\Big\{(\Delta u_n,\Delta w_n)_Y-(\Delta u_{n-1},\Delta w_{n-1})_Y\\ &+(\Delta(u_n-u_{n-1}),\Delta(w_n-w_{n-1}))_Y\Big\},\\ \mu_1(\nabla(u_n-u_{n-1}),\nabla(u_n-u_{n-1}))_H&+\mu_2(\nabla(w_n-w_{n-1}),\nabla(w_n-w_{n-1}))_H\\ &+2\mu(\nabla(u_n-u_{n-1}),\nabla(w_n-w_{n-1}))_H\geq 0,\\ \gamma_1(\Delta(u_n-u_{n-1}),\Delta(u_n-u_{n-1}))_Y&+\gamma_2(\Delta(w_n-w_{n-1}),\Delta(w_n-w_{n-1}))_Y\\ &+2\gamma(\Delta(u_n-u_{n-1}),\Delta(w_n-w_{n-1}))_Y\geq 0,\end{aligned}$$

where we have used assumptions (2), and combining the previous estimates, it follows that

$$\begin{aligned}&\frac{\rho_1}{2k}\left\{\|v_n\|_Y^2-\|v_{n-1}\|_Y^2\right\}+\frac{\rho_2}{2k}\left\{\|e_n\|_Y^2-\|e_{n-1}\|_Y^2\right\}\\ &+\frac{\mu_1}{2k}\left\{\|\nabla u_n\|_H^2-\|\nabla u_{n-1}\|_H^2\right\}+\frac{\gamma_1}{2k}\left\{\|\Delta u_n\|_Y^2-\|\Delta u_{n-1}\|_Y^2\right\}\\ &+\frac{a}{2k}\left\{\|u_n\|_Y^2-\|u_{n-1}\|_Y^2\right\}+\frac{a}{2k}\left\{\|w_n\|_Y^2-\|w_{n-1}\|_Y^2\right\}\\ &+\frac{\mu_2}{2k}\left\{\|\nabla w_n\|_H^2-\|\nabla w_{n-1}\|_H^2\right\}+\frac{\gamma_2}{2k}\left\{\|\Delta w_n\|_Y^2-\|\Delta w_{n-1}\|_Y^2\right\}\\ &+\frac{\mu}{k}\Big\{(\nabla u_n,\nabla w_n)_H-(\nabla u_{n-1},\nabla w_{n-1})_H\Big\}\\ &+\frac{\gamma}{k}\Big\{(\Delta u_n,\Delta w_n)_Y-(\Delta u_{n-1},\Delta w_{n-1})_Y\Big\}\\ \leq& C\Big(\|v_n\|_Y^2+\|w_n\|_Y^2+\|u_n\|_Y^2+\|e_n\|_Y^2\Big).\end{aligned}$$

If we multiply these estimates by $k$ and we sum the corresponding estimates up to $n$, we have

$$\begin{aligned}&\rho_1\|v_n\|^2+\rho_2\|e_n\|_Y^2+\mu_1\|\nabla u_n\|_H^2+\gamma_1\|\Delta u_n\|_Y^2\\ &+a\|u_n\|^2+a\|w_n\|_Y^2+\mu_2\|\nabla w_n\|_H^2\\ &+\gamma_2\|\Delta w_n\|_Y^2+2\mu(\nabla u_n,\nabla w_n)_H+2\gamma(\Delta u_n,\Delta w_n)_Y\\ \leq& Ck\sum_{j=1}^n\Big(\|v_j\|_Y^2+\|w_j\|_Y^2+\|u_j\|_Y^2+\|e_j\|_Y^2\Big)\\ &+C\Big(\|v_0\|_Y^2+\|u_0\|_{H^2(B)}^2+\|e_0\|_Y^2+\|w_0\|_{H^2(B)}^2\Big).\end{aligned}$$

Again, by using assumptions (2) we find that

$$\begin{aligned}\mu_1\|\nabla u_n\|_H^2+\mu_2\|\nabla w_n\|_H^2+2\mu(\nabla u_n,\nabla w_n)_H\geq C(\|\nabla u_n\|_H^2+\|\nabla w_n\|_H^2),\\ \gamma_1\|\Delta u_n\|_Y^2+\gamma_2\|\Delta w_n\|_Y^2+2\gamma(\Delta u_n,\Delta w_n)_Y\geq C(\|\Delta u_n\|_Y^2+\|\Delta w_n\|_Y^2),\end{aligned}$$

and so, using the discrete Gronwall's inequality (see [9]), it leads to the desired stability property. □

In what follows, we will show some a priori error estimates for the fully discrete approximation of problem $VP$.

First, let us obtain the error estimates on the velocity of the first constituent. If we subtract (6) at time $t = t_n$ for $r = r^h \in V^h \subset V$ and discrete variational equation (11), we find that

$$\begin{aligned}
&\rho_1(\dot{v}_n - \delta v_n^{hk}, r^h)_Y + \mu_1(\nabla(u_n - u_n^{hk}), \nabla r^h)_H + \mu(\nabla(w_n - w_n^{hk}), \nabla r^h)_H \\
&+ \gamma_1(\Delta(u_n - u_n^{hk}), \Delta r^h)_Y + \gamma(\Delta(w_n - w_n^{hk}), \Delta r^h)_Y \\
&+ a((u_n - u_n^{hk}) - (w_n - w_n^{hk}), r^h)_Y + \mu^*(\nabla(v_n - v_n^{hk}), \nabla r^h)_H \\
&+ \gamma^*(\Delta(v_n - v_n^{hk}), \Delta r^h)_Y + a^*(v_n - v_n^{hk} - (w_n - w_n^{hk}), r^h)_Y = 0.
\end{aligned}$$

Therefore, we have, for all $r^h \in V^h$,

$$\begin{aligned}
&\rho_1(\dot{v}_n - \delta v_n^{hk}, v_n - v_n^{hk})_Y + \mu_1(\nabla(u_n - u_n^{hk}), \nabla(v_n - v_n^{hk}))_H \\
&+ \mu(\nabla(w_n - w_n^{hk}), \nabla(v_n - v_n^{hk}))_H + \gamma_1(\Delta(u_n - u_n^{hk}), \Delta(v_n - v_n^{hk}))_Y \\
&+ \gamma(\Delta(w_n - w_n^{hk}), \Delta(v_n - v_n^{hk}))_Y \\
&+ a(u_n - u_n^{hk} - (w_n - w_n^{hk}), v_n - v_n^{hk})_Y \\
&+ \mu^*(\nabla(v_n - v_n^{hk}), \nabla(v_n - v_n^{hk}))_H + \gamma^*(\Delta(v_n - v_n^{hk}), \Delta(v_n - v_n^{hk}))_Y \\
&+ a^*(v_n - v_n^{hk} - (w_n - w_n^{hk}), v_n - v_n^{hk})_Y \\
=&\rho_1(\dot{v}_n - \delta v_n^{hk}, v_n - r^h)_Y + \mu_1(\nabla(u_n - u_n^{hk}), \nabla(v_n - r^h))_H \\
&+ \mu(\nabla(w_n - w_n^{hk}), \nabla(v_n - r^h))_H + \gamma_1(\Delta(u_n - u_n^{hk}), \Delta(v_n - r^h))_Y \\
&+ \gamma(\Delta(w_n - w_n^{hk}), \Delta(v_n - r^h))_Y + a(u_n - u_n^{hk} - (w_n - w_n^{hk}), v_n - r^h)_Y \\
&+ \mu^*(\nabla(v_n - v_n^{hk}), \nabla(v_n - r^h))_H + \gamma^*(\Delta(v_n - v_n^{hk}), \Delta(v_n - r^h))_Y \\
&+ a^*(v_n - v_n^{hk} - (w_n - w_n^{hk}), v_n - r^h)_Y.
\end{aligned}$$

Using the estimates

$$\begin{aligned}
\left(\delta v_n - \delta v_n^{hk}, v_n - v_n^{hk}\right)_Y \geq& \frac{1}{2k}\Big\{\|v_n - v_n^{hk}\|_Y^2 - \|v_{n-1} - v_{n-1}^{hk}\|_Y^2\Big\}, \\
a(u_n - u_n^{hk}, v_n - v_n^{hk})_Y \geq& a(u_n - u_n^{hk}, \dot{u}_n - \delta u_n)_Y \\
&+ \frac{a}{2k}\Big\{\|u_n - u_n^{hk}\|_Y^2 - \|u_{n-1} - u_{n-1}^{hk}\|_Y^2\Big\}, \\
\mu_1(\nabla(u_n - u_n^{hk}), \nabla(v_n - v_n^{hk}))_H \geq& \mu_1(\nabla(u_n - u_n^{hk}), \nabla(\dot{u}_n - \delta u_n))_H \\
&+ \frac{\mu_1}{2k}\Big\{\|\nabla(u_n - u_n^{hk})\|_H^2 - \|\nabla(u_{n-1} - u_{n-1}^{hk})\|_H^2 \\
&+ \|\nabla(u_n - u_n^{hk} - (u_{n-1} - u_n^{hk}))\|_H^2\Big\}, \\
\gamma_1(\Delta(u_n - u_n^{hk}), \Delta(v_n - v_n^{hk}))_Y \geq& \gamma_1(\Delta(u_n - u_n^{hk}), \Delta(\dot{u}_n - \delta u_n))_Y \\
&+ \frac{\gamma_1}{2k}\Big\{\|\Delta(u_n - u_n^{hk})\|_Y^2 - \|\Delta(u_{n-1} - u_{n-1}^{hk})\|_Y^2 \\
&+ \|\Delta(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk}))\|_Y^2\Big\},
\end{aligned}$$

we obtain, for all $r^h \in V^h$,

$$\begin{aligned}
&\frac{\rho_1}{2k}\Big\{\|v_n - v_n^{hk}\|_Y^2 - \|v_{n-1} - v_{n-1}^{hk}\|_Y^2\Big\} + \mu(\nabla(w_n - w_n^{hk}), \nabla(\delta u_n - \delta u_n^{hk}))_H \\
&+ \frac{a}{2k}\Big\{\|u_n - u_n^{hk}\|_Y^2 - \|u_{n-1} - u_{n-1}^{hk}\|_Y^2\Big\} + \gamma(\Delta(w_n - w_n^{hk}), \Delta(\delta u_n - \delta u_n^{hk}))_Y \\
&+ \frac{\mu_1}{2k}\Big\{\|\nabla(u_n - u_n^{hk})\|_H^2 - \|\nabla(u_{n-1} - u_{n-1}^{hk})\|_H^2 \\
&+ \|\nabla(u_n - u_n^{hk} - (u_{n-1} - u_n^{hk}))\|_H^2\Big\} \\
&+ \frac{\gamma_1}{2k}\Big\{\|\Delta(u_n - u_n^{hk})\|_Y^2 - \|\Delta(u_{n-1} - u_{n-1}^{hk})\|_Y^2 \\
&+ \|\Delta(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk}))\|_Y^2\Big\} \\
\leq & C\Big(\|\dot{v}_n - \delta v_n\|_Y^2 + \|\dot{u}_n - \delta u_n\|_{H^2(B)}^2 + \|v_n - r^h\|_{H^2(B)}^2 \\
&+ \|u_n - u_n^{hk}\|_Y^2 + \|e_n - e_n^{hk}\|_Y^2 \\
&+ \|v_n - v_n^{hk}\|_Y^2 + \|w_n - w_n^{hk}\|_Y^2 + \|\nabla(w_n - w_n^{hk})\|_H^2 + \|\Delta(w_n - w_n^{hk})\|_Y^2 \\
&+ \|\nabla(u_n - u_n^{hk})\|_H^2 + \|\Delta(u_n - u_n^{hk})\|_Y^2 + (\delta v_n - \delta v_n^{hk}, v_n - r^h)_Y\Big).
\end{aligned}$$

Proceeding in a similar way, we have the following estimates for the velocity of the second constituent, for all $z^h \in V^h$,

$$\begin{aligned}
&\frac{\rho_2}{2k}\Big\{\|e_n - e_n^{hk}\|_Y^2 - \|e_{n-1} - e_{n-1}^{hk}\|_Y^2\Big\} + \mu(\nabla(u_n - u_n^{hk}), \nabla(\delta w_n - \delta w_n^{hk}))_H \\
&+ \frac{a}{2k}\Big\{\|w_n - w_n^{hk}\|_Y^2 - \|w_{n-1} - w_{n-1}^{hk}\|_Y^2\Big\} \\
&+ \gamma(\Delta(u_n - u_n^{hk}), \Delta(\delta w_n - \delta w_n^{hk}))_Y \\
&+ \frac{\mu_2}{2k}\Big\{\|\nabla(w_n - w_n^{hk})\|_H^2 - \|\nabla(w_{n-1} - w_{n-1}^{hk})\|_H^2 \\
&+ \|\nabla(w_n - w_n^{hk} - (w_{n-1} - w_n^{hk}))\|_H^2\Big\} \\
&+ \frac{\gamma_2}{2k}\Big\{\|\Delta(w_n - w_n^{hk})\|_Y^2 - \|\Delta(w_{n-1} - w_{n-1}^{hk})\|_Y^2 \\
&+ \|\Delta(w_n - w_n^{hk} - (w_{n-1} - w_n^{hk}))\|_Y^2\Big\} \\
\leq & C\Big(\|\dot{e}_n - \delta e_n\|_Y^2 + \|\dot{w}_n - \delta w_n\|_{H^2(B)}^2 + \|e_n - z^h\|_{H^2(B)}^2 + \|w_n - w_n^{hk}\|_Y^2 \\
&+ \|e_n - e_n^{hk}\|_Y^2 + \|u_n - u_n^{hk}\|_Y^2 + \|\nabla(u_n - u_n^{hk})\|_H^2 \\
&+ \|\Delta(u_n - u_n^{hk})\|_Y^2 + \|v_n - v_n^{hk}\|_Y^2 \\
&+ \|\nabla(w_n - w_n^{hk})\|_H^2 + \|\Delta(w_n - w_n^{hk})\|_Y^2 + (\delta e_n - \delta e_n^{hk}, e_n - z^h)_Y\Big).
\end{aligned}$$

Thanks again to (2) we find that

$$\begin{aligned}
&\mu(\nabla(w_n - w_n^{hk}), \nabla(\delta u_n - \delta u_n^{hk}))_H + \mu(\nabla(u_n - u_n^{hk}), \nabla(\delta w_n - \delta w_n^{hk}))_H \\
=&\frac{\mu}{k}\Big\{(\nabla(u_n - u_n^{hk}), \nabla(w_n - w_n^{hk}))_H - (\nabla(u_{n-1} - u_{n-1}^{hk}), \nabla(w_{n-1} - w_{n-1}^{hk}))_H \\
&+ (\nabla(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk})), \nabla(w_n - w_n^{hk} - (w_{n-1} - w_{n-1}^{hk})))_H\Big\},
\end{aligned}$$

$$
\begin{aligned}
&\gamma(\Delta(w_n - w_n^{hk}), \Delta(\delta u_n - \delta u_n^{hk}))_Y + \gamma(\Delta(u_n - u_n^{hk}), \Delta(\delta w_n - \delta w_n^{hk}))_Y \\
=&\frac{\gamma}{k}\Big\{(\Delta(u_n - u_n^{hk}), \Delta(w_n - w_n^{hk}))_Y - (\Delta(u_{n-1} - u_{n-1}^{hk}), \Delta(w_{n-1} - w_{n-1}^{hk}))_Y \\
&+ (\Delta(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk})), \Delta(w_n - w_n^{hk} - (w_{n-1} - w_{n-1}^{hk})))_Y\Big\}, \\
&\mu_1(\nabla(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk})), \nabla(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk})))_H \\
&+ \mu_2(\nabla(w_n - w_n^{hk} - (w_{n-1} - w_{n-1}^{hk})), \nabla(w_n - w_n^{hk} - (w_{n-1} - w_{n-1}^{hk})))_H \\
&+ 2\mu(\nabla(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk})), \nabla(w_n - w_n^{hk} - (w_{n-1} - w_{n-1}^{hk})))_H \geq 0, \\
&\gamma_1(\Delta(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk})), \Delta(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk})))_Y \\
&+ \gamma_2(\Delta(w_n - w_n^{hk} - (w_{n-1} - w_{n-1}^{hk})), \Delta(w_n - w_n^{hk} - (w_{n-1} - w_{n-1}^{hk})))_Y \\
&+ 2\gamma(\Delta(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk})), \Delta(w_n - w_n^{hk} - (w_{n-1} - w_{n-1}^{hk})))_Y \geq 0,
\end{aligned}
$$

and so, combining the previous estimates, we obtain

$$
\begin{aligned}
&\frac{\rho_1}{2k}\Big\{\|v_n - v_n^{hk}\|_Y^2 - \|v_{n-1} - v_{n-1}^{hk}\|_Y^2\Big\} + \frac{a}{2k}\Big\{\|u_n - u_n^{hk}\|_Y^2 - \|u_{n-1} - u_{n-1}^{hk}\|_Y^2\Big\} \\
&+ \frac{\mu_1}{2k}\Big\{\|\nabla(u_n - u_n^{hk})\|_H^2 - \|\nabla(u_{n-1} - u_{n-1}^{hk})\|_H^2\Big\} \\
&+ \frac{\gamma_1}{2k}\Big\{\|\Delta(u_n - u_n^{hk})\|_Y^2 - \|\Delta(u_{n-1} - u_{n-1}^{hk})\|_Y^2\Big\} \\
&+ \frac{\rho_2}{2k}\Big\{\|e_n - e_n^{hk}\|_Y^2 - \|e_{n-1} - e_{n-1}^{hk}\|_Y^2\Big\} \\
&+ \frac{a}{2k}\Big\{\|w_n - w_n^{hk}\|_Y^2 - \|w_{n-1} - w_{n-1}^{hk}\|_Y^2\Big\} \\
&+ \frac{\mu_2}{2k}\Big\{\|\nabla(w_n - w_n^{hk})\|_H^2 - \|\nabla(w_{n-1} - w_{n-1}^{hk})\|_H^2\Big\} \\
&+ \frac{\gamma_2}{2k}\Big\{\|\Delta(w_n - w_n^{hk})\|_Y^2 - \|\Delta(w_{n-1} - w_{n-1}^{hk})\|_Y^2\Big\} \\
&+ \frac{\mu}{k}\Big\{(\nabla(u_n - u_n^{hk}), \nabla(w_n - w_n^{hk}))_H - (\nabla(u_{n-1} - u_{n-1}^{hk}), \nabla(w_{n-1} - w_{n-1}^{hk}))_H\Big\} \\
&+ \frac{\gamma}{k}\Big\{(\Delta(u_n - u_n^{hk}), \Delta(w_n - w_n^{hk}))_Y - (\Delta(u_{n-1} - u_{n-1}^{hk}), \Delta(w_{n-1} - w_{n-1}^{hk}))_Y\Big\} \\
\leq& C\Big(\|\dot{v}_n - \delta v_n\|_Y^2 + \|\dot{u}_n - \delta u_n\|_{H^2(B)}^2 + \|v_n - r^h\|_{H^2(B)}^2 + \|u_n - u_n^{hk}\|_Y^2 \\
&+ \|v_n - v_n^{hk}\|_Y^2 + \|w_n - w_n^{hk}\|_Y^2 + \|\nabla(w_n - w_n^{hk})\|_H^2 + \|\Delta(w_n - w_n^{hk})\|_Y^2 \\
&+ \|\nabla(u_n - u_n^{hk})\|_H^2 + \|\Delta(u_n - u_n^{hk})\|_Y^2 + (\delta v_n - \delta v_n^{hk}, v_n - r^h)_Y \\
&+ \|\dot{e}_n - \delta e_n\|_Y^2 + \|\dot{w}_n - \delta w_n\|_{H^2(B)}^2 + \|e_n - z^h\|_{H^2(B)}^2 + \|e_n - e_n^{hk}\|_Y^2 \\
&+ (\delta e_n - \delta e_n^{hk}, e_n - z^h)_Y\Big).
\end{aligned}
$$

If we multiply these estimates by $k$ and we sum up to $n$, it follows, for all $\{r_j^h\}_{j=1}^n$, $\{z_j^h\}_{j=1}^n \subset V^h$,

$$
\begin{aligned}
&\rho_1\|v_n - v_n^{hk}\|_Y^2 + a\|u_n - u_n^{hk}\|_Y^2 + \mu_1\|\nabla(u_n - u_n^{hk})\|_H^2 + \gamma_1\|\Delta(u_n - u_n^{hk})\|_Y^2 \\
&+ \rho_2\|e_n - e_n^{hk}\|_Y^2 + a\|w_n - w_n^{hk}\|_Y^2 + \mu_2\|\nabla(w_n - w_n^{hk})\|_H^2 + \gamma_2\|\Delta(w_n - w_n^{hk})\|_Y^2 \\
&+ 2\mu(\nabla(u_n - u_n^{hk}), \nabla(w_n - w_n^{hk}))_H + 2\gamma(\Delta(u_n - u_n^{hk}), \Delta(w_n - w_n^{hk}))_Y \\
\leq& Ck\sum_{j=1}^n \Big[\|\dot{v}_j - \delta v_j\|_Y^2 + \|\dot{u}_j - \delta u_j\|_{H^2(B)}^2 + \|v_j - r_j^h\|_{H^2(B)}^2 + \|u_j - u_j^{hk}\|_Y^2
\end{aligned}
$$

$$
\begin{aligned}
&+\|v_j-v_j^{hk}\|_Y^2+\|w_j-w_j^{hk}\|_Y^2+\|\nabla(w_j-w_j^{hk})\|_H^2+\|\Delta(w_j-w_j^{hk})\|_Y^2\\
&+\|\nabla(u_j-u_j^{hk})\|_H^2+\|\Delta(u_j-u_j^{hk})\|_Y^2+(\delta v_j-\delta v_j^{hk},v_j-r_j^h)_Y\\
&+\|\dot e_j-\delta e_j\|_Y^2+\|\dot w_j-\delta w_j\|_{H^2(B)}^2+\|e_j-z_j^h\|_{H^2(B)}^2+\|e_j-e_j^{hk}\|_Y^2\\
&+(\delta e_j-\delta e_j^{hk},e_j-z_j^h)_Y\Big]+C\Big(\|v_0-v_0^h\|_Y^2+\|u_0-u_0^h\|_{H^2(B)}^2\\
&+\|e_0-e_0^h\|_Y^2+\|w_0-w_0^h\|_{H^2(B)}^2\Big).
\end{aligned}
$$

Keeping in mind that

$$
\begin{aligned}
&\mu_1\|\nabla(u_n-u_n^{hk})\|_H^2+\mu_2\|\nabla(w_n-w_n^{hk})\|_H^2+2\mu(\nabla(u_n-u_n^{hk}),\nabla(w_n-w_n^{hk}))_H\\
\ge& C(\|\nabla(u_n-u_n^{hk})\|_H^2+\|\nabla(w_n-w_n^{hk})\|_H^2),\\
&\gamma_1\|\Delta(u_n-u_n^{hk})\|_Y^2+\gamma_2\|\Delta(w_n-w_n^{hk})\|_Y^2+2\gamma(\Delta(u_n-u_n^{hk}),\Delta(w_n-w_n^{hk}))_Y\\
\ge& C(\|\Delta(u_n-u_n^{hk})\|_Y^2+\|\Delta(w_n-w_n^{hk})\|_Y^2),
\end{aligned}
$$

where we have used again assumptions (2), we find that, for all $\{r_j^h\}_{j=1}^n, \{z_j^h\}_{j=1}^n\subset V^h$,

$$
\begin{aligned}
&\|v_n-v_n^{hk}\|_Y^2+\|u_n-u_n^{hk}\|_Y^2+\|\nabla(u_n-u_n^{hk})\|_H^2+\|\Delta(u_n-u_n^{hk})\|_Y^2\\
&+\|e_n-e_n^{hk}\|_Y^2+\|w_n-w_n^{hk}\|_Y^2+\|\nabla(w_n-w_n^{hk})\|_H^2+\|\Delta(w_n-w_n^{hk})\|_Y^2\\
\le& Ck\sum_{j=1}^n\Big[\|\dot v_j-\delta v_j\|_Y^2+\|\dot u_j-\delta u_j\|_{H^2(B)}^2+\|v_j-r_j^h\|_{H^2(B)}^2+\|u_j-u_j^{hk}\|_Y^2\\
&+\|v_j-v_j^{hk}\|_Y^2+\|w_j-w_j^{hk}\|_Y^2+\|\nabla(w_j-w_j^{hk})\|_H^2+\|\Delta(w_j-w_j^{hk})\|_Y^2\\
&+\|\nabla(u_j-u_j^{hk})\|_H^2+\|\Delta(u_j-u_j^{hk})\|_Y^2+(\delta v_j-\delta v_j^{hk},v_j-r_j^h)_Y\\
&+\|\dot e_j-\delta e_j\|_Y^2+\|\dot w_j-\delta w_j\|_{H^2(B)}^2+\|e_j-z_j^h\|_{H^2(B)}^2+\|e_j-e_j^{hk}\|_Y^2\\
&+(\delta e_j-\delta e_j^{hk},e_j-z_j^h)_Y\Big]+C\Big(\|v_0-v_0^h\|_Y^2+\|u_0-u_0^h\|_{H^2(B)}^2\\
&+\|e_0-e_0^h\|_Y^2+\|w_0-w_0^h\|_{H^2(B)}^2\Big).
\end{aligned}
$$

Finally, if we take into account that

$$
\begin{aligned}
&k\sum_{j=1}^n(\delta v_j-\delta v_j^{hk},v_j-r_j^h)_Y=\sum_{j=1}^n(v_j-v_j^{hk}-(v_{j-1}-v_{j-1}^{hk}),v_j-r_j^h)_Y\\
=&(v_n-v_n^{hk},v_n-r_n^h)_Y+(v_0^h-v_0,v_1-r_1^h)_Y\\
&+\sum_{j=1}^{n-1}(v_j-v_j^{hk},v_j-r_j^h-(v_{j+1}-r_{j+1}^h))_Y,\\
&k\sum_{j=1}^n(\delta e_j-\delta e_j^{hk},e_j-z_j^h)_Y=\sum_{j=1}^n(e_j-e_j^{hk}-(e_{j-1}-e_{j-1}^{hk}),e_j-z_j^h)_Y\\
=&(e_n-e_n^{hk},e_n-z_n^h)_Y+(e_0^h-e_0,e_1-z_1^h)_Y\\
&+\sum_{j=1}^{n-1}(e_j-e_j^{hk},e_j-z_j^h-(e_{j+1}-z_{j+1}^h))_Y,
\end{aligned}
$$

and we use the discrete Gronwall's inequality (see, for instance, [9]), we conclude some a priori error estimates that we state in the following theorem.

**Theorem 3.** *Under the assumptions of Theorem 1, if we denote by* $(u, v, w, e)$ *the solution to problem* $VP$ *and by* $(u^{hk}, v^{hk}, w^{hk}, e^{hk})$ *the solution to problem* $VP^{hk}$*, then it follows that, for all* $r^h = \{r_j^h\}_{j=0}^N$*,* $z^h = \{z_j^h\}_{j=0}^N \subset V^h$*,*

$$
\begin{aligned}
&\max_{0\le n\le N} \Big\{ \|v_n - v_n^{hk}\|_Y^2 + \|u_n - u_n^{hk}\|_Y^2 + \|\nabla(u_n - u_n^{hk})\|_H + \|\Delta(u_n - u_n^{hk})\|_Y^2 \\
&\quad + \|e_n - e_n^{hk}\|_Y^2 + \|w_n - w_n^{hk}\|_Y^2 + \|\nabla(w_n - w_n^{hk})\|_H^2 + \|\Delta(w_n - w_n^{hk})\|_Y^2 \Big\} \\
&\le Ck \sum_{j=1}^N \Big[ \|\dot v_j - \delta v_j\|_Y^2 + \|\dot u_j - \delta u_j\|_{H^2(B)}^2 + \|v_j - r_j^h\|_{H^2(B)}^2 + \|\dot e_j - \delta e_j\|_Y^2 \\
&\quad + \|\dot w_j - \delta w_j\|_{H^2(B)}^2 + \|e_j - z_j^h\|_{H^2(B)}^2 \Big] \\
&\quad + \frac{C}{k} \sum_{j=1}^{N-1} \Big[ \|v_j - r_j^h - (v_{j+1} - r_{j+1}^h)\|_Y^2 + \|e_j - z_j^h - (e_{j+1} - z_{j+1}^h)\|_Y^2 \Big] \\
&\quad + C \max_{0\le n\le N} \|v_n - r_n^h\|_Y^2 + C \max_{0\le n\le N} \|e_n - z_n^h\|_Y^2 \\
&\quad + C\Big( \|v_0 - v_0^h\|_Y^2 + \|u_0 - u_0^h\|_{H^2(B)}^2 + \|e_0 - e_0^h\|_Y^2 + \|w_0 - w_0^h\|_{H^2(B)}^2 \Big), \qquad (15)
\end{aligned}
$$

*where the positive constant* $C$ *is independent of the discretization parameters* $h$ *and* $k$*.*

It is worth noting that, from estimates (15), if we assume some regularity conditions on the continuous solution, we can obtain the convergence order of the approximations which we summarize in the following.

**Corollary 4.** *Under the conditions of Theorem 3 and the additional regularity:*

$$u, w \in H^3(0, T; Y) \cap W^{1,\infty}(0, T; H^3(\Omega)) \cap H^2(0, T; H^1(\Omega)),$$

*we conclude the linear convergence of the approximations obtained from Problem* $VP^{hk}$*; that is, there exists a constant* $C > 0$*, independent of parameters* $h$ *and* $k$*, such that*

$$
\begin{aligned}
&\max_{0\le n\le N} \Big\{ \|v_n - v_n^{hk}\|_Y + \|u_n - u_n^{hk}\|_{H^2(B)} + \|e_n - e_n^{hk}\|_Y + \|w_n - w_n^{hk}\|_{H^2(B)} \Big\} \\
&\le C(h + k).
\end{aligned}
$$

## 4. Numerical results

In this final section, we present some numerical examples which demonstrate the accuracy of the approximations and the behavior of the solution.

We note that, given the solution $u_{n-1}^{hk}$ and $w_{n-1}^{hk}$ at time $t_{n-1}$, variables $v_n^{hk}$ and $e_n^{hk}$ are obtained by solving the discrete linear system, for all $r^h$, $z^h \in V^h$,

$$
\begin{aligned}
&\frac{\rho_1}{k}(v_n^{hk}, r^h)_Y + k\mu_1(\nabla v_n^{hk}, \nabla r^h)_H + k\gamma_1(\Delta v_n^{hk}, \Delta r^h)_Y + ka(v_n^{hk}, r^h)_Y \\
&\quad + \mu^*(\nabla v_n^{hk}, \nabla r^h)_H + \gamma^*(\Delta v_n^{hk}, \Delta r^h)_Y + a^*(v_n^{hk}, r^h)_Y \\
&= \frac{\rho_1}{k}(v_{n-1}^{hk}, r^h)_Y - \mu_1(\nabla u_{n-1}^{hk}, \nabla r^h)_H - \mu(\nabla w_n^{hk}, \nabla r^h)_H - \gamma_1(\Delta u_{n-1}^{hk}, \Delta r^h)_Y \\
&\quad - \gamma(\Delta w_n^{hk}, \Delta r^h)_Y - a(u_{n-1}^{hk} - w_n^{hk}, r^h)_Y + a^*(e_n^{hk}, r^h)_Y,
\end{aligned}
$$

$$\begin{aligned}
&\frac{\rho_2}{k}(e_n^{hk}, z^h)_Y + \mu_2 k(\nabla e_n^{hk}, \nabla z^h)_H + \gamma_2 k(\Delta e_n^{hk}, \Delta z^h)_Y \\
&+ka(e_n^{hk}, z^h)_Y + a^*(e_n^{hk}, z^h)_Y \\
=&\frac{\rho_2}{k}(e_{n-1}^{hk}, z^h)_Y - \mu(\nabla u_n^{hk}, \nabla z^h)_H - \mu_2(\nabla w_{n-1}^{hk}, \nabla z^h)_H - \gamma(\Delta u_n^{hk}, \Delta z^h)_Y \\
&-\gamma_2(\Delta w_{n-1}^{hk}, \Delta z^h)_Y + a(w_{n-1}^{hk} - u_n^{hk}, z^h)_Y + a^*(v_n^{hk}, r^h)_Y.
\end{aligned}$$

This numerical scheme was implemented by using MATLAB, and, regarding the CPU time, we note that a typical one-dimensional run ($h = k = 0.001$) took about 0.86 seconds.

**4.1. First example: numerical convergence in a one-dimensional problem.** As a simple example, in order to show the accuracy of the approximations the one-dimensional version of problem (1), (4) and (3) is solved with the following data:

$$\begin{aligned}
&T = 1, \quad B = (0,1), \quad \rho_1 = 1, \quad \rho_2 = 1, \quad \mu = 1, \quad \mu_1 = 2, \quad \mu_2 = 2, \\
&\gamma = 1, \quad \gamma_1 = 2, \quad \gamma_2 = 2, \quad a = 1, \quad \gamma^* = 0, \quad \mu^* = 0, \quad a^* = 0.
\end{aligned}$$

By using the initial conditions, for all $x \in [0,1]$,

$$u_0(x) = v_0(x) = w_0(x) = e_0(x) = x^3(x-1)^3,$$

homogeneous null boundary conditions and the (artificial) supply terms, for all $(x,t) \in (0,1) \times (0,1)$,

$$\begin{aligned}
F_1(x,t) &= e^t(x^6 - 3x^5 - 87x^4 + 179x^3 + 972x^2 - 1062x + 216), \\
F_2(x,t) &= e^t(x^6 - 3x^5 - 87x^4 + 179x^3 + 972x^2 - 1062x + 216),
\end{aligned}$$

the exact solution to this problem can be easily calculated and it has the form, for $(x,t) \in [0,1] \times [0,1]$:

$$u(x,t) = w(x,t) = e^t x^3 (x-1)^3.$$

Therefore, if we estimate the approximation errors by

$$\max_{0 \le n \le N} \left\{ \|v_n - v_n^{hk}\|_Y + \|u_n - u_n^{hk}\|_{H^2(B)} + \|e_n - e_n^{hk}\|_Y + \|w_n - w_n^{hk}\|_{H^2(B)} \right\},$$

we present them in TABLE 1 for several values of the discretization parameters $h$ and $k$. Moreover, the evolution of the error depending on the parameter $h + k$ is plotted in FIGURE 1. As can be clearly seen, the convergence of the algorithm is shown, and we also found the linear convergence proved in Corollary 4.

TABLE 1. Example 1: Numerical errors for some values of $h$ and $k$.

| $h \downarrow k \rightarrow$ | 0.01 | 0.005 | 0.002 | 0.001 | 0.0005 | 0.0002 | 0.0001 |
|---|---|---|---|---|---|---|---|
| $1/2^3$ | 0.900869 | 0.902092 | 0.917193 | 0.931667 | 0.939786 | 0.943989 | 0.947605 |
| $1/2^4$ | 0.453140 | 0.453514 | 0.458135 | 0.462674 | 0.465456 | 0.469465 | 0.474052 |
| $1/2^5$ | 0.224097 | 0.224100 | 0.225168 | 0.226371 | 0.227407 | 0.230378 | 0.233664 |
| $1/2^6$ | 0.111218 | 0.111059 | 0.111211 | 0.111477 | 0.111955 | 0.113852 | 0.115790 |
| $1/2^7$ | 0.055673 | 0.055323 | 0.055255 | 0.055298 | 0.055546 | 0.056613 | 0.057650 |
| $1/2^8$ | 0.028447 | 0.027762 | 0.027567 | 0.027553 | 0.027678 | 0.028236 | 0.028769 |
| $1/2^9$ | 0.015449 | 0.014205 | 0.013819 | 0.013767 | 0.013821 | 0.014102 | 0.014372 |
| $1/2^{10}$ | 0.009715 | 0.007726 | 0.007013 | 0.006910 | 0.006908 | 0.007048 | 0.007189 |
| $1/2^{11}$ | 0.007528 | 0.004812 | 0.003707 | 0.003519 | 0.003466 | 0.003479 | 0.003558 |
| $1/2^{12}$ | 0.006808 | 0.003791 | 0.000329 | 0.000244 | 0.000225 | 0.001704 | 0.002698 |
| $1/2^{13}$ | 0.018605 | 0.011654 | 0.006018 | 0.007580 | 0.000174 | 0.009922 | 0.001363 |

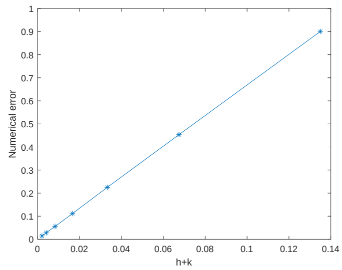


FIGURE 1. Example 1: Asymptotic constant error.

**4.2. Second example: Energy decay in a one-dimensional example.** In this case, we are going to compare the energy decay of problem (1), (3) and (4) if we consider the three dissipation mechanisms studied in this work.

We assume now that there are not supply terms, and we use the final time $T = 10$ and the data:

$$B = (0,1), \quad \rho_1 = 1, \quad \rho_2 = 1, \quad \mu = 1, \quad \mu_1 = 2, \quad \mu_2 = 2,$$
$$\gamma = 1, \quad \gamma_1 = 2, \quad \gamma_2 = 2, \quad a = 1.$$

The values of parameters $\gamma^*, \mu^*, a^*$ depend on the mechanisms we are considering. We recall that the three mechanisms are hyperviscosity ($\gamma^* = 1$ and $a^* = \mu^* = 0$), the weak viscosity ($a^* = 1$ and $\gamma^* = \mu^* = 0$), and the viscosity ($\mu^* = 1$ and $a^* = \gamma^* = 0$). The initial conditions are the same for the three cases and they are defined as, for all $x \in (0,1)$,

$$u_0(x) = v_0(x) = w_0(x) = e_0(x) = x^3(x-1)^3.$$

Thus, taking the discretization parameters $h = 0.001$ and $k = 0.001$, the evolution in time of the discrete energy of problem (1), (3) and (4), obtained from the expression (9), is plotted in FIGURE 2 (in both natural and semi-log scales) for the three dissipation mechanisms. As we can see, the energy decay seems to be exponential in all the cases but the best dissipation is found for the viscosity case (the second-order mechanism), being the worst case, as expected, the weak viscosity.

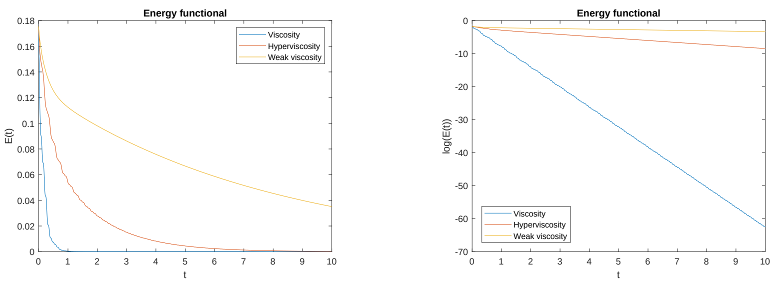


FIGURE 2. Example 2: Evolution in time of the discrete energy for the three dissipation mechanisms (natural and semi-log scales).

**4.3. Third example: Numerical results in a two-dimensional problem.** For this third example, we restrict ourselves to the two-dimensional setting and we consider the square domain $B = [0,1] \times [0,1]$ which is assumed to be fixed on its boundary. Our aim is to study the dependence of the solution with respect to the coupling parameter $\gamma$.

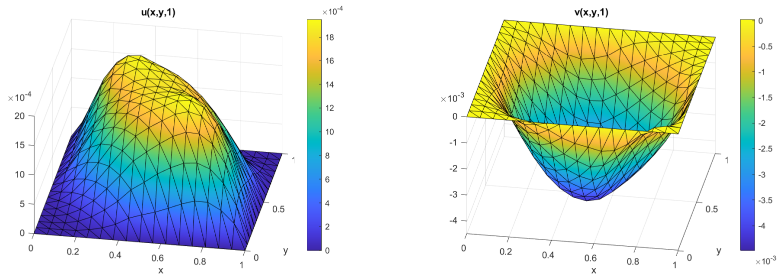


FIGURE 3. Example 2: Displacement (left) and velocity (right) of the first constituent at final time for $\gamma = 1$.

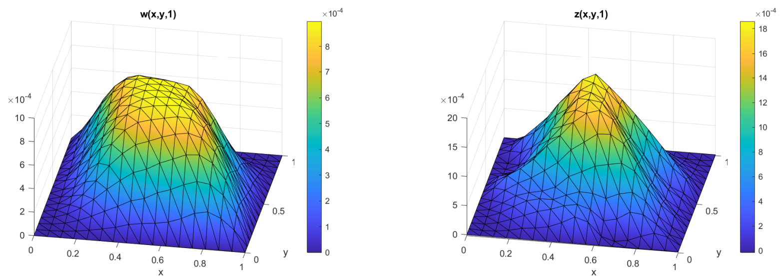


FIGURE 4. Example 2: Displacement (left) and velocity (right) of the second constituent at final time for $\gamma = 1$.

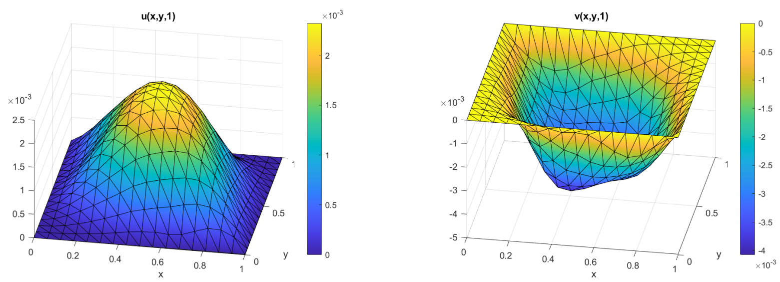


FIGURE 5. Example 2: Displacement (left) and velocity (right) of the first constituent at final time for $\gamma = 5$.

The following data have been employed in this simulation:

$$T = 1, \quad \rho_1 = 1, \quad \rho_2 = 1, \quad \mu = 1, \quad \mu_1 = 3, \quad \mu_2 = 1,$$
$$\gamma = 1, \quad \gamma_1 = 5, \quad \gamma_2 = 6, \quad a = 1, \quad \gamma^* = 1, \quad \mu^* = 0, \quad a^* = 0,$$

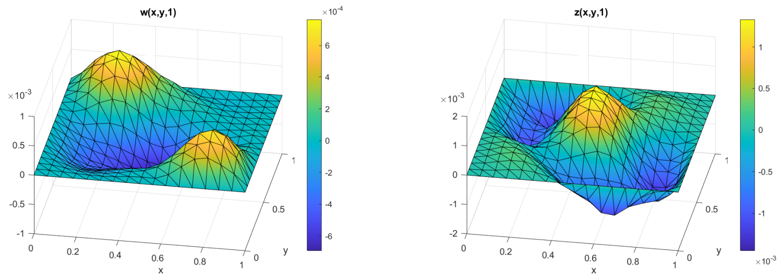


FIGURE 6. Example 2: Displacement (left) and velocity (right) of the second constituent at final time for $\gamma = 5$.

and the initial conditions, for $(x, y) \in (0, 1) \times (0, 1)$,

$$u_0(x, y) = v_0(x, y) = w_0(x, y) = e_0(x, y) = x^2(x-1)^2y^2(y-1)^2.$$

If we take now the time discretization parameter $k = 0.01$ and the mesh size $h = \frac{\sqrt{2}}{20}$, in FIGURES 3 and 4 we plot the displacements (left) and velocities (right) at final time for the first and second constituents, respectively, for $\gamma = 1$. Finally, we show the displacements (left) and velocities (right) at final time for the first and second constituents, respectively, for $\gamma = 5$ in FIGURES 5 and 6. We observe that for the greater value of the parameter the differences are huge because the dependence between the functions is greater.

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