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A SHARP α-ROBUST $L1$ SCHEME ON GRADED MESHES FOR TWO-DIMENSIONAL TIME TEMPERED FRACTIONAL FOKKER-PLANCK EQUATION

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Abstract. In this paper, we are concerned with the numerical solution for the two-dimensional time fractional Fokker-Planck equation with the tempered fractional derivative of order α. Although some of its variants are considered in many recent numerical analysis works, there are still some significant differences. Here we first provide the regularity estimates of the solution. Then a modified $L1$ scheme inspired by the middle rectangle quadrature formula on graded meshes is employed to compensate for the singularity of the solution at $t \to 0^+$, while the five-point difference scheme is used in space. Stability and convergence are proved in the sense of L^∞ norm, getting a sharp error estimate $\mathscr{O}(\tau^{\min\{2-\alpha, r\alpha\}})$ on graded meshes. Furthermore, the constant multipliers in the analysis do not blow up as the order of Caputo fractional derivative α approaches the classical value of 1. Finally, we perform the numerical experiments to verify the effectiveness and convergence orders of the presented schemes.

Key words. Fractional diffusion equation, weak singularity, middle rectangle quadrature formula, modified $L1$ scheme, five-point difference scheme, graded mesh, α-robust.

1. Introduction

Anomalous diffusion with mean squared displacement (MSD) $\langle x^2(t)\rangle \simeq t^\alpha$, including subdiffusion and superdiffusion, is ubiquitously observed in a wide range of complex systems [1, 2, 3], and its anomalous diffusion exponent differs from the value $\alpha = 1$ of Brownian motion. Subdiffusion with $0 < \alpha < 1$ often occurs in cytoplasm of biological cells [4], amorphous semiconductors [5], or in hydrology [6]. Superdiffusion with $\alpha > 1$ is observed in some active systems such as molecular motor transport in cells [7] or in turbulence [8]. The continuous time random walk (CTRW) is one of the central stochastic models for both regimes of anomalous diffusion, which is based on two identically distributed random variables of the waiting times τ between any two jumps and the single jump lengths x. In fact, based on the fractional Fourier law and conservation law, the fractional diffusion equations (FDEs) can be also derived.

In this paper, we consider the generalized two-dimensional time fractional Fokker-Planck equation [9, 10]

$$(1) \qquad \frac{\partial}{\partial t}u(x,y,t) = \frac{\partial}{\partial t}\int_0^t K(t-s,\mu)\Delta u(x,y,s)\,\mathrm{d}s,$$

which is derived from a CTRW model with the tempered α-stable waiting times. Here the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is solved in a rectangular domain $\Omega = (0, L_1) \times (0, L_2)$ and the Laplace transform of the memory kernel is given by $\widehat{K}(\lambda,\mu) = \frac{1}{(\lambda+\mu)^\alpha - \mu^\alpha}$ with the tempering index $\mu > 0$ and stability index $0 < \alpha < 1$. In fact, Ref. [10] indicates that the second moment of the CTRW model corresponding to (1) is $\langle x^2(t)\rangle \simeq t^\alpha$ as $t \to 0$, while $\langle x^2(t)\rangle \simeq t$ as $t \to \infty$. We first

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transform Eq. (1) into an equivalent Eq. (A.1) (see Appendix A). Without loss of generality, we consider Eq. (A.1) with a source term $f(x,y,t)$. Then under initial condition and homogeneous Dirichlet boundary condition, we discuss the time fractional Fokker-Planck initial-boundary value problem:

$$\begin{cases} \partial_t^{\alpha,\mu} u(x,y,t) - \Delta u(x,y,t) = f(x,y,t), & (x,y,t) \in Q := \Omega \times (0,T], \\ u(x,y,t) = 0, & (x,y,t) \in \partial\Omega \times (0,T], \\ u(x,y,0) = \phi(x,y), & (x,y) \in \bar{\Omega}, \end{cases} \tag{2}$$

where Ω is a bounded domain, $\bar{Q} := \bar{\Omega} \times [0,T]$, and f and ϕ are given functions; the time fractional derivative $\partial_t^{\alpha,\mu} u(x,y,t)$ is the tempered Caputo fractional derivative of order α, defined by

$$\partial_t^{\alpha,\mu} u(x,y,t) = \int_0^t w_\mu^\alpha(t-s) \frac{\partial}{\partial s} u(x,y,s)\, \mathrm{d}s \tag{3}$$

with

$$w_\mu^\alpha(t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_t^\infty e^{-\mu s} s^{-1-\alpha}\, \mathrm{d}s. \tag{4}$$

Here $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t}\, \mathrm{d}t$ is the Gamma function. Clearly, when $\mu = 0$, $\partial_t^{\alpha,\mu} u$ is just the classical Caputo fractional derivative of order α.

However, since the nonlocal properties of fractional operators, it is more challenging or sometimes even impossible to obtain the analytical solutions of FDEs, or the obtained analytical solutions are less practicable (expressed by the transcendental functions or infinite series). Efficiently solving FDEs naturally has always been an urgent topic. So far there have been many works, including the finite difference method, finite element method, spectral method, and so on [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. In particular, for the fractional derivatives in Caputo sense, the $L1$-type scheme [20, 21] and k-step backward difference formulae [15, 16, 22, 23] on uniform meshes are two popular and predominant discretization techniques. To the best of our knowledge, the smoothness of all the known data of (2) does not imply smoothness of the solution u in the closed domain $\bar{Q}$ [24, 25, 26]. In the early research, most papers ignored the possible presence of an initial layer in the solution at $t = 0$, and the corresponding convergence analyses make an unrealistic assumption that u is smooth in the closed domain $\bar{Q}$. Until later, the nonuniform time meshes were successfully employed to compensate for the singularity of the solution at $t \to 0^+$ [28, 29, 30, 31], which are flexible and reasonably convenient for practical implementation. Such graded meshes were originally used in the context of Volterra integral equations with weakly singular kernels [32, 33]. In particular, the $L1$ schemes on graded meshes for discretizing the fractional derivatives in Caputo sense with the optimal rate of convergence $\mathscr{O}(\tau^{\min\{2-\alpha, r\alpha\}})$ have been detailedly discussed in [17, 28]. However, it seems not easy to extend to the tempered Caputo fractional derivative $\partial_t^{\alpha,\mu} u(x,y,t)$ in our Fokker-Planck equation (2) because the kernel function $w_\mu^\alpha(t)$ is an improper integral. In this paper, a modified $L1$ scheme is designed to discretize the tempered Caputo fractional derivative $\partial_t^{\alpha,\mu} u(x,y,t)$, which seems to be the first time to be considered, and the classical five-point finite difference scheme is used to approximate Δu. After verifying the regularity of the solution to (2), a precise stability result and sharp α-robust error estimate $\mathscr{O}(\tau^{\min\{2-\alpha, r\alpha\}} + h^2)$ are obtained.

The structure of the paper is as follows. In Section 2, the regularity of the solution u of (2) is investigated and the bounds of those derivatives of u are derived, which are needed for the subsequent numerical analyses. In Section 3, the kernel function in $\partial_t^{\alpha,\mu} u(x,y,t)$ is first explicitly written as four parts; then the discretization scheme of (2) is presented. Based on the regularity assumptions for u in the previous section, we mainly analyze the truncation error bound for the tempered Caputo derivative, while the space approximation accuracy is obvious. Stability and convergence analyses are carried out in Section 4, where a sharp α-robust error estimate is derived. To verify the theoretical results, the numerical experiments are performed in Section 5.

Notation: Let $\|\cdot\|$ denote the $L^2(\Omega)$ norm defined by $\|v\|^2 = (v,v)$, where $(\cdot,\cdot)$ is the $L^2(\Omega)$ inner product, and $\|\cdot\|_\infty$ denotes the $L^\infty(\Omega)$ norm. Throughout this paper, "$'$" represents the time partial derivative and C a generic constant that can take different values in different places. Also C may depend on the data of the boundary value problem (2) but is independent of any mesh used to solve (2) numerically.

2. Regularity of the solution to (2)

The two-parameter Mittag-Leffler function is defined by [34]

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \quad \text{for } \alpha>0, \beta\in\mathbb{R}. \tag{5}$$

Lemma 2.1 ([34, Section 4.10.2])**.** *The Mittag-Leffler function $E_{\alpha,\beta}(-x) : [0,\infty) \to \mathbb{R}$ possesses the complete monotonicity property for $0\le\alpha\le1$, $\beta\ge\alpha$, that is*

$$(-1)^n \frac{d^n}{dx^n} E_{\alpha,\beta}(-x) \ge 0, \ \forall x\in(0,\infty), n=0,1,\cdots. \tag{6}$$

Lemma 2.2 ([34, Eq. (4.4.4), p. 61])**.** *For $\alpha,\beta>0$, one has*

$$\int_0^t s^{\beta-1} E_{\alpha,\beta}(\lambda s^\alpha)\,\mathrm{d}s = t^\beta E_{\alpha,\beta+1}(\lambda t^\alpha). \tag{7}$$

Lemma 2.3 ([35])**.** *Recall that $0<\alpha<1$. Let $\lambda>0$ be a constant. Then one has the identity*

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha). \tag{8}$$

Lemma 2.4 ([36, Theorem 1.5 and Theorem 1.6])**.** *If $0<\alpha<2$, $\beta\in\mathbb{R}$, $\pi\alpha/2<\mu<\min\{\pi,\pi\alpha\}$, then there exist positive constants C_1, C_2, and C_3 such that for $|\arg(z)|\le\mu$,*

$$|E_{\alpha,\beta}(z)| \le C_1\left(1+|z|\right)^{(1-\beta)/\alpha} \exp\left(\Re(z^{1/\alpha})\right) + \frac{C_2}{1+|z|} \tag{9}$$

and for $\mu\le|\arg(z)|\le\pi$,

$$|E_{\alpha,\beta}(z)| \le \frac{C_3}{1+|z|}. \tag{10}$$

Let $\{\lambda_j\}_{j=1}^\infty$ and $\{\psi_j\}_{j=1}^\infty$ be respectively the eigenvalues (ordered nondecreasingly with multiplicity counted) and the $L^2(\Omega)$-orthogonal eigenfunctions of the negative Laplace operator $-\Delta$ on the domain Ω with homogeneous Dirichlet boundary condition. Then $\{\psi_j\}_{j=1}^\infty$ forms an orthogonal basis in $L^2(\Omega)$. In fact, by separation of variables, one can construct an infinite series solution to (2) in the form

$$u(x,y,t) = \sum_{i=1}^{\infty} u_i(t)\psi_i(x,y),$$

where

$$u_i(t) = (u(\cdot,\cdot,t), \psi_i(\cdot,\cdot)) .$$

Further denote $\phi_i = (\phi(\cdot,\cdot), \psi_i(\cdot,\cdot))$ and $f_i(t) = (f(\cdot,\cdot,t), \psi_i(\cdot,\cdot))$. Then we have

$$\partial_t^{\alpha,\mu} u_i(t) = -\lambda_i u_i(t) + f_i(t),$$

the Laplace transform of which leads to

$$[(s+\mu)^\alpha - \mu^\alpha]\, \widehat{u_i}(s) - \frac{(s+\mu)^\alpha - \mu^\alpha}{s}\phi_i = -\lambda_i \widehat{u_i}(s) + \widehat{f_i}(s),$$

equivalently,

$$\begin{aligned}\widehat{u_i}(s) &= \frac{s^{-1}\left[(s+\mu)^\alpha - \mu^\alpha\right]\phi_i + \widehat{f_i}(s)}{(s+\mu)^\alpha - \mu^\alpha + \lambda_i} \\ &= \frac{\left[(s+\mu)^{\alpha-1} + \mu s^{-1}(s+\mu)^{\alpha-1} - s^{-1}\mu^\alpha\right]\phi_i + \widehat{f_i}(s)}{(s+\mu)^\alpha - \mu^\alpha + \lambda_i}.\end{aligned}$$

Hence, with the inverse Laplace transform and the two parameter Mittag-Leffler function (5), there exists

$$\begin{aligned}u_i(t) = \phi_i \Big[e^{-\mu t} E_{\alpha,1}(-a_i t^\alpha) &+ \mu \int_0^t e^{-\mu s} E_{\alpha,1}(-a_i s^\alpha)\, \mathrm{d}s \\ &-\mu^\alpha \int_0^t e^{-\mu s} s^{\alpha-1} E_{\alpha,\alpha}(-a_i s^\alpha)\, \mathrm{d}s \Big] \\ &+ \int_0^t e^{-\mu s} s^{\alpha-1} E_{\alpha,\alpha}(-a_i s^\alpha) f_i(t-s)\, \mathrm{d}s, \qquad (11)\end{aligned}$$

where $a_i = \lambda_i - \mu^\alpha$. Since the eigenvalue sequence $\{\lambda_i\}_{i=1}^\infty$ is non-decreasing, there exists an integer i_0 such that $a_i > 0$ for all $i \geq i_0$ and $a_i \leq 0$ for all $i < i_0$.

Then under suitable additional assumptions on the data (cf. [35]), the solution of (2) can be denoted by

$$u(x,y,t) = \sum_{i=1}^\infty \left[\phi_i G_i(t) + F_i(t)\right] \psi_i(x,y) \qquad (12)$$

with

$$\begin{aligned}G_i(t) = e^{-\mu t} E_{\alpha,1}(-a_i t^\alpha) &+ \mu \int_0^t e^{-\mu s} E_{\alpha,1}(-a_i s^\alpha)\, \mathrm{d}s \\ &- \mu^\alpha \int_0^t e^{-\mu s} s^{\alpha-1} E_{\alpha,\alpha}(-a_i s^\alpha)\, \mathrm{d}s\end{aligned}$$

and

$$F_i(t) = \int_0^t e^{-\mu s} s^{\alpha-1} E_{\alpha,\alpha}(-a_i s^\alpha) f_i(t-s)\, \mathrm{d}s.$$

The identity (12) will be used several times when proving the regularity of the solution u to (2). Using the theory of sectorial operator [37], for each $\gamma \in \mathbb{R}$ the fractional power $(-\Delta)^\gamma$ of the operator $-\Delta$ is defined with the domain

$$\dot{H}^{2\gamma}(\Omega) = D\left((-\Delta)^\gamma\right) := \left\{ g \in L^2(\Omega) : \sum_{i=1}^\infty \lambda_i^{2\gamma} \left|(g,\psi_i)\right|^2 < \infty \right\},$$

and the norm of $\dot{H}^{2\gamma}(\Omega)$ is

$$\|g\|_{\dot{H}^{2\gamma}(\Omega)} := \left(\sum_{i=1}^{\infty} \lambda_i^{2\gamma} |(g, \psi_i)|^2 \right)^{1/2}.$$

Theorem 2.1. *Let u be the solution of (2). Then, for arbitrary $\epsilon > 0$, there exist constants C independent of t such that*

(i) If $\phi \in \dot{H}^{1+\epsilon}(\Omega)$, $f(\cdot,\cdot,t) \in \dot{H}^{1+\epsilon}(\Omega)$ for $t \in [0,T]$, then $|u(x,y,t)| \le C$ for $(x,y,t) \in \bar{Q}$;

(ii) If $\phi \in \dot{H}^{3+\epsilon}(\Omega)$, $f(\cdot,\cdot,t) \in \dot{H}^{1+\epsilon}(\Omega)$ for $t \in [0,T]$, and $\| f_t(\cdot,\cdot,t) \|_{\dot{H}^{1+\epsilon}(\Omega)} \le Ct^{-\rho}$ $(\rho < 1)$ for all $t \in (0,T]$, then $|u_t(x,y,t)| \le Ct^{\alpha-1}$ for $(x,y,t) \in Q$.

Proof. (i) According to (12), by the triangle inequality, one has

$$\begin{aligned}
|u(x,y,t)| &\le \sum_{i=1}^{\infty} |\phi_i G_i(t) + F_i(t)| \, |\psi_i(x,y)| \\
&= \sum_{i=1}^{\infty} \Big| \phi_i e^{-\mu t} E_{\alpha,1}(-a_i t^\alpha) + \phi_i \mu \int_0^t e^{-\mu s} E_{\alpha,1}(-a_i s^\alpha)\,\mathrm{d}s \\
&\qquad - \phi_i \mu^\alpha \int_0^t e^{-\mu s} s^{\alpha-1} E_{\alpha,\alpha}(-a_i s^\alpha)\,\mathrm{d}s \\
&\qquad + \int_0^t e^{-\mu s} s^{\alpha-1} E_{\alpha,\alpha}(-a_i s^\alpha) f_i(t-s)\,\mathrm{d}s \Big| \, |\psi_i(x,y)| \\
&\le \sum_{i=1}^{\infty} \Big[\left| \phi_i e^{-\mu t} E_{\alpha,1}(-a_i t^\alpha) \right| |\psi_i(x,y)| \\
&\qquad + \left| \phi_i \mu \int_0^t e^{-\mu s} E_{\alpha,1}(-a_i s^\alpha)\,\mathrm{d}s \right| |\psi_i(x,y)| \\
&\qquad + \left| \phi_i \mu^\alpha \int_0^t e^{-\mu s} s^{\alpha-1} E_{\alpha,\alpha}(-a_i s^\alpha)\,\mathrm{d}s \right| |\psi_i(x,y)| \\
&\qquad + \left| \int_0^t e^{-\mu s} s^{\alpha-1} E_{\alpha,\alpha}(-a_i s^\alpha) f_i(t-s)\,\mathrm{d}s \right| |\psi_i(x,y)| \Big].
\end{aligned} \tag{13}$$

Since $\phi \in \dot{H}^{1+\epsilon}(\Omega)$, $f(\cdot,\cdot,t) \in \dot{H}^{1+\epsilon}(\Omega)$, there exist $\|\phi(\cdot,\cdot,t)\|_{\dot{H}^{1+\epsilon}(\Omega)} \le C$ and $\|f(\cdot,\cdot,t)\|_{\dot{H}^{1+\epsilon}(\Omega)} \le C$. In addition, $\lambda_i \approx i$ (cf. [38]) and $|\psi_i(x,y)| \le C$.

Consider the terms in (13). Using the Cauchy-Schwarz inequality and Lemma 2.4, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} \left| \phi_i e^{-\mu t} E_{\alpha,1}(-a_i t^\alpha) \right| &\le C \sum_{i=1}^{\infty} |\phi_i| \, |E_{\alpha,1}(-a_i t^\alpha)| \\
&\le C \left(\sum_{i=1}^{i_0-1} + \sum_{i=i_0}^{\infty} \right) |\phi_i| \, |E_{\alpha,1}(-a_i t^\alpha)| \\
&\le C \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i^{1+\epsilon}} \right)^{1/2} \left(\sum_{i=1}^{\infty} \lambda_i^{1+\epsilon} |\phi_i|^2 \right)^{1/2} \\
&\le C \|\phi\|_{\dot{H}^{1+\epsilon}(\Omega)}.
\end{aligned}$$

By the Cauchy-Schwarz inequality, Lemma 2.1, Lemma 2.2, and Lemma 2.4, one has

$$\begin{aligned}\sum_{i=1}^{\infty}\left|\phi_i\mu\int_0^t e^{-\mu s}E_{\alpha,1}(-a_is^{\alpha})\,\mathrm{d}s\right| &\leq C\sum_{i=1}^{\infty}|\phi_i|\left|\int_0^t e^{-\mu s}E_{\alpha,1}(-a_is^{\alpha})\,\mathrm{d}s\right|\\ &\leq C\sum_{i=1}^{\infty}|\phi_i|\left(\int_0^t |E_{\alpha,1}(-a_is^{\alpha})|\,\mathrm{d}s\right)\\ &\leq C\left(\sum_{i=1}^{i_0-1}+\sum_{i=i_0}^{\infty}\right)|\phi_i|\,t\,(E_{\alpha,2}(-a_it^{\alpha}))\\ &\leq C\left(\sum_{i=1}^{\infty}\frac{1}{\lambda_i^{1+\epsilon}}\right)^{1/2}\left(\sum_{i=1}^{\infty}\lambda_i^{1+\epsilon}|\phi_i|^2\right)^{1/2}\\ &\leq C\|\phi\|_{\dot{H}^{1+\epsilon}(\Omega)}.\end{aligned}$$

Again according to the Cauchy-Schwarz inequality, Lemma 2.1, Lemma 2.2, and Lemma 2.4, it yields

$$\begin{aligned}\sum_{i=1}^{\infty}\left|\phi_i\mu^{\alpha}\int_0^t e^{-\mu s}s^{\alpha-1}E_{\alpha,\alpha}(-a_is^{\alpha})\,\mathrm{d}s\right| &\leq C\sum_{i=1}^{\infty}|\phi_i|\left|\int_0^t e^{-\mu s}s^{\alpha-1}E_{\alpha,\alpha}(-a_is^{\alpha})\,\mathrm{d}s\right|\\ &\leq C\sum_{i=1}^{\infty}|\phi_i|\left(\int_0^t \left|s^{\alpha-1}E_{\alpha,\alpha}(-a_is^{\alpha})\right|\,\mathrm{d}s\right)\\ &\leq C\left(\sum_{i=1}^{i_0-1}+\sum_{i=i_0}^{\infty}\right)|\phi_i|\,t^{\alpha}\,(E_{\alpha,\alpha+1}(-a_it^{\alpha}))\\ &\leq C\left(\sum_{i=1}^{\infty}\frac{1}{\lambda_i^{1+\epsilon}}\right)^{1/2}\left(\sum_{i=1}^{\infty}\lambda_i^{1+\epsilon}|\phi_i|^2\right)^{1/2}\\ &\leq C\|\phi\|_{\dot{H}^{1+\epsilon}(\Omega)}.\end{aligned}$$

By the Cauchy-Schwarz inequality and Lemma 2.4, one has

$$\begin{aligned}&\sum_{i=1}^{\infty}\left|\int_0^t e^{-\mu s}s^{\alpha-1}E_{\alpha,\alpha}(-a_is^{\alpha})f_i(t-s)\,\mathrm{d}s\right|\\ &\leq C\int_0^t\left|s^{\alpha-1}\sum_{i=1}^{\infty}E_{\alpha,\alpha}(-a_is^{\alpha})f_i(t-s)\right|\,\mathrm{d}s\\ &\leq C\int_0^t s^{\alpha-1}\left(\sum_{i=1}^{\infty}\frac{1}{\lambda_i^{1+\epsilon}}\left(E_{\alpha,\alpha}(-a_is^{\alpha})\right)^2\right)^{1/2}\left(\sum_{i=1}^{\infty}\lambda_i^{1+\epsilon}f_i^2(t-s)\right)^{1/2}\,\mathrm{d}s\\ &\leq C.\end{aligned}$$

Hence the series (13) is absolutely and uniformly convergent on $\bar{Q}$, and

$$|u(x,y,t)|\leq C \text{ for } (x,y,t)\in\bar{Q}. \tag{14}$$

(ii) Differentiating (12) term by term with respect to t for $(x,y,t)\in Q$ yields infinite series formulas for the derivatives that we desire to bound

$$\begin{aligned} u_t(x,y,t) = \sum_{i=1}^{\infty} \Big[& -\phi_i e^{-\mu t} a_i t^{\alpha-1} E_{\alpha,\alpha}(-a_i t^\alpha) - \phi_i \mu^\alpha e^{-\mu t} t^{\alpha-1} E_{\alpha,\alpha}(-a_i t^\alpha) \\ & + e^{-\mu t} t^{\alpha-1} E_{\alpha,\alpha}(-a_i t^\alpha) f_i(0) \\ & + \int_0^t e^{-\mu s} s^{\alpha-1} E_{\alpha,\alpha}(-a_i s^\alpha) f_i'(t-s)\,\mathrm{d}s \Big] \psi_i(x,y), \end{aligned} \tag{15}$$

where (8) is used in Lemma 2.3 to differentiate $E_{\alpha,1}(\cdot)$. Based on the proof of (i), it is easy to check that

$$|u_t(x,y,t)| \le C t^{\alpha-1} \text{ for } (x,y,t)\in Q. \tag{16}$$

This completes the proof. □

Similarly, one can carry out the calculations to bound $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^p u}{\partial x^i y^j}$ (for $i+j=p$ and $i,j,p=1,2,3,4$) on Q. Moreover, one can similarly show (cf. [35]) that $\partial_t^{\alpha,\mu} u$ exists and u is the solution of (2). The maximum principle guarantees the uniqueness of solution.

Now we summarize all the above activities in the following result.

Theorem 2.2. *Assume that* $\phi\in \dot H^{5+\epsilon}(\Omega)$, $f(\cdot,t)\in \dot H^{5+\epsilon}(\Omega)$, $f_t(\cdot,\cdot,t)$ *and* $f_{tt}(\cdot,\cdot,t)$ *are in* $\dot H^{1+\epsilon}(\Omega)$ *for any* $t\in(0,T]$ *with*

$$\|f(\cdot,\cdot,t)\|_{\dot H^{5+\epsilon}(\Omega)} + \|f_t(\cdot,\cdot,t)\|_{\dot H^{1+\epsilon}(\Omega)} + t^\rho \|f_{tt}(\cdot,\cdot,t)\|_{\dot H^{1+\epsilon}(\Omega)} \le C_1$$

for all $t\in(0,T]$, $\forall\epsilon>0$, *and some constant* $\rho<1$, *where* C_1 *is a constant independent of* t. *Then* (2) *has an unique solution* u, *and there exists a constant* C *such that*

$$\left|\frac{\partial^p u}{\partial x^i y^j}(x,y,t)\right| \le C \quad \textit{for } i+j=p \text{ and } i,j,p=0,1,2,3,4, \tag{17}$$

$$\left|\frac{\partial^q u}{\partial t^q}(x,y,t)\right| \le C\left(1+t^{\alpha-q}\right) \quad \textit{for } q=0,1,2, \tag{18}$$

for all $(x,y,t)\in\bar\Omega\times(0,T]$.

3. The discrete problem

In this section, we construct the finite difference scheme for the problem (2). To address the weak singularity at $t=0$, we use a graded mesh in time and uniform one in space. Let N_x, N_y, and N be positive integers. Set the mesh points $x_i := ih_x$ for $i=0,1,2,\cdots,N_x$ and $y_j := jh_y$ for $j=0,1,2,\cdots,N_y$ with the uniform spacesizes $h_x = L_1/N_x$ and $h_y = L_2/N_y$, respectively. Set $t_n := T(\frac{n}{N})^r$ for $n=0,1,2,\cdots,N$, $\tau_n = t_n - t_{n-1}$ for $n=1,2,\cdots,N$, where the constant $r\ge 1$ is the mesh grading. It is easy to show that such a time-graded mesh has the following property [28]: for $k=0,1,\cdots,N-1$,

$$\tau_{k+1} = T\left(\frac{k+1}{N}\right)^r - T\left(\frac{k}{N}\right)^r \le CTN^{-r}k^{r-1}. \tag{19}$$

The numerical solution and source term at each mesh point (x_i,y_j,t_n) are denoted by u_{ij}^n and f_{ij}^n, respectively.

3.1. Spatial and temporal discretization. We use five-point difference scheme in space

(20)
$$\Delta u(x_i,y_j,t_n) \approx \delta_x^2 u_{ij}^n + \delta_y^2 u_{ij}^n := \frac{u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n}{h_x^2} + \frac{u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n}{h_y^2}.$$

Before discretizing the tempered Caputo derivative (3), we first make integration by parts for the kernel function

$$\begin{aligned} w_\mu^\alpha(t) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_t^\infty e^{-\mu s} s^{-1-\alpha}\, \mathrm{d}s \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \int_t^\infty e^{-\mu s} \left(\frac{s^{-\alpha}}{-\alpha}\right)' \mathrm{d}s \\ &= \frac{e^{-\mu t} t^{-\alpha}}{\Gamma(1-\alpha)} - \frac{\mu}{\Gamma(1-\alpha)} \left(\int_0^\infty e^{-\mu s} s^{-\alpha}\, \mathrm{d}s - \int_0^t e^{-\mu s} s^{-\alpha}\, \mathrm{d}s \right) \\ &= \frac{e^{-\mu t} t^{-\alpha}}{\Gamma(1-\alpha)} - \frac{\mu^\alpha}{\Gamma(1-\alpha)} + \frac{\mu}{\Gamma(1-\alpha)} \int_0^t e^{-\mu s} s^{-\alpha}\, \mathrm{d}s, \end{aligned}$$

where the last equality uses the fact that $\Gamma(\lambda) = \int_0^\infty s^{\lambda-1} e^{-s}\, \mathrm{d}s$. Then

$$\begin{aligned} \int_0^t w_\mu^\alpha(s)\, \mathrm{d}s &= \int_0^t \left[\frac{e^{-\mu s} s^{-\alpha}}{\Gamma(1-\alpha)} - \frac{\mu^\alpha}{\Gamma(1-\alpha)} + \frac{\mu}{\Gamma(1-\alpha)} \int_0^s e^{-\mu\tau} \tau^{-\alpha}\, \mathrm{d}\tau \right] \mathrm{d}s \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t e^{-\mu s} s^{-\alpha}\, \mathrm{d}s - \frac{1}{\Gamma(1-\alpha)} \int_0^t \mu^\alpha\, \mathrm{d}s \\ &\quad + \frac{\mu}{\Gamma(1-\alpha)} \int_0^t \int_\tau^t e^{-\mu\tau} \tau^{-\alpha}\, \mathrm{d}s\, \mathrm{d}\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t e^{-\mu s} s^{-\alpha}\, \mathrm{d}s - \frac{1}{\Gamma(1-\alpha)} \int_0^t \mu^\alpha\, \mathrm{d}s \\ &\quad + \frac{\mu}{\Gamma(1-\alpha)} \int_0^t (t-s) e^{-\mu s} s^{-\alpha}\, \mathrm{d}s. \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(\frac{e^{-\mu s}}{s^\alpha} - \mu^\alpha + \frac{\mu t e^{-\mu s}}{s^\alpha} - \frac{\mu e^{-\mu s}}{s^{\alpha-1}} \right) \mathrm{d}s. \end{aligned}$$

The tempered Caputo derivative $\partial_t^{\alpha,\mu} u(x,y,t)$ in (3) at each mesh point (x_i, y_j, t_n) can be written as

$$\begin{aligned} \partial_t^{\alpha,\mu} u(x_i,y_j,t_n) &= \int_0^{t_n} w_\mu^\alpha(t_n - s) \frac{\partial}{\partial s} u(x_i,y_j,s)\, \mathrm{d}s \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left[\frac{e^{-\mu(t_n-s)}}{(t_n-s)^\alpha} - \mu^\alpha \right. \\ &\quad \left. + \frac{\mu e^{-\mu(t_n-s)} t_n}{(t_n-s)^\alpha} - \frac{\mu e^{-\mu(t_n-s)}}{(t_n-s)^{\alpha-1}} \right] \frac{\partial}{\partial s} u(x_i,y_j,s)\, \mathrm{d}s. \end{aligned} \tag{21}$$

Our technique is to use the following modified $L1$ scheme inspired by the middle rectangle quadrature formula to approximate $\partial_t^{\alpha,\mu} u(x_i,y_j,t_n)$, which has some resemblance to the discretisations discussed in [17], i.e.,

$$D_N^{\alpha,\mu} u_{ij}^n = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \left(e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)} \frac{u_{ij}^{k+1} - u_{ij}^k}{\tau_{k+1}} \right)$$
$$\cdot \int_{t_k}^{t_{k+1}} \left[\frac{1}{(t_n-s)^\alpha} - \mu^\alpha e^{\mu(t_n-s)} + \frac{\mu t_n}{(t_n-s)^\alpha} - \frac{\mu}{(t_n-s)^{\alpha-1}} \right] \mathrm{d}s. \tag{22}$$

Thus we approximate problem (2) by the discrete scheme

(23)
$$\begin{cases} L_{N_x,N_y}{}^N u_{ij}^n := D_N^{\alpha,\mu} u_{ij}^n - \left(\delta_x^2 + \delta_y^2\right) u_{ij}^n = f_{ij}^n, \text{for } 1 \le i(j) \le N_x(N_y) - 1,\ 1 \le n \le N, \\ u_{0j}^n = 0,\ u_{N_x,j}^n = 0,\ u_{i0}^n = 0,\ u_{i,N_y}^n = 0, \text{for } 0 \le i(j) \le N_x(N_y),\ 0 < n \le N, \\ u_{ij}^0 = \phi(x_i, y_j), \text{for } 0 \le i(j) \le N_x(N_y). \end{cases}$$

3.2. Truncation error. We now consider the truncation error of the difference scheme (23). In fact, using (17) yields the truncation error estimate for space approximation

$$\Delta u(x_i, y_j, t_n) = \left(\delta_x^2 + \delta_y^2\right) u(x_i, y_j, t_n) + \mathcal{O}(h_x^2 + h_y^2) \quad \text{for } (x_i, y_j, t_n) \in Q, \tag{24}$$

which is a well-known conclusion. Below we mainly analyze the truncation error bounds for the tempered Caputo derivative. The notation relating to space mesh points (x_i, y_j) is omitted here, since the bounds are based on (18), independent of x and y. Then from (21) and (22), we have

$$D_N^{\alpha,\mu} u(t_n) - \partial_t^{\alpha,\mu} u(t_n) = \sum_{k=0}^{n-1} \left[\mathbb{J}_{n,k}^1 + \mathbb{J}_{n,k}^2 + \mathbb{J}_{n,k}^3 + \mathbb{J}_{n,k}^4 \right]$$

with

$$\mathbb{J}_{n,k}^1 := \int_{s=t_k}^{t_{k+1}} \frac{(t_n-s)^{-\alpha}}{\Gamma(1-\alpha)} \left[e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)} \frac{u(t_{k+1}) - u(t_k)}{\tau_{k+1}} - e^{-\mu(t_n-s)} \frac{\partial}{\partial s} u(s) \right] \mathrm{d}s,$$
$$\mathbb{J}_{n,k}^2 := \int_{s=t_k}^{t_{k+1}} \frac{\mu^\alpha}{\Gamma(1-\alpha)} \left[\frac{\partial}{\partial s} u(s) - e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)} e^{\mu(t_n-s)} \frac{u(t_{k+1}) - u(t_k)}{\tau_{k+1}} \right] \mathrm{d}s,$$
$$\mathbb{J}_{n,k}^3 := \int_{s=t_k}^{t_{k+1}} \frac{\mu t_n (t_n-s)^{-\alpha}}{\Gamma(1-\alpha)} \left[e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)} \frac{u(t_{k+1}) - u(t_k)}{\tau_{k+1}} - e^{-\mu(t_n-s)} \frac{\partial}{\partial s} u(s) \right] \mathrm{d}s,$$
$$\mathbb{J}_{n,k}^4 := \int_{s=t_k}^{t_{k+1}} \frac{\mu (t_n-s)^{1-\alpha}}{\Gamma(1-\alpha)} \left[e^{-\mu(t_n-s)} \frac{\partial}{\partial s} u(s) - e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)} \frac{u(t_{k+1}) - u(t_k)}{\tau_{k+1}} \right] \mathrm{d}s.$$

Theorem 3.1. *Under the regularity results of the solution of Eq. (2), there exists a constant C such that for $1 \le n \le N$,*

$$|D_N^{\alpha,\mu} u(t_n) - \partial_t^{\alpha,\mu} u(t_n)| \le C n^{-\min\{2-\alpha, r\alpha\}}. \tag{25}$$

Proof. Using the triangle inequality leads to

$$|\ D_N^{\alpha,\mu} u(t_n) - \partial_t^{\alpha,\mu} u(t_n)\ | \le \sum_{k=0}^{n-1} \left[|\mathbb{J}_{n,k}^1| + |\mathbb{J}_{n,k}^2| + |\mathbb{J}_{n,k}^3| + |\mathbb{J}_{n,k}^4| \right].$$

Next, the corresponding bounds of $\mathbb{J}_{n,k}^1$, $\mathbb{J}_{n,k}^2$, $\mathbb{J}_{n,k}^3$, $\mathbb{J}_{n,k}^4$ are given respectively in the following three steps.

Step 1: For $\mathbb{J}^1_{n,k}$, we prove the desired results in the following three cases, i.e., $k = 0$, $1 \le k \le n-2$, and $k = n-1$,

$$\sum_{k=0}^{n-1} |\mathbb{J}^1_{n,k}| = |\mathbb{J}^1_{n,0}| + \sum_{k=1}^{n-2} |\mathbb{J}^1_{n,k}| + |\mathbb{J}^1_{n,n-1}|.$$

Case 1: For $k = 0$, if $n = 1$, from (18), one has

$$\begin{aligned}
\left|\mathbb{J}^1_{1,0}\right| &\le \frac{1}{\Gamma(1-\alpha)} \int_{s=0}^{t_1} (t_1 - s)^{-\alpha} e^{-\mu\left(t_1 - \frac{t_0+t_1}{2}\right)} \left| \frac{u(t_1) - u(t_0)}{\tau_1} \right| \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(1-\alpha)} \int_{s=0}^{t_1} (t_1 - s)^{-\alpha} e^{-\mu(t_1 - s)} \left| \frac{\partial}{\partial s} u(s) \right| \mathrm{d}s \\
&\le \frac{e^{-\mu\frac{\tau_1}{2}} \tau_1^{-\alpha}}{\Gamma(2-\alpha)} \int_{s=0}^{t_1} \left| \frac{\partial}{\partial s} u(s) \right| \mathrm{d}s + \frac{1}{\Gamma(1-\alpha)} \int_{s=0}^{t_1} (t_1 - s)^{-\alpha} \left| \frac{\partial}{\partial s} u(s) \right| \mathrm{d}s \\
&\le C \frac{\tau_1^{-\alpha}}{\Gamma(2-\alpha)} \int_{s=0}^{t_1} s^{\alpha-1} \,\mathrm{d}s + \frac{C}{\Gamma(1-\alpha)} \int_{s=0}^{t_1} (t_1 - s)^{-\alpha} s^{\alpha-1} \,\mathrm{d}s \\
&\le C \frac{\tau_1^{-\alpha}}{\Gamma(2-\alpha)} \frac{\tau_1^{\alpha}}{\alpha} + \frac{C}{\Gamma(1-\alpha)} B(\alpha, 1-\alpha) \\
&\le C;
\end{aligned}$$

if $n > 1$, applying $t_n = T(\frac{n}{N})^r$, (18), and the mean value theorem gives

$$\begin{aligned}
\left|\mathbb{J}^1_{n,0}\right| &\le \frac{1}{\Gamma(1-\alpha)} \int_{s=0}^{t_1} (t_n - s)^{-\alpha} e^{-\mu\left(t_n - \frac{t_0+t_1}{2}\right)} \left| \frac{u(t_1) - u(t_0)}{\tau_1} \right| \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(1-\alpha)} \int_{s=0}^{t_1} (t_n - s)^{-\alpha} e^{-\mu(t_n - s)} \left| \frac{\partial}{\partial s} u(s) \right| \mathrm{d}s \\
&\le \frac{e^{-\mu\left(t_n - \frac{t_0+t_1}{2}\right)} \tau_1^{-1}}{\Gamma(2-\alpha)} \left[t_n^{1-\alpha} - (t_n - t_1)^{1-\alpha}\right] \int_{s=0}^{t_1} \left| \frac{\partial}{\partial s} u(s) \right| \mathrm{d}s \\
&\quad + \frac{e^{-\mu(t_n - t_1)} (t_n - t_1)^{-\alpha}}{\Gamma(1-\alpha)} \int_{s=0}^{t_1} \left| \frac{\partial}{\partial s} u(s) \right| \mathrm{d}s \\
&\le \frac{C\tau_1^{-1}}{\Gamma(2-\alpha)} \left[t_n^{1-\alpha} - (t_n - t_1)^{1-\alpha}\right] \int_{s=0}^{t_1} s^{\alpha-1} \,\mathrm{d}s \\
&\quad + \frac{C(t_n - t_1)^{-\alpha}}{\Gamma(1-\alpha)} \int_{s=0}^{t_1} s^{\alpha-1} \,\mathrm{d}s \\
&\le C \left[t_n^{1-\alpha} - (t_n - t_1)^{1-\alpha}\right] t_1^{\alpha-1} + C(t_n - t_1)^{-\alpha} t_1^{\alpha} \\
&\le 2C \left(\frac{t_n - t_1}{t_1}\right)^{-\alpha} \\
&\le C n^{-r\alpha}.
\end{aligned}$$

It follows that

$$\left|\mathbb{J}^1_{n,0}\right| \le C n^{-r\alpha} \text{ for } n \ge 1. \tag{26}$$

Case 2: For $1 \le k \le n-2$, using integration by parts yields

$$\begin{aligned}
\mathbb{J}^1_{n,k} =& \frac{e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)}}{\Gamma(1-\alpha)} \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha} \left(\frac{u\,(t_{k+1}) - u\,(t_k)}{\tau_{k+1}} - \frac{\partial}{\partial s} u(s) \right) \mathrm{d}s \\
&+ \frac{1}{\Gamma(1-\alpha)} \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha} \left(e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)} - e^{-\mu(t_n - s)} \right) \frac{\partial}{\partial s} u(s)\,\mathrm{d}s \\
=& \frac{e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)}}{\Gamma(1-\alpha)} \left[\frac{u\,(t_{k+1}) - u\,(t_k)}{\tau_{k+1}} \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha}\,\mathrm{d}\,(s - t_k) \right. \\
&\left. - \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha}\,\mathrm{d}\,(u(s) - u\,(t_k)) \right] \\
&+ \frac{1}{\Gamma(1-\alpha)} \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha} \left(e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)} - e^{-\mu(t_n - s)} \right) \frac{\partial}{\partial s} u(s)\,\mathrm{d}s \\
:=& \mathbb{K}^1_{n,k} + \mathbb{K}^2_{n,k}.
\end{aligned}$$

Here

$$\begin{aligned}
\mathbb{K}^1_{n,k} =& \frac{\alpha e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)}}{\Gamma(1-\alpha)} \\
&\cdot \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha-1} \left[(u(s) - u\,(t_k)) - \frac{u\,(t_{k+1}) - u\,(t_k)}{\tau_{k+1}} (s - t_k) \right] \mathrm{d}s
\end{aligned}$$

and

$$\mathbb{K}^2_{n,k} = \frac{1}{\Gamma(1-\alpha)} \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha} \left(e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)} - e^{-\mu(t_n - s)} \right) \frac{\partial}{\partial s} u(s)\,\mathrm{d}s.$$

Applying the mean value theorem to $\mathbb{K}^1_{n,k}$ gives

$$\begin{aligned}
\mathbb{K}^1_{n,k} =& \frac{\alpha e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)}}{\Gamma(1-\alpha)} \\
&\cdot \left[(u(\xi_1) - u\,(t_k)) - \frac{u\,(t_{k+1}) - u\,(t_k)}{\tau_{k+1}} (\xi_1 - t_k) \right] \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha-1}\,\mathrm{d}s \\
=& \frac{\alpha e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)}}{\Gamma(1-\alpha)} (\xi_1 - t_k) \left[u_t(\xi_2) - u_t(\xi_3) \right] \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha-1}\,\mathrm{d}s \\
=& \frac{\alpha e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)}}{\Gamma(1-\alpha)} (\xi_1 - t_k)(\xi_2 - \xi_3) u_{tt}(\xi_4) \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha-1}\,\mathrm{d}s
\end{aligned}$$

for some $\xi_1, \xi_2, \xi_3, \xi_4 \in (t_k, t_{k+1})$. Thus for $1 \le k \le n-2$, one has

$$\left| \mathbb{K}^1_{n,k} \right| \le C \tau^2_{k+1} \max_{t_k \le s \le t_{k+1}} |u_{tt}(s)| \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha-1}\,\mathrm{d}s$$

and

$$\begin{aligned}
\left| \mathbb{K}^2_{n,k} \right| \le& C (t_n - t_{k+1})^{-\alpha} \max_{t_k \le s \le t_{k+1}} |u_t(s)| \\
&\cdot \int_{s=t_k}^{t_{k+1}} \left| e^{-\mu\left(t_n - \frac{t_k+t_{k+1}}{2}\right)} - e^{-\mu(t_n - s)} \right| \mathrm{d}s \\
\le& C \tau^3_{k+1} t_k^{\alpha-1} (t_n - t_{k+1})^{-\alpha},
\end{aligned}$$

where the last inequality is guaranteed by the middle rectangle quadrature formula and (18).

On the one hand, since $\left(1-\frac{3^r}{4^r}\right) n^r \leq n^r-(k+1)^r \leq n^r$ for $\forall 1 \leq k \leq \lceil n/2\rceil-1$ and $n \geq 2$, using the triangle inequality yields

$$\sum_{k=1}^{\lceil n/2\rceil-1} |\mathbb{J}_{n,k}^1| \leq \sum_{k=1}^{\lceil n/2\rceil-1} |\mathbb{K}_{n,k}^1| + \sum_{k=1}^{\lceil n/2\rceil-1} |\mathbb{K}_{n,k}^2|.$$

By $t_n = T(\frac{n}{N})^r$, (18), (19), and the mean value theorem, one has

$$\begin{aligned}
\sum_{k=1}^{\lceil n/2\rceil-1} |\mathbb{K}_{n,k}^1| &\leq C \sum_{k=1}^{\lceil n/2\rceil-1} \tau_{k+1}^2 \max_{t_k\leq s\leq t_{k+1}} |u_{tt}(s)| \int_{s=t_k}^{t_{k+1}} (t_n-s)^{-\alpha-1}\,\mathrm{d}s \\
&\leq C \sum_{k=1}^{\lceil n/2\rceil-1} \tau_{k+1}^2 t_k^{\alpha-2} \tau_{k+1}(t_n-t_{k+1})^{-\alpha-1} \\
&\leq C \sum_{k=1}^{\lceil n/2\rceil-1} \left(N^{-r}k^{r-1}\right)^3 \left[\left(\frac{k}{N}\right)^r\right]^{\alpha-2} \left[\left(\frac{n}{N}\right)^r - \left(\frac{k+1}{N}\right)^r\right]^{-\alpha-1} \\
&\leq C \sum_{k=1}^{\lceil n/2\rceil-1} k^{(\alpha+1)r-3}\left[n^r-(k+1)^r\right]^{-\alpha-1} \\
&\leq Cn^{-r(\alpha+1)} \sum_{k=1}^{\lceil n/2\rceil-1} k^{(\alpha+1)r-3} \\
&\leq \begin{cases} Cn^{-r(\alpha+1)} & \text{if } r(\alpha+1)<2, \\ Cn^{-2}\ln n & \text{if } r(\alpha+1)=2, \\ Cn^{-2} & \text{if } r(\alpha+1)>2, \end{cases}
\end{aligned} \tag{27}$$

where the last inequality is derived in [28]. Also by $t_n = T(\frac{n}{N})^r$, (18), and (19), one has

$$\begin{aligned}
\sum_{k=1}^{\lceil n/2\rceil-1} |\mathbb{K}_{n,k}^2| &\leq C \sum_{k=1}^{\lceil n/2\rceil-1} \tau_{k+1}^3 t_k^{\alpha-1}(t_n-t_{k+1})^{-\alpha} \\
&\leq CT^2 \sum_{k=1}^{\lceil n/2\rceil-1} \left(N^{-r}k^{r-1}\right)^3 \left[\left(\frac{k}{N}\right)^r\right]^{\alpha-1} \left[\left(\frac{n}{N}\right)^r - \left(\frac{k+1}{N}\right)^r\right]^{-\alpha} \\
&\leq CT^2 \sum_{k=1}^{\lceil n/2\rceil-1} N^{-2r}k^{(\alpha+2)r-3}\left[n^r-(k+1)^r\right]^{-\alpha} \\
&\leq C \sum_{k=1}^{\lceil n/2\rceil-1} N^{-2r}k^{(\alpha+2)r-3}n^{-\alpha r} \\
&\leq Cn^{-2} \sum_{k=1}^{\lceil n/2\rceil-1} \left(\frac{n}{N}\right)^{2r} \left(\frac{k}{n}\right)^{(\alpha+2)r-3} \frac{1}{n} \\
&\leq Cn^{-2}.
\end{aligned} \tag{28}$$

On the other hand, for $\lceil n/2\rceil \le k \le n-2$, using the triangle inequality yields

$$\sum_{k=\lceil n/2\rceil}^{n-2} |\mathbb{J}_{n,k}^1| \le \sum_{k=\lceil n/2\rceil}^{n-2} |\mathbb{K}_{n,k}^1| + \sum_{k=\lceil n/2\rceil}^{n-2} |\mathbb{K}_{n,k}^2|.$$

By $t_n = T(\frac{n}{N})^r$, (18), and (19), one has

$$\begin{aligned}
\sum_{k=\lceil n/2\rceil}^{n-2} |\mathbb{K}_{n,k}^1| &\le C \sum_{k=\lceil n/2\rceil}^{n-2} \tau_{k+1}^2 \max_{t_k\le s\le t_{k+1}} |u_{tt}(s)| \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha-1}\,\mathrm{d}s \\
&\le C \sum_{k=\lceil n/2\rceil}^{n-2} \tau_{k+1}^2 t_k^{\alpha-2} \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha-1}\,\mathrm{d}s \\
&\le C 2^{r(2-\alpha)} t_n^{\alpha-2} \sum_{k=\lceil n/2\rceil}^{n-2} \tau_{k+1}^2 \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha-1}\,\mathrm{d}s \\
&\le C T^\alpha \left[\left(\frac{n}{N}\right)^r\right]^{\alpha-2} \sum_{k=\lceil n/2\rceil}^{n-2} \left(N^{-r} k^{r-1}\right)^2 \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha-1}\,\mathrm{d}s \\
&\le C T^\alpha N^{-r\alpha} n^{r\alpha-2} \sum_{k=\lceil n/2\rceil}^{n-2} \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha-1}\,\mathrm{d}s \\
&\le C N^{-r\alpha} n^{r\alpha-2} \int_{s=t_{\lceil n/2\rceil}}^{t_{n-1}} (t_n - s)^{-\alpha-1}\,\mathrm{d}s \\
&\le C N^{-r\alpha} n^{r\alpha-2} \left(t_n - t_{n-1}\right)^{-\alpha} \\
&\le C n^{-(2-\alpha)}, \qquad (29)
\end{aligned}$$

where we use the properties of time-graded mesh: $t_n \le 2^r t_{n-1}$ and $rT^{\frac{1}{r}} N^{-1} t_{n-1}^{1-\frac{1}{r}} \le \tau_n \le rT^{\frac{1}{r}} N^{-1} t_n^{1-\frac{1}{r}}$ for $n \ge 2$ (cf. [27]). Also by $t_n = T(\frac{n}{N})^r$, (18), and (19), one has

$$\begin{aligned}
\sum_{k=\lceil n/2\rceil}^{n-2} |\mathbb{K}_{n,k}^2| &\le C \sum_{k=\lceil n/2\rceil}^{n-2} \tau_{k+1}^3 t_k^{\alpha-1} (t_n - t_{k+1})^{-\alpha} \\
&\le C 2^{r(1-\alpha)} t_n^{\alpha-1} \sum_{k=\lceil n/2\rceil}^{n-2} \tau_{k+1}^3 (t_n - t_{k+1})^{-\alpha} \\
&\le C T^2 \left[\left(\frac{n}{N}\right)^r\right]^{\alpha-1} \sum_{k=\lceil n/2\rceil}^{n-2} \left(N^{-r} k^{r-1}\right)^3 \left[\left(\frac{n}{N}\right)^r - \left(\frac{k+1}{N}\right)^r\right]^{-\alpha} \\
&\le C \sum_{k=\lceil n/2\rceil}^{n-2} N^{-2r} n^{2r-3} \left[1 - \left(\frac{k+1}{n}\right)^r\right]^{-\alpha} \\
&\le C n^{-2} \sum_{k=\lceil n/2\rceil}^{n-2} \left(\frac{n}{N}\right)^{2r} \left[1 - \left(\frac{k+1}{n}\right)^r\right]^{-\alpha} \frac{1}{n} \\
&\le C n^{-2}. \qquad (30)
\end{aligned}$$

Case 3: For $k = n-1$, applying the mean value theorem, $t_n = T(\frac{n}{N})^r$, $t_n \le 2^r t_{n-1}$ for $n \ge 2$ (cf. [27]), (18), and (19) gives

$$\begin{aligned}
|\mathbb{J}^1_{n,n-1}| &\le \frac{e^{-\mu(t_n - \frac{t_{n-1}+t_n}{2})}}{\Gamma(1-\alpha)} \int_{s=t_{n-1}}^{t_n} (t_n - s)^{-\alpha} \left| \frac{u(t_n) - u(t_{n-1})}{\tau_n} - \frac{\partial}{\partial s} u(s) \right| \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(1-\alpha)} \int_{s=t_{n-1}}^{t_n} (t_n - s)^{-\alpha} \left| e^{-\mu(t_n - \frac{t_{n-1}+t_n}{2})} - e^{-\mu(t_n - s)} \right| \left| \frac{\partial}{\partial s} u(s) \right| \mathrm{d}s \\
&\le \frac{e^{-\mu \frac{\tau_n}{2}} \tau_n}{\Gamma(1-\alpha)} \max_{t_{n-1} \le s \le t_n} |u_{tt}(s)| \int_{s=t_{n-1}}^{t_n} (t_n - s)^{-\alpha} \mathrm{d}s \\
&\quad + \frac{C\tau_n}{\Gamma(1-\alpha)} \max_{t_{n-1} \le s \le t_n} |u_t(s)| \int_{s=t_{n-1}}^{t_n} (t_n - s)^{-\alpha} \mathrm{d}s \\
&\le C\tau_n^{2-\alpha} t_{n-1}^{\alpha-2} + C\tau_n^{2-\alpha} t_{n-1}^{\alpha-1} \\
&\le C\tau_n^{2-\alpha} \left(2^{-r} t_n\right)^{\alpha-2} + C\tau_n^{2-\alpha} \left(2^{-r} t_n\right)^{\alpha-1} \\
&\le C \left[N^{-r}(n-1)^{r-1}\right]^{2-\alpha} \left[\left(\frac{n}{N}\right)^{r(\alpha-2)} + \left(\frac{n}{N}\right)^{r(\alpha-1)} \right] \\
&\le Cn^{-(2-\alpha)}.
\end{aligned}$$

It follows that

$$\left|\mathbb{J}^1_{n,n-1}\right| \le Cn^{-(2-\alpha)} \text{ for } n \ge 2. \tag{31}$$

Combining (26)–(31), we have

$$\sum_{k=0}^{n-1} |\mathbb{J}^1_{n,k}| \le Cn^{-\min\{2-\alpha, r\alpha\}}. \tag{32}$$

In fact, $\mathbb{J}^3_{n,k} = \mu t_n \mathbb{J}^1_{n,k}$. Using $t_n = T(\frac{n}{N})^r$ yields

$$\begin{aligned}
\sum_{k=0}^{n-1} |\mathbb{J}^3_{n,k}| &= \mu t_n \sum_{k=0}^{n-1} |\mathbb{J}^1_{n,k}| \\
&\le C \left(\frac{n}{N}\right)^r n^{-\min\{2-\alpha, r\alpha\}} \\
&\le Cn^{-\min\{2-\alpha, r\alpha\}}.
\end{aligned} \tag{33}$$

Step 2: For $\mathbb{J}^2_{n,k}$, we prove the desired results by the following two cases, i.e., $k = 0$ and $1 \le k \le n-1$,

$$\sum_{k=0}^{n-1} |\mathbb{J}^2_{n,k}| = |\mathbb{J}^2_{n,0}| + \sum_{k=1}^{n-1} |\mathbb{J}^2_{n,k}|.$$

Case 1: For $k=0$, according to (18), $t_n = T(\frac{n}{N})^r$, and the middle rectangle quadrature formula, we obtain

$$
\begin{aligned}
\left|\mathbb{J}^2_{n,0}\right| &\le \frac{\mu^\alpha}{\Gamma(1-\alpha)} \int_{s=0}^{t_1} \left[\left|\frac{\partial}{\partial s}u(s)\right| + \left(e^{-\mu s}\left|e^{\mu\frac{t_0+t_1}{2}} - e^{\mu s}\right| + 1\right)\left|\frac{u(t_1)-u(t_0)}{\tau_1}\right|\right] \mathrm{d}s \\
&\le \frac{2C\mu^\alpha}{\Gamma(1-\alpha)}\int_{s=0}^{t_1} s^{\alpha-1}\,\mathrm{d}s + \frac{\mu^\alpha}{\Gamma(1-\alpha)}\int_{s=0}^{t_1}\left|e^{\mu(\frac{t_0+t_1}{2})} - e^{\mu s}\right|\left|\frac{u(t_1)-u(t_0)}{\tau_1}\right| \mathrm{d}s \\
&\le C\int_{s=0}^{t_1} s^{\alpha-1}\,\mathrm{d}s + C\tau_1^2\int_{s=0}^{t_1} s^{\alpha-1}\,\mathrm{d}s \\
&\le Ct_1^\alpha + Ct_1^{\alpha+2} \\
&\le Cn^{-r\alpha}.
\end{aligned}
$$

It follows that

$$
\left|\mathbb{J}^2_{n,0}\right| \le Cn^{-r\alpha} \text{ for } n \ge 1. \tag{34}
$$

Case 2: For $1 \le k \le n-1$, using integration by parts yields

$$
\begin{aligned}
\mathbb{J}^2_{n,k} &= \frac{\mu^\alpha}{\Gamma(1-\alpha)}\int_{s=t_k}^{t_{k+1}}\left(1 - e^{\mu\left(\frac{t_k+t_{k+1}}{2}-s\right)}\right)\frac{\partial}{\partial s}u(s)\,\mathrm{d}s \\
&\quad + \frac{\mu^\alpha}{\Gamma(1-\alpha)}\int_{s=t_k}^{t_{k+1}} e^{\mu\left(\frac{t_k+t_{k+1}}{2}-s\right)}\left(\frac{\partial}{\partial s}u(s) - \frac{u(t_{k+1})-u(t_k)}{\tau_{k+1}}\right)\mathrm{d}s \\
&= \frac{\mu^\alpha}{\Gamma(1-\alpha)}\int_{s=t_k}^{t_{k+1}} e^{-\mu s}\left(e^{\mu s} - e^{\mu\frac{t_k+t_{k+1}}{2}}\right)\frac{\partial}{\partial s}u(s)\,\mathrm{d}s \\
&\quad + \frac{\mu^\alpha}{\Gamma(1-\alpha)}\left[\int_{s=t_k}^{t_{k+1}} e^{\mu\left(\frac{t_k+t_{k+1}}{2}-s\right)}\,\mathrm{d}(u(s)-u(t_k)) \right.\\
&\qquad\left. - \frac{u(t_{k+1})-u(t_k)}{\tau_{k+1}}\int_{s=t_k}^{t_{k+1}} e^{\mu\left(\frac{t_k+t_{k+1}}{2}-s\right)}\,\mathrm{d}(s-t_k)\right] \\
&:= \mathbb{M}^1_{n,k} + \mathbb{M}^2_{n,k}.
\end{aligned}
$$

Here

$$
\mathbb{M}^1_{n,k} = \frac{\mu^\alpha}{\Gamma(1-\alpha)}\int_{s=t_k}^{t_{k+1}} e^{-\mu s}\left(e^{\mu s} - e^{\mu\frac{t_k+t_{k+1}}{2}}\right)\frac{\partial}{\partial s}u(s)\,\mathrm{d}s
$$

and

$$
\mathbb{M}^2_{n,k} = \frac{\mu^{\alpha+1}}{\Gamma(1-\alpha)}\int_{s=t_k}^{t_{k+1}} e^{\mu\left(\frac{t_k+t_{k+1}}{2}-s\right)}\left[(u(s)-u(t_k)) - \frac{u(t_{k+1})-u(t_k)}{\tau_{k+1}}(s-t_k)\right]\mathrm{d}s.
$$

Applying the mean value theorem to $\mathbb{M}^2_{n,k}$ gives

$$
\begin{aligned}
\mathbb{M}^2_{n,k} &= \frac{\mu^{\alpha+1}}{\Gamma(1-\alpha)}\left[(u(\gamma_1)-u(t_k)) - \frac{u(t_{k+1})-u(t_k)}{\tau_{k+1}}(\gamma_1-t_k)\right]\int_{s=t_k}^{t_{k+1}} e^{\mu\left(\frac{t_k+t_{k+1}}{2}-s\right)}\mathrm{d}s \\
&= \frac{\mu^{\alpha+1}}{\Gamma(1-\alpha)}(\gamma_1-t_k)\left[u_t(\gamma_2) - u_t(\gamma_3)\right]\int_{s=t_k}^{t_{k+1}} e^{\mu\left(\frac{t_k+t_{k+1}}{2}-s\right)}\mathrm{d}s \\
&= \frac{\mu^{\alpha+1}}{\Gamma(1-\alpha)}(\gamma_1-t_k)(\gamma_2-\gamma_3)u_{tt}(\gamma_4)\int_{s=t_k}^{t_{k+1}} e^{\mu\left(\frac{t_k+t_{k+1}}{2}-s\right)}\mathrm{d}s
\end{aligned}
$$

for some $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in (t_k, t_{k+1})$. Thus for $1 \le k \le n-1$, by the middle rectangle quadrature formula and (18), there exists

$$\begin{aligned}\left|\mathbb{M}^1_{n,k}\right| &\le \frac{\mu^\alpha}{\Gamma(1-\alpha)} \int_{s=t_k}^{t_{k+1}} e^{-\mu s} \left|e^{\mu s} - e^{\mu \frac{t_k+t_{k+1}}{2}}\right| \left|\frac{\partial}{\partial s} u(s)\right| \mathrm{d}s \\ &\le C \max_{t_k \le s \le t_{k+1}} |u_t(s)| \int_{s=t_k}^{t_{k+1}} \left|e^{\mu s} - e^{\mu \frac{t_k+t_{k+1}}{2}}\right| \mathrm{d}s \\ &\le C\tau_{k+1}^3 t_k^{\alpha-1}\end{aligned}$$

and

$$\left|\mathbb{M}^2_{n,k}\right| \le C\tau_{k+1}^2 \max_{t_k \le s \le t_{k+1}} |u_{tt}(s)| \left|\int_{s=t_k}^{t_{k+1}} e^{\mu\left(\frac{t_k+t_{k+1}}{2} - s\right)} \mathrm{d}s\right|.$$

Then for $1 \le k \le n-1$, using the triangle inequality yields

$$\sum_{k=1}^{n-1} |\mathbb{J}^2_{n,k}| \le \sum_{k=1}^{n-1} |\mathbb{M}^1_{n,k}| + \sum_{k=1}^{n-1} |\mathbb{M}^2_{n,k}|.$$

From $t_n = T(\frac{n}{N})^r$, (18), and (19), one has

$$\sum_{k=1}^{n-1} \left|\mathbb{M}^1_{n,k}\right| \le C \sum_{k=1}^{n-1} \tau_{k+1}^3 t_k^{\alpha-1} \le Cn^{-2} \sum_{k=1}^{n-1} \left(\frac{k}{n}\right)^{(\alpha+2)r-3} \frac{1}{n} \le Cn^{-2} \tag{35}$$

and

$$\begin{aligned}\sum_{k=1}^{n-1} \left|\mathbb{M}^2_{n,k}\right| &\le C \sum_{k=1}^{n-1} \tau_{k+1}^2 \max_{t_k \le s \le t_{k+1}} |u_{tt}(s)| \left|\int_{s=t_k}^{t_{k+1}} e^{\mu\left(\frac{t_k+t_{k+1}}{2} - s\right)} \mathrm{d}s\right| \\ &\le C \sum_{k=1}^{n-1} \tau_{k+1}^3 t_k^{\alpha-2} \\ &\le Cn^{-r(\alpha+1)} \sum_{k=1}^{n-1} k^{(\alpha+1)r-3} \\ &\le \begin{cases} Cn^{-r(\alpha+1)} & \text{if } r(\alpha+1) < 2, \\ Cn^{-2}\ln n & \text{if } r(\alpha+1) = 2, \\ Cn^{-2} & \text{if } r(\alpha+1) > 2. \end{cases}\end{aligned} \tag{36}$$

Combining (34)–(36), we have

$$\sum_{k=0}^{n-1} |\mathbb{J}^2_{n,k}| \le Cn^{-\min\{2, r\alpha\}}. \tag{37}$$

Step 3: For $\mathbb{J}^4_{n,k}$, we prove the desired results in the following three cases, i.e., $k = 0$, $1 \le k \le n-2$, and $k = n-1$,

$$\sum_{k=0}^{n-1} |\mathbb{J}^4_{n,k}| = |\mathbb{J}^4_{n,0}| + \sum_{k=1}^{n-2} |\mathbb{J}^4_{n,k}| + |\mathbb{J}^4_{n,n-1}|.$$

Case 1: For $k=0$, if $n=1$, by $t_n = T(\frac{n}{N})^r$, (18), and the mean value theorem, one has

$$\begin{aligned}
\left|\mathbb{J}_{1,0}^4\right| &\le \frac{\mu}{\Gamma(1-\alpha)}\int_{s=0}^{t_1}(t_1-s)^{1-\alpha}e^{-\mu(t_1-s)}\left|\frac{\partial}{\partial s}u(s)\right|\mathrm{d}s \\
&\quad + \frac{\mu}{\Gamma(1-\alpha)}\int_{s=0}^{t_1}(t_1-s)^{1-\alpha}e^{-\mu\left(t_1-\frac{t_0+t_1}{2}\right)}\left|\frac{u(t_1)-u(t_0)}{\tau_1}\right|\mathrm{d}s \\
&\le \frac{\mu}{\Gamma(1-\alpha)}\int_{s=0}^{t_1}(t_1-s)^{1-\alpha}\left|\frac{\partial}{\partial s}u(s)\right|\mathrm{d}s + \frac{\mu e^{-\mu\frac{\tau_1}{2}}\tau_1^{1-\alpha}}{(2-\alpha)\Gamma(1-\alpha)}\int_{s=0}^{\tau_1}\left|\frac{\partial}{\partial s}u(s)\right|\mathrm{d}s \\
&\le \frac{\mu\tau_1^{1-\alpha}}{\Gamma(1-\alpha)}\int_{s=0}^{\tau_1}s^{\alpha-1}\,\mathrm{d}s + \frac{C\tau_1^{1-\alpha}}{(2-\alpha)\Gamma(1-\alpha)}\int_{s=0}^{\tau_1}s^{\alpha-1}\,\mathrm{d}s \\
&\le C;
\end{aligned}$$

if $n>1$, applying $t_n = T(\frac{n}{N})^r$, (18), (19), and the mean value theorem gives

$$\begin{aligned}
\left|\mathbb{J}_{n,0}^4\right| &\le \frac{\mu}{\Gamma(1-\alpha)}\int_{s=0}^{t_1}(t_n-s)^{1-\alpha}e^{-\mu(t_n-s)}\left|\frac{\partial}{\partial s}u(s)\right|\mathrm{d}s \\
&\quad + \frac{\mu}{\Gamma(1-\alpha)}\int_{s=0}^{t_1}(t_n-s)^{1-\alpha}e^{-\mu\left(t_n-\frac{t_0+t_1}{2}\right)}\left|\frac{u(t_1)-u(t_0)}{\tau_1}\right|\mathrm{d}s \\
&\le \frac{\mu e^{-\mu(t_n-t_1)}t_n^{1-\alpha}}{\Gamma(1-\alpha)}\int_{s=0}^{t_1}\left|\frac{\partial}{\partial s}u(s)\right|\mathrm{d}s \\
&\quad + \frac{\mu e^{-\mu\left(t_n-\frac{t_0+t_1}{2}\right)}\tau_1^{-1}}{(2-\alpha)\Gamma(1-\alpha)}\left[t_n^{2-\alpha}-(t_n-t_1)^{2-\alpha}\right]\int_{s=0}^{t_1}\left|\frac{\partial}{\partial s}u(s)\right|\mathrm{d}s \\
&\le Ct_n^{1-\alpha}\int_{s=0}^{t_1}s^{\alpha-1}\,\mathrm{d}s + C\tau_1^{-1}\left[t_n^{2-\alpha}-(t_n-t_1)^{2-\alpha}\right]\int_{s=0}^{t_1}s^{\alpha-1}\,\mathrm{d}s \\
&\le Ct_n^{1-\alpha}t_1^{\alpha} + C\left[t_n^{2-\alpha}-(t_n-t_1)^{2-\alpha}\right]t_1^{\alpha-1} \\
&\le 2Ct_n^{1-\alpha}t_1^{\alpha} \\
&\le Cn^{-r\alpha}.
\end{aligned}$$

It follows that

$$\left|\mathbb{J}_{n,0}^4\right| \le Cn^{-r\alpha} \text{ for } n\ge 1. \tag{38}$$

Case 2: For $1\le k\le n-2$, using integration by parts yields

$$\begin{aligned}
\mathbb{J}_{n,k}^4 &= \frac{\mu}{\Gamma(1-\alpha)}\int_{s=t_k}^{t_{k+1}}(t_n-s)^{1-\alpha}\left(e^{-\mu(t_n-s)}-e^{-\mu\left(t_n-\frac{t_k+t_{k+1}}{2}\right)}\right)\frac{\partial}{\partial s}u(s)\,\mathrm{d}s \\
&\quad + \frac{\mu e^{-\mu\left(t_n-\frac{t_k+t_{k+1}}{2}\right)}}{\Gamma(1-\alpha)}\int_{s=t_k}^{t_{k+1}}(t_n-s)^{1-\alpha}\left(\frac{\partial}{\partial s}u(s)-\frac{u(t_{k+1})-u(t_k)}{\tau_{k+1}}\right)\mathrm{d}s \\
&= \frac{\mu}{\Gamma(1-\alpha)}\int_{s=t_k}^{t_{k+1}}(t_n-s)^{1-\alpha}\left(e^{-\mu(t_n-s)}-e^{-\mu\left(t_n-\frac{t_k+t_{k+1}}{2}\right)}\right)\frac{\partial}{\partial s}u(s)\,\mathrm{d}s \\
&\quad + \frac{\mu e^{-\mu\left(t_n-\frac{t_k+t_{k+1}}{2}\right)}}{\Gamma(1-\alpha)}\Bigg[\int_{s=t_k}^{t_{k+1}}(t_n-s)^{1-\alpha}\,\mathrm{d}\left(u(s)-u(t_k)\right) \\
&\qquad\qquad - \frac{u(t_{k+1})-u(t_k)}{\tau_{k+1}}\int_{s=t_k}^{t_{k+1}}(t_n-s)^{1-\alpha}\,\mathrm{d}\left(s-t_k\right)\Bigg] \\
&:= \mathbb{H}_{n,k}^1 + \mathbb{H}_{n,k}^2.
\end{aligned}$$

Here

$$\mathbb{H}_{n,k}^{1}=\frac{\mu}{\Gamma(1-\alpha)}\int_{s=t_k}^{t_{k+1}}(t_n-s)^{1-\alpha}\left(e^{-\mu(t_n-s)}-e^{-\mu\left(t_n-\frac{t_k+t_{k+1}}{2}\right)}\right)\frac{\partial}{\partial s}u(s)\,\mathrm{d}s$$

and

$$\begin{aligned}\mathbb{H}_{n,k}^{2}=&\frac{(\alpha-1)\mu e^{-\mu\left(t_n-\frac{t_k+t_{k+1}}{2}\right)}}{\Gamma(1-\alpha)}\\&\cdot\int_{s=t_k}^{t_{k+1}}(t_n-s)^{-\alpha}\left[\frac{u\left(t_{k+1}\right)-u\left(t_k\right)}{\tau_{k+1}}(s-t_k)-(u(s)-u\left(t_k\right))\right]\mathrm{d}s.\end{aligned}$$

Applying the mean value theorem to $\mathbb{H}_{n,k}^{2}$ gives

$$\begin{aligned}\mathbb{H}_{n,k}^{2}=&\frac{(\alpha-1)\mu e^{-\mu\left(t_n-\frac{t_k+t_{k+1}}{2}\right)}}{\Gamma(1-\alpha)}\\&\cdot\left[\frac{u\left(t_{k+1}\right)-u\left(t_k\right)}{\tau_{k+1}}(\zeta_1-t_k)-(u(\zeta_1)-u\left(t_k\right))\right]\int_{s=t_k}^{t_{k+1}}(t_n-s)^{-\alpha}\,\mathrm{d}s\\=&\frac{(\alpha-1)\mu e^{-\mu\left(t_n-\frac{t_k+t_{k+1}}{2}\right)}}{\Gamma(1-\alpha)}(\zeta_1-t_k)\left[u_t(\zeta_2)-u_t(\zeta_3)\right]\int_{s=t_k}^{t_{k+1}}(t_n-s)^{-\alpha}\,\mathrm{d}s\\=&\frac{(\alpha-1)\mu e^{-\mu\left(t_n-\frac{t_k+t_{k+1}}{2}\right)}}{\Gamma(1-\alpha)}(\zeta_1-t_k)(\zeta_2-\zeta_3)u_{tt}(\zeta_4)\int_{s=t_k}^{t_{k+1}}(t_n-s)^{-\alpha}\,\mathrm{d}s\end{aligned}$$

for some $\zeta_1,\zeta_2,\zeta_3,\zeta_4\in(t_k,t_{k+1})$. Thus for $1\le k\le n-2$, by the middle rectangle quadrature formula and (18), one has

$$\begin{aligned}\left|\mathbb{H}_{n,k}^{1}\right|&\le C(t_n-t_k)^{1-\alpha}\max_{t_k\le s\le t_{k+1}}|u_t(s)|\int_{s=t_k}^{t_{k+1}}\left|e^{-\mu\left(t_n-\frac{t_k+t_{k+1}}{2}\right)}-e^{-\mu(t_n-s)}\right|\mathrm{d}s\\&\le C\tau_{k+1}^{3}t_k^{\alpha-1}(t_n-t_k)^{1-\alpha}\end{aligned}$$

and

$$\left|\mathbb{H}_{n,k}^{2}\right|\le C\tau_{k+1}^{2}\max_{t_k\le s\le t_{k+1}}|u_{tt}(s)|\int_{s=t_k}^{t_{k+1}}(t_n-s)^{-\alpha}\,\mathrm{d}s.$$

Then for $1\le k\le n-2$, using the triangle inequality yields

$$\sum_{k=1}^{n-2}|\mathbb{J}_{n,k}^{4}|\le\sum_{k=1}^{n-2}|\mathbb{H}_{n,k}^{1}|+\sum_{k=1}^{n-2}|\mathbb{H}_{n,k}^{2}|.$$

By $t_n = T(\frac{n}{N})^r$, (18), and (19), one has

$$
\begin{aligned}
\sum_{k=1}^{n-2} |\mathbb{H}_{n,k}^1| &\le C \sum_{k=1}^{n-2} \tau_{k+1}^3 t_k^{\alpha-1} (t_n - t_k)^{1-\alpha} \\
&\le C \sum_{k=1}^{n-2} \left(TN^{-r}k^{r-1}\right)^3 \left[T\left(\frac{k}{N}\right)^r\right]^{\alpha-1} \left[T\left(\frac{n}{N}\right)^r - T\left(\frac{k}{N}\right)^r\right]^{1-\alpha} \\
&\le CT^3 \sum_{k=1}^{n-2} N^{-3r} k^{(\alpha+2)r-3} \left[n^r - k^r\right]^{1-\alpha} \\
&\le Cn^{-2} \sum_{k=1}^{n-2} \left(\frac{n}{N}\right)^{3r} \left(\frac{k}{n}\right)^{(\alpha+2)r-3} \frac{1}{n} \\
&\le Cn^{-2}. \qquad (39)
\end{aligned}
$$

On the one hand, since $\left(1 - \frac{3^r}{4^r}\right) n^r \le n^r - (k+1)^r \le n^r$ for $\forall 1 \le k \le \lceil n/2 \rceil - 1$ and $n \ge 2$, using $t_n = T(\frac{n}{N})^r$, (18), (19), and the mean value theorem yields

$$
\begin{aligned}
\sum_{k=1}^{\lceil n/2 \rceil - 1} |\mathbb{H}_{n,k}^2| &\le C \sum_{k=1}^{\lceil n/2 \rceil - 1} \tau_{k+1}^2 \max_{t_k \le s \le t_{k+1}} |u_{tt}(s)| \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha} \, \mathrm{d}s \\
&\le C \sum_{k=1}^{\lceil n/2 \rceil - 1} \tau_{k+1}^2 t_k^{\alpha-2} \tau_{k+1} (t_n - t_{k+1})^{-\alpha} \\
&\le CT \sum_{k=1}^{\lceil n/2 \rceil - 1} \left(N^{-r}k^{r-1}\right)^3 \left[\left(\frac{k}{N}\right)^r\right]^{\alpha-2} \left[\left(\frac{n}{N}\right)^r - \left(\frac{k+1}{N}\right)^r\right]^{-\alpha} \\
&\le CT \sum_{k=1}^{\lceil n/2 \rceil - 1} N^{-r} k^{(\alpha+1)r-3} \left[n^r - (k+1)^r\right]^{-\alpha} \\
&\le Cn^{-r(\alpha+1)} \sum_{k=1}^{\lceil n/2 \rceil - 1} k^{(\alpha+1)r-3} \\
&\le \begin{cases} Cn^{-r(\alpha+1)} & \text{if } r(\alpha+1) < 2, \\ Cn^{-2} \ln n & \text{if } r(\alpha+1) = 2, \\ Cn^{-2} & \text{if } r(\alpha+1) > 2. \end{cases} \qquad (40)
\end{aligned}
$$

On the other hand, for $\lceil n/2 \rceil \le k \le n-2$, by $t_n = T(\frac{n}{N})^r$, (18), (19), and the mean value theorem, one has

(41)
$$
\begin{aligned}
\sum_{k=\lceil n/2 \rceil}^{n-2} |\mathbb{H}_{n,k}^2| &\le C \sum_{k=\lceil n/2 \rceil}^{n-2} \tau_{k+1}^2 \max_{t_k \le s \le t_{k+1}} |u_{tt}(s)| \int_{s=t_k}^{t_{k+1}} (t_n - s)^{-\alpha} \, \mathrm{d}s \\
&\le C \sum_{k=\lceil n/2 \rceil}^{n-2} \tau_{k+1}^2 t_k^{\alpha-2} \tau_{k+1} (t_n - t_{k+1})^{-\alpha} \\
&\le CT \sum_{k=\lceil n/2 \rceil}^{n-2} \left(N^{-r}k^{r-1}\right)^3 \left[\left(\frac{k}{N}\right)^r\right]^{\alpha-2} \left[\left(\frac{n}{N}\right)^r - \left(\frac{k+1}{N}\right)^r\right]^{-\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&\le CT \sum_{k=\lceil n/2\rceil}^{n-2} N^{-r} k^{(\alpha+1)r-3} n^{-\alpha r}\left[1-\left(\frac{k+1}{n}\right)^{r}\right]^{-\alpha}\\
&\le C \sum_{k=\lceil n/2\rceil}^{n-2} N^{-r} n^{(\alpha+1)r-3} n^{-\alpha r}\left[1-\left(\frac{k+1}{n}\right)^{r}\right]^{-\alpha}\\
&\le Cn^{-2} \sum_{k=\lceil n/2\rceil}^{n-2} \left(\frac{n}{N}\right)^{r}\left[1-\left(\frac{k+1}{n}\right)^{r}\right]^{-\alpha}\frac{1}{n}\\
&\le Cn^{-2}.
\end{aligned}
$$

Case 3: For $k = n-1$, applying the mean value theorem, $t_n = T(\frac{n}{N})^r$, $t_n \le 2^r t_{n-1}$ for $n \ge 2$ (cf. [27]), (18), and (19) yields

$$
\begin{aligned}
|\mathbb{J}^4_{n,n-1}| \le{}& \frac{\mu}{\Gamma(1-\alpha)}\int_{s=t_{n-1}}^{t_n}(t_n-s)^{1-\alpha}\left|e^{-\mu(t_n-s)}-e^{-\mu\left(t_n-\frac{t_{n-1}+t_n}{2}\right)}\right|\left|\frac{\partial}{\partial s}u(s)\right|\mathrm{d}s\\
&+\frac{\mu e^{-\mu\left(t_n-\frac{t_{n-1}+t_n}{2}\right)}}{\Gamma(1-\alpha)}\int_{s=t_{n-1}}^{t_n}(t_n-s)^{1-\alpha}\left|\frac{\partial}{\partial s}u(s)-\frac{u(t_n)-u(t_{n-1})}{\tau_n}\right|\mathrm{d}s\\
\le{}& \frac{C\tau_n}{\Gamma(1-\alpha)}\max_{t_{n-1}\le s\le t_n}|u_t(s)|\int_{s=t_{n-1}}^{t_n}(t_n-s)^{1-\alpha}\,\mathrm{d}s\\
&+\frac{Ce^{-\mu\frac{\tau_n}{2}}\tau_n}{\Gamma(1-\alpha)}\max_{t_{n-1}\le s\le t_n}|u_{tt}(s)|\int_{s=t_{n-1}}^{t_n}(t_n-s)^{1-\alpha}\,\mathrm{d}s\\
\le{}& C\tau_n^{3-\alpha}t_{n-1}^{\alpha-1}+C\tau_n^{3-\alpha}t_{n-1}^{\alpha-2}\\
\le{}& C\tau_n^{3-\alpha}\left(2^{-r}t_n\right)^{\alpha-1}+C\tau_n^{3-\alpha}\left(2^{-r}t_n\right)^{\alpha-2}\\
\le{}& C\left[N^{-r}(n-1)^{r-1}\right]^{3-\alpha}\left[\left(\frac{n}{N}\right)^{r(\alpha-1)}+\left(\frac{n}{N}\right)^{r(\alpha-2)}\right]\\
\le{}& Cn^{-(3-\alpha)}.
\end{aligned}
$$

It follows that

$$
\left|\mathbb{J}^4_{n,n-1}\right| \le Cn^{-(3-\alpha)} \text{ for } n \ge 2. \tag{42}
$$

Combining (38)–(42), we have

$$
\sum_{k=0}^{n-1}|\mathbb{J}^4_{n,k}| \le Cn^{-\min\{2,r\alpha\}}. \tag{43}
$$

To finish the proof, we use the facts, i.e., if $r(\alpha+1) \le 2$, one has $n^{-r\alpha} \ge Cn^{-r(\alpha+1)}\ln n$ and if $r(\alpha+1) > 2$, then $n^{-(2-\alpha)} \ge n^{-2}$ in the previous estimates. Finally, according to (32), (33), (37), and (43), we have

$$
|D_N^{\alpha,\mu}u(t_n) - \partial_t^{\alpha,\mu}u(t_n)| \le Cn^{-\min\{2-\alpha,r\alpha\}} \quad \text{for } 1 \le n \le N.
$$

This completes the proof. □

4. Stability and convergence analysis

In this section, we prove the discrete scheme (23) is unconditionally stable and converges in L^∞ norm, and a sharp α-robust error bound for the solution of (23) is given later.

4.1. Stability analysis. From (3) and the definition $w_\mu^\alpha(t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_t^\infty e^{-\mu s} s^{-1-\alpha} \mathrm{d}s$, we rewrite $\partial_t^{\alpha,\mu} u(x_i, y_j, t_n)$ in (21) as follows

$$\begin{aligned}\partial_t^{\alpha,\mu} u(x_i, y_j, t_n) = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{\alpha}{\Gamma(1-\alpha)} e^{-\mu(t_n - s)} \\ \cdot \left(\int_{t_n - s}^\infty e^{\mu(t_n - s - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau \right) \frac{\partial}{\partial s} u(x_i, y_j, s) \,\mathrm{d}s.\end{aligned} \tag{44}$$

It is easy to see that $e^{-\mu(t_n - s)}$ in (44) is non-negative and monotonically increasing with respect to $s \in [0, t_n]$. Now we prove the following lemma.

Lemma 4.1. *The improper integral* $\int_{t_n-s}^\infty e^{\mu(t_n - s - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau$ *in* (44) *is non-negative and monotonically increasing with respect to* $s \in [0, t_n]$.

Proof. The non-negative property is obvious and we omit it here. Let $\tilde{t} = t_n - s$. For arbitrary $M \in (\tilde{t}, \infty)$, one has

$$\int_{\tilde{t}}^\infty e^{\mu(\tilde{t} - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau - \int_M^\infty e^{\mu(\tilde{t} - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau = \int_{\tilde{t}}^M e^{\mu(\tilde{t} - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau;$$

and for arbitrary constant $\delta \in (0, \tilde{t})$, there exists

$$\int_{\tilde{t}-\delta}^\infty e^{\mu(\tilde{t} - \delta - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau - \int_{M-\delta}^\infty e^{\mu(\tilde{t} - \delta - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau = \int_{\tilde{t}-\delta}^{M-\delta} e^{\mu(\tilde{t} - \delta - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau.$$

It is easy to verify that

$$\int_{\tilde{t}-\delta}^{M-\delta} e^{\mu(\tilde{t} - \delta - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau = \int_{\tilde{t}}^M e^{\mu(\tilde{t} - \tau)} (\tau - \delta)^{-1-\alpha} \,\mathrm{d}\tau > \int_{\tilde{t}}^M e^{\mu(\tilde{t} - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau.$$

Now by the sign-preserving property of limit operations, it yields

$$\lim_{M \to \infty} \int_{\tilde{t}}^M e^{\mu(\tilde{t} - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau \le \lim_{M \to \infty} \int_{\tilde{t}-\delta}^{M-\delta} e^{\mu(\tilde{t} - \delta - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau,$$

which is equivalent to

$$\begin{aligned}&\lim_{M \to \infty} \left[\int_{\tilde{t}}^\infty e^{\mu(\tilde{t} - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau - \int_M^\infty e^{\mu(\tilde{t} - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau \right] \\ &\le \lim_{M \to \infty} \left[\int_{\tilde{t}-\delta}^\infty e^{\mu(\tilde{t} - \delta - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau - \int_{M-\delta}^\infty e^{\mu(\tilde{t} - \delta - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau \right].\end{aligned}$$

In fact, using the arbitrariness of M leads to

$$\lim_{M \to \infty} \int_M^\infty e^{\mu(\tilde{t} - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau = 0$$

and

$$\lim_{M \to \infty} \int_{M-\delta}^\infty e^{\mu(\tilde{t} - \delta - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau = 0.$$

Thus we arrive at

$$\int_{\tilde{t}}^\infty e^{\mu(\tilde{t} - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau \le \int_{\tilde{t}-\delta}^\infty e^{\mu(\tilde{t} - \delta - \tau)} \tau^{-1-\alpha} \,\mathrm{d}\tau.$$

This completes the proof. □

Remark 4.1. *We note that this lemma can also be proved by making the substitution* $r = \tau - (t_n - s)$, *which gives the improper integral* $\int_0^\infty e^{-\mu r} (r + t_n - s)^{-1-\alpha} \,\mathrm{d}r$, *and this is a non-negative and monotonically increasing function of* $s \in [0, t_n]$.

Remark 4.2. *In fact, based on the previous analysis, it is easy to conclude that* $w_\mu^\alpha(t)$ *is a non-negative decreasing function on* $(0,\infty)$ *and blows up at* $t=0$.

According to (21) and (44), we can rewrite the discretization (22) of the tempered Caputo fractional derivative as

$$D_N^{\alpha,\mu} u_{ij}^n = \frac{d_{n,1}}{\Gamma(1-\alpha)} u_{ij}^n - \frac{d_{n,n}}{\Gamma(1-\alpha)} u_{ij}^0 + \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n-1} \left[d_{n,k+1} - d_{n,k} \right] u_{ij}^{n-k},$$

where

$$d_{n,k} := \frac{\alpha e^{-\mu\left(t_n - \frac{t_{n-k}+t_{n-k+1}}{2}\right)}}{\tau_{n-k+1}} \int_{t_{n-k}}^{t_{n-k+1}} \int_{t_n-s}^{\infty} e^{\mu(t_n-s-\tau)} \tau^{-1-\alpha} \, \mathrm{d}\tau \, \mathrm{d}s \tag{45a}$$

$$= \frac{e^{-\mu\left(t_n - \frac{t_{n-k}+t_{n-k+1}}{2}\right)}}{\tau_{n-k+1}} \int_{t_{n-k}}^{t_{n-k+1}} \left[\frac{1}{(t_n-s)^\alpha} - \mu^\alpha e^{\mu(t_n-s)} + \frac{\mu t_n}{(t_n-s)^\alpha} - \frac{\mu}{(t_n-s)^{\alpha-1}} \right] \mathrm{d}s. \tag{45b}$$

From (45a), using the mean value theorem and Lemma 4.1 yields

$$0 < d_{n,k+1} \le d_{n,k}. \tag{46}$$

From (45b), one has

$$d_{n,1} = \frac{e^{-\mu\left(t_n - \frac{t_{n-1}+t_n}{2}\right)}}{\tau_n} \int_{t_{n-1}}^{t_n} \left[\frac{1}{(t_n-s)^\alpha} - \mu^\alpha e^{\mu(t_n-s)} + \frac{\mu t_n}{(t_n-s)^\alpha} - \frac{\mu}{(t_n-s)^{\alpha-1}} \right] \mathrm{d}s$$

$$= \frac{e^{-\mu\frac{\tau_n}{2}} \left[1 + (1-\alpha)\mu^{\alpha-1} \left(1 - e^{\mu\tau_n}\right) \tau_n^{\alpha-1} + \mu t_n - \frac{(1-\alpha)\mu\tau_n}{2-\alpha} \right]}{1-\alpha} \tau_n^{-\alpha}. \tag{47}$$

Now, the scheme can be written as

$$\left[\frac{d_{n,1}}{\Gamma(1-\alpha)} + \frac{2}{h_x^2} + \frac{2}{h_y^2} \right] u_{ij}^n = \frac{u_{i+1,j}^n}{h_x^2} + \frac{u_{i-1,j}^n}{h_x^2} + \frac{u_{i,j+1}^n}{h_y^2} + \frac{u_{i,j-1}^n}{h_y^2} + f_{ij}^n + \frac{1}{\Gamma(1-\alpha)} \left[d_{n,n} u_{ij}^0 + \sum_{k=1}^{n-1} \left(d_{n,k} - d_{n,k+1} \right) u_{ij}^{n-k} \right],$$

for $i = 1,2,\cdots,N_x-1$, $j = 1,2,\cdots,N_y-1$ and $n = 1,2,\cdots,N$. The stability result will be presented in a general framework, which imitates the analysis of [28, 39]. Suppose that

$$L_{N_x,N_y}^N v_{ij}^n = g_{ij}^n \quad \text{for } 1 \le i \le N_x-1,\ 1 \le j \le N_y-1,\ 1 \le n \le N, \tag{48}$$

with v_{ij}^0 being given for $0 \le i \le N_x$, $0 \le j \le N_y$ and $v_{0j}^n = v_{N_x,j}^n = v_{i0}^n = v_{i,N_y}^n = 0$ for $0 \le i \le N_x$, $0 \le j \le N_y$ and $1 \le n \le N$. For any mesh function $\{v_{ij}^n : i = 0,1,2,\cdots,N_x, j = 0,1,2,\cdots,N_y, n = 0,1,2,\cdots,N\}$, set $\|v^n\|_\infty = \max\{|v_{ij}^n| : i = 0,1,2,\cdots,N_x, j = 0,1,2,\cdots,N_y\}$ for each n. We seek to bound $\|v^n\|_\infty$ in terms of the data of (48) for each n. Therefore, a preliminary result will be given first.

Lemma 4.2. *For $1 \le n \le N$, the solution of the discrete problem* (48) *satisfies*

(49)
$$\|v^n\|_\infty \le d_{n,1}^{-1}\left[d_{n,n}\|v^0\|_\infty + \sum_{k=1}^{n-1}(d_{n,k}-d_{n,k+1})\|v^{n-k}\|_\infty + \Gamma(1-\alpha)\|g^n\|_\infty\right].$$

Proof. Fix $n \in \{1, 2, \cdots, N\}$. The equation associated with the mesh point (x_i, y_j, t_n) is

$$\left[\frac{d_{n,1}}{\Gamma(1-\alpha)} + \frac{2}{h_x^2} + \frac{2}{h_y^2}\right] v_{ij}^n = \frac{v_{i+1,j}^n}{h_x^2} + \frac{v_{i-1,j}^n}{h_x^2} + \frac{v_{i,j+1}^n}{h_y^2} + \frac{v_{i,j-1}^n}{h_y^2} + g_{ij}^n + \frac{1}{\Gamma(1-\alpha)}\left[d_{n,n}v_{ij}^0 + \sum_{k=1}^{n-1}(d_{n,k}-d_{n,k+1})\, v_{ij}^{n-k}\right].$$

Using the property $\|v^n\|_\infty = \max\{|v_{ij}^n| : i = 0, 1, 2, \cdots, N_x, j = 0, 1, 2, \cdots, N_y\}$ for each n and recalling (46), it follows that

$$\left[\frac{d_{n,1}}{\Gamma(1-\alpha)} + \frac{2}{h_x^2} + \frac{2}{h_y^2}\right]\|v^n\|_\infty \le \left(\frac{2}{h_x^2} + \frac{2}{h_y^2}\right)\|v^n\|_\infty + \left|g_{ij}^n + \frac{1}{\Gamma(1-\alpha)}\left[d_{n,n}\|v^0\|_\infty + \sum_{k=1}^{n-1}(d_{n,k}-d_{n,k+1})\|v^{n-k}\|_\infty\right]\right|.$$

Then

$$\frac{d_{n,1}\|v^n\|_\infty}{\Gamma(1-\alpha)} \le \|g^n\|_\infty + \frac{1}{\Gamma(1-\alpha)}\left[d_{n,n}\|v^0\|_\infty + \sum_{k=1}^{n-1}(d_{n,k}-d_{n,k+1})\|v^{n-k}\|_\infty\right],$$

which leads to the desired result. □

The lemma above will be used in an inductive argument to give a weighted bound for $\|v^n\|_\infty$ in terms of the given data $\|v^0\|_\infty$ and $\|g^\theta\|_\infty$ for $\theta = 1, 2, \ldots, n$. From (47), letting $c_n := d_{n,1}$, for each $n \in \{1, 2, \cdots, N\}$, the stability multipliers $m_{n,\theta}$ for $\theta = 1, 2, \cdots, n-1$ are defined by

$$m_{n,n} = 1,\ m_{n,\theta} = \sum_{k=1}^{n-\theta} c_{n-k}^{-1}(d_{n,k}-d_{n,k+1})\, m_{n-k,\theta}. \tag{50}$$

Recalling (46), it's easy to see that $m_{n,\theta} > 0$ for all n, θ. We will use $m_{n,\theta} > 0$ in the following stability result.

Theorem 4.1. *For $1 \le n \le N$, the solution of the discrete problem* (48) *satisfies*

$$\|v^n\|_\infty \le \|v^0\|_\infty + d_{n,1}^{-1}\Gamma(1-\alpha)\sum_{\theta=1}^{n} m_{n,\theta}\|g^\theta\|_\infty. \tag{51}$$

Proof. We use induction on n. The case $n = 1$ of (51) is

$$\|v^1\|_\infty \le d_{1,1}^{-1}d_{1,1}\|v^0\|_\infty + d_{1,1}^{-1}\Gamma(1-\alpha)m_{1,1}\|g^1\|_\infty,$$

which is the same as (49) according to (50). Fix $n \in \{2, 3, \cdots, N\}$. Assume that (51) is valid for $k = 1, 2, \cdots, n-1$. Then (49) and the inductive hypothesis yield

$$\begin{aligned}
\|v^n\|_\infty \leq\ & d_{n,1}^{-1} \Big\{\Gamma(1-\alpha)\|g^n\|_\infty + d_{n,n}\|v^0\|_\infty \\
& + \sum_{k=1}^{n-1} (d_{n,k} - d_{n,k+1}) \left[\|v^0\|_\infty + d_{n-k,1}^{-1}\Gamma(1-\alpha) \sum_{\theta=1}^{n-k} m_{n-k,\theta}\|g^\theta\|_\infty \right] \Big\} \\
=\ & d_{n,1}^{-1} \Big\{\Gamma(1-\alpha)\|g^n\|_\infty + d_{n,1}\|v^0\|_\infty \\
& + \Gamma(1-\alpha) \sum_{k=1}^{n-1} \left[(d_{n,k} - d_{n,k+1})\, d_{n-k,1}^{-1} \sum_{\theta=1}^{n-k} m_{n-k,\theta}\|g^\theta\|_\infty \right] \Big\} \\
=\ & d_{n,1}^{-1} \Big\{ d_{n,1}\|v^0\|_\infty \\
& + \Gamma(1-\alpha) \left[\sum_{\theta=1}^{n-1} \left(\sum_{k=1}^{n-\theta} c_{n-k}^{-1} (d_{n,k} - d_{n,k+1})\, m_{n-k,\theta} \right) \|g^\theta\|_\infty + m_{n,n}\|g^n\|_\infty \right] \Big\} \\
=\ & \|v^0\|_\infty + d_{n,1}^{-1}\Gamma(1-\alpha) \sum_{\theta=1}^{n} m_{n,\theta}\|g^\theta\|_\infty.
\end{aligned}$$

Thus we have proved that (51) holds for $k = n$. By the principle of induction, the theorem has been proved. $\square$

4.2. Convergence analysis. In our error analyses we shall need some estimates for the stability multipliers $m_{n,\theta}$. We first introduce a concept: if an error bound does not blows up as $\alpha \to 1^-$, then it is called α-robust [39], otherwise it is called α-nonrobust. The following results are critical for deriving an estimate for the temporal error of our numerical scheme, where $m_{n,n}$ and $m_{n,\theta}$ are defined in (50).

Lemma 4.3. *For $n = 1, 2, \cdots, N$ and $0 \leq k \leq n-1$, there exists*

$$\sum_{\theta=k+1}^{n} m_{n,\theta} d_{\theta,\theta-k} = d_{n,1} = c_n. \tag{52}$$

Proof. If $n = 1$, then $k = 0$ and $m_{1,1}d_{1,1} = c_1$ as desired. Now assume that $i \geq 2$, and (52) holds for $n = 1, 2, \cdots, i-1$ and $0 \leq k \leq n-1$. We need to prove that (52) still holds for $n = i$ and $0 \leq k \leq i-1$. For $n = i$ and $k = i-1$, one has $m_{i,i}d_{i,1} = d_{i,1} = c_i$. So (52) holds. Then considering $n = i$ and $0 \leq k \leq i-2$, by (50) and the inductive hypothesis, one has

$$\begin{aligned}
\sum_{\theta=k+1}^{i} m_{i,\theta} d_{\theta,\theta-k} &= \sum_{\theta=k+1}^{i-1} m_{i,\theta} d_{\theta,\theta-k} + m_{i,i} d_{i,i-k} \\
&= \sum_{\theta=k+1}^{i-1} d_{\theta,\theta-k} \left[\sum_{\chi=1}^{i-\theta} c_{i-\chi}^{-1} (d_{i,\chi} - d_{i,\chi+1})\, m_{i-\chi,\theta} \right] + m_{i,i} d_{i,i-k} \\
&= \sum_{\chi=1}^{i-k-1} c_{i-\chi}^{-1} (d_{i,\chi} - d_{i,\chi+1}) \left[\sum_{\theta=k+1}^{i-\chi} m_{i-\chi,\theta} d_{\theta,\theta-k} \right] + m_{i,i} d_{i,i-k} \\
&= \sum_{\chi=1}^{i-k-1} c_{i-\chi}^{-1} (d_{i,\chi} - d_{i,\chi+1})\, c_{i-\chi} + m_{i,i} d_{i,i-k} \\
&= d_{i,1},
\end{aligned}$$

and $d_{i,1} = c_i$, which completes the proof of (52) for $n = i$ and $0 \le k \le i-1$. By the principle of induction, we are done. $\square$

Lemma 4.4. *Let $\gamma \in (0,1)$ be a constant. Then for $n = 1, 2, \cdots, N$, there exists*

$$d_{n,1}^{-1} \sum_{\theta=1}^{n} \theta^{r(\gamma-\alpha)} m_{n,\theta} \le \frac{C\Gamma(1+\gamma-\alpha)}{\gamma\Gamma(1-\alpha)} T^\alpha \left(\frac{t_n}{T}\right)^\gamma N^{r(\gamma-\alpha)}, \tag{53}$$

where C is some constant independent of t and as $\alpha \to 1$, C is bounded.

Proof. First, we have

$$\partial_t^{\alpha,\mu} e^{-\mu t} t^\gamma E_{\alpha,\gamma+1}(\mu^\alpha t^\alpha) = \frac{e^{-\mu t} t^{\gamma-\alpha}}{\Gamma(1+\gamma-\alpha)}; \tag{54}$$

see Appendix B for the proof of Eq. (54). Hence, for $\theta \ge 1$, after performing the derivative operation, we discard a negative term. Then by using the equation $\frac{\partial}{\partial s}\left[s^\gamma E_{\alpha,\gamma+1}(\mu^\alpha s^\alpha)\right] = s^{\gamma-1} E_{\alpha,\gamma}(\mu^\alpha s^\alpha)$ (cf. [34, Eq. (4.3.1), p. 58]), it yields

$$\begin{aligned}
&\frac{e^{-\mu(t_\theta - \frac{t_k+t_{k+1}}{2})}}{\Gamma(1+\gamma-\alpha)} (t_\theta)^{\gamma-\alpha} \\
&= e^{\mu\frac{t_k+t_{k+1}}{2}} \int_0^{t_\theta} w_\mu^\alpha(t_\theta - s) \frac{\partial}{\partial s}\left[e^{-\mu s} s^\gamma E_{\alpha,\gamma+1}(\mu^\alpha s^\alpha)\right] \mathrm{d}s \\
&\le e^{\mu\frac{t_k+t_{k+1}}{2}} \int_0^{t_\theta} w_\mu^\alpha(t_\theta - s) e^{-\mu s} s^{\gamma-1} E_{\alpha,\gamma}(\mu^\alpha s^\alpha)\,\mathrm{d}s \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{\theta-1} e^{-\mu(t_\theta - \frac{t_k+t_{k+1}}{2})} \int_{t_k}^{t_{k+1}} \left[\frac{1}{(t_\theta - s)^\alpha} - \mu^\alpha e^{\mu(t_\theta - s)} \right. \\
&\qquad \left. + \frac{\mu t_\theta}{(t_\theta - s)^\alpha} - \frac{\mu}{(t_\theta - s)^{\alpha-1}}\right] s^{\gamma-1} E_{\alpha,\gamma}(\mu^\alpha s^\alpha)\,\mathrm{d}s \\
&\le \frac{C_1}{\Gamma(1-\alpha)} \sum_{k=0}^{\theta-1} e^{-\mu(t_\theta - \frac{t_k+t_{k+1}}{2})} \int_{t_k}^{t_{k+1}} \left[\frac{1}{(t_\theta - s)^\alpha} - \mu^\alpha e^{\mu(t_\theta - s)} \right. \\
&\qquad \left. + \frac{\mu t_\theta}{(t_\theta - s)^\alpha} - \frac{\mu}{(t_\theta - s)^{\alpha-1}}\right] s^{\gamma-1}\,\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\le \frac{C_1}{\gamma\Gamma(1-\alpha)} \sum_{k=0}^{\theta-1} e^{-\mu(t_\theta - \frac{t_k+t_{k+1}}{2})} \int_{t_k}^{t_{k+1}} \left[\frac{1}{(t_\theta - s)^\alpha} - \mu^\alpha e^{\mu(t_\theta - s)} \right. \\
&\qquad \left. + \frac{\mu t_\theta}{(t_\theta - s)^\alpha} - \frac{\mu}{(t_\theta - s)^{\alpha-1}}\right] \mathrm{d}s \frac{(t_{k+1})^\gamma - (t_k)^\gamma}{\tau_{k+1}} \\
&= \frac{C_1}{\gamma\Gamma(1-\alpha)} \sum_{k=0}^{\theta-1} d_{\theta,\theta-k}\left[(t_{k+1})^\gamma - (t_k)^\gamma\right],
\end{aligned}$$

where we use Chebyshev's integral inequality (cf. [39, Lemma 5.2]) in the calculation, and C_1 is some constant independent of t, as $\alpha \to 1$, C_1 is bounded. Multiplying this inequality by $m_{n,\theta}$ then summing from $\theta = 1$ to n, and changing the order of summation then invoking Lemma 4.3, it yields

$$\begin{aligned}\frac{e^{-\mu(t_\theta-\frac{t_k+t_{k+1}}{2})}}{\Gamma(1+\gamma-\alpha)}\sum_{\theta=1}^{n}(t_\theta)^{\gamma-\alpha}m_{n,\theta} &\le \frac{C_1}{\gamma\Gamma(1-\alpha)}\sum_{\theta=1}^{n}m_{n,\theta}\sum_{k=0}^{\theta-1}d_{\theta,\theta-k}\left[(t_{k+1})^\gamma-(t_k)^\gamma\right]\\ &= \frac{C_1}{\gamma\Gamma(1-\alpha)}\sum_{k=0}^{n-1}\left[(t_{k+1})^\gamma-(t_k)^\gamma\right]\sum_{\theta=k+1}^{n}m_{n,\theta}d_{\theta,\theta-k}\\ &= \frac{C_1}{\gamma\Gamma(1-\alpha)}\cdot c_n\sum_{k=0}^{n-1}\left[(t_{k+1})^\gamma-(t_k)^\gamma\right]\\ &= \frac{C_1}{\gamma\Gamma(1-\alpha)}\cdot c_n(t_n)^\gamma,\end{aligned}$$

which is equivalent to

$$c_n^{-1}\sum_{\theta=1}^{n}(t_\theta)^{\gamma-\alpha}m_{n,\theta} \le \frac{C_1\Gamma(1+\gamma-\alpha)e^{\mu\left(t_\theta-\frac{t_k+t_{k+1}}{2}\right)}}{\gamma\Gamma(1-\alpha)}(t_n)^\gamma.$$

Then we have

$$d_{n,1}^{-1}\sum_{\theta=1}^{n}\theta^{r(\gamma-\alpha)}m_{n,\theta} \le \frac{C\Gamma(1+\gamma-\alpha)}{\gamma\Gamma(1-\alpha)}T^\alpha\left(\frac{t_n}{T}\right)^\gamma N^{r(\gamma-\alpha)}.$$

This completes the proof. □

In fact, according to [39], we can choose $\gamma = \frac{1}{\ln N}+\alpha-\frac{\beta}{r}$, where $\beta=\min\{2-\alpha, r\alpha\}$. Then the following useful conclusion can be derived by using $1 \le \theta^{\frac{r}{\ln N}} \le N^{\frac{r}{\ln N}} = e^r$ and Lemma 4.4. For simplicity, we omit the proof here.

Lemma 4.5. *Let the parameter $\beta=\min\{2-\alpha, r\alpha\}$. Assume that $r\le 2(2-\alpha)/\alpha$ and $N\ge 8$. Then for $n=1,2,\cdots,N$, one has*

(55)
$$d_{n,1}^{-1}\Gamma(1-\alpha)\sum_{\theta=1}^{n}\theta^{-\beta}m_{n,\theta} \le \frac{Ce^r\Gamma(1+1/(\ln N)-\beta/r)T^\alpha}{\left(\frac{1}{\ln N}+\alpha-\frac{\beta}{r}\right)}\left(\frac{t_n}{T}\right)^{1/(\ln N)+\alpha-\beta/r}N^{-\beta},$$

where C is some constant independent of t and as $\alpha\to 1$, C is bounded.

Remark 4.3. *The constant multiplier in* (55) *is α-robust, which can be seen from an inspection of the proof of Lemma* 4.4 *and the discussion in* [39].

Now we prove a sharp α-robust convergence result of the difference scheme (23). The error function is defined by $e_{ij}^n := u_{ij}^n - u(x_i,y_j,t_n)$ for all mesh points (x_i,y_j,t_n).

Theorem 4.2. *The solution u_{ij}^n of the scheme* (23) *satisfies*

$$\max_{(x_i,y_j,t_n)\in\bar{Q}}\left|e_{i,j}^n\right| \le C\left(h^2+T^\alpha N^{-\min\{2-\alpha,r\alpha\}}\right), \tag{56}$$

for some constant C not blowing up as $\alpha\to 1^-$.

Proof. The error function satisfies $L_{N_x,N_y}^N e_{ij}^n = q_{ij}^n$ by (48) for $i=0,1,\cdots,N_x-1$, $j=0,1,\cdots,N_y-1$, and $n=1,2,\cdots,N$, where L_{N_x,N_y}^N is defined in (23) and q_{ij}^n is the truncation error of the scheme. Note that $e_{0j}^n = e_{N_x,j}^n = e_{i0}^n = e_{i,N_y}^n = 0$ for

$0 \leq i \leq N_x$, $0 \leq j \leq N_y$ and $0 < n \leq N$, and $e_{ij}^0 = 0$ for all i and j. For fixed $(x_i, y_j, t_n) \in \bar{Q}$, by Eq. (24) and Theorem 3.1, one has

$$\left|q_{ij}^n\right| \leq C\left(h^2 + n^{-\min\{2-\alpha, r\alpha\}}\right).$$

Then from Lemma 4.1 and Lemma 4.5, we obtain

$$\begin{aligned}\max_{(x_i,y_j,t_n)\in\bar{Q}} \left|e_{i,j}^n\right| &\leq C d_{n,1}^{-1}\Gamma(1-\alpha)\sum_{\theta=1}^{n} m_{n,\theta}\|q^\theta\|_\infty \\ &\leq C d_{n,1}^{-1}\Gamma(1-\alpha)\sum_{\theta=1}^{n} m_{n,\theta}\left(h^2 + \theta^{-\min\{2-\alpha, r\alpha\}}\right) \\ &\leq C\left(h^2 + T^\alpha N^{-\min\{2-\alpha, r\alpha\}}\right).\end{aligned}$$

The constant C in this bound does not blow up as $\alpha \to 1^-$, because the constants C in the bounds of Theorem 3.1 and Lemma 4.5 are α-robust, which can be seen from an inspection of the proofs of these theorems and lemmas. □

Remark 4.4. *In fact, there is another important parameter μ in the tempered Caputo fractional derivative in this paper. However, the parameter μ doesn't affect the truncation error, the stability and convergence results of the obtained scheme, even if μ is very large, as long as it is bounded. This can also be seen from an inspection of the proofs of these theorems and lemmas above.*

5. Numerical results

In this section, we present some numerical experiments to validate the predicted temporal convergence rate of the numerical scheme. In the computational process, since all the previous outputs are needed in the current time level, we use two-dimensional array. That is, we use a matrix of order $\{(N_x+1)(N_y+1)\} \times (N+1)$ in the program. There are $(N_x+1)(N_y+1)$ elements for the spatial variables including all the boundary points per time level and the number $N+1$ corresponds to the total time levels. Due to the unknown exact solution, the temporal errors can be tested by

$$E_n = \max_{0\leq n\leq N} \max_{\substack{0\leq i\leq N_x \\ 0\leq j\leq N_y}} \left|u_{ij}^{n/2} - u_{ij}^n\right|,$$

where u_{ij}^n are the numerical solutions of $u(x_i, y_j)$ at the fixed time t_n. The order of the convergence of the numerical results are calculated by the two-mesh principle

$$Rate = \frac{\ln(E_n/E_{2n})}{\ln 2}.$$

Example 5.1: In this example, we consider the boundary value problem (2) on a finite rectangular domain $(x, y) \in (0, \pi) \times (0, \pi)$ for $t \in (0, 1]$ with the forcing function $f(x, y, t) = 0$. Also the initial condition is $u(x, y, 0) = \sin(x)\sin(y)$ with the zero boundary conditions. Here we take $\alpha = 0.4, 0.6, 0.8$, $N = 2^6, 2^7, 2^8, 2^9, 2^{10}$. We use small spatial mesh sizes $h_x = \pi/2^{10}$ and $h_y = \pi/2^{10}$ to guarantee that the spatial discretization error is relatively negligible. Table 1, Table 2, and Table 3 present the L^∞ errors and convergence rates for $\mu = 1$, $\mu = 0.01$, and $\mu = 10$, respectively, which verify the predictions of Theorem 4.2 and Remark 4.4. In Example 5.1, we take the initial data $u(x, y, 0) = \sin(x)\sin(y)$ to be sufficiently smooth, then the numerical results agree precisely with the theoretical rate of convergence of Theorem 4.2 for various values of α and μ. But typical solutions of (2)

Table 1. Maximum errors and convergence orders of scheme (23) for Example 5.1 with $\mu = 1$.

l_∞ norm	τ	$r = 1$	Rate	$r = \frac{2-\alpha}{\alpha}$	Rate
	1/64	0.0339		9.21e-04	
	1/128	0.0270	0.33	3.24e-04	1.51
$\alpha = 0.4$	1/256	0.0213	0.34	1.12e-04	1.53
	1/512	0.0166	0.36	3.82e-05	1.55
	1/1024	0.0129	0.37	1.30e-05	1.56
	1/64	0.0153		0.0016	
	1/128	0.0102	0.58	6.61e-04	1.31
$\alpha = 0.6$	1/256	0.0068	0.59	2.61e-04	1.34
	1/512	0.0045	0.60	1.02e-04	1.36
	1/1024	0.0030	0.60	3.94e-05	1.37
	1/64	0.0056		0.0024	
	1/128	0.0032	0.80	0.0011	1.08
$\alpha = 0.8$	1/256	0.0019	0.81	5.26e-04	1.11
	1/512	0.0011	0.81	2.41e-04	1.13
	1/1024	6.07e-04	0.81	1.09e-04	1.14

Table 2. Maximum errors and convergence orders of scheme (23) for Example 5.1 with $\mu = 0.01$.

l_∞ norm	τ	$r = 1$	Rate	$r = \frac{2-\alpha}{\alpha}$	Rate
	1/64	0.0305		8.97e-04	
	1/128	0.0250	0.30	3.18e-04	1.50
$\alpha = 0.4$	1/256	0.0201	0.31	1.10e-04	1.53
	1/512	0.0159	0.33	3.78e-05	1.54
	1/1024	0.0125	0.35	1.29e-05	1.55
	1/64	0.0150		0.0016	
	1/128	0.0102	0.57	6.52e-04	1.30
$\alpha = 0.6$	1/256	0.0068	0.58	2.59e-04	1.33
	1/512	0.0045	0.59	1.01e-04	1.35
	1/1024	0.0030	0.59	3.92e-05	1.37
	1/64	0.0057		0.0024	
	1/128	0.0033	0.81	0.0011	1.09
$\alpha = 0.8$	1/256	0.0019	0.81	5.30e-04	1.11
	1/512	0.0011	0.81	2.42e-04	1.13
	1/1024	6.09e-04	0.81	1.10e-04	1.14

are nonsmooth, as we can see in Section 2. Thus we now test the finite difference scheme (23) on Example 5.2 to see how it performs for the initial data with slight smoothness, i.e., $u_0 \in \dot{H}^1(\Omega)$.

Example 5.2: In this example, we consider (2) on a finite rectangular domain $(x, y) \in (0, \pi) \times (0, \pi)$ for $t \in (0, 1]$ and $\mu = 1$ with the forcing function $f(x, y, t) = 0$. The initial condition is $u(x, y, 0) = \sin(x)^{\frac{3}{4}} \sin(y)^{\frac{3}{4}}$ with the zero boundary conditions. Here we take $\alpha = 0.4, 0.6, 0.8$, $N = 2^6, 2^7, 2^8, 2^9, 2^{10}$. We use small spatial

TABLE 3. Maximum errors and convergence orders of scheme (23) for Example 5.1 with $\mu = 10$.

l_∞ norm	τ	$r=1$	Rate	$r=\frac{2-\alpha}{\alpha}$	Rate
	1/64	0.0432		0.0167	
	1/128	0.0322	0.43	0.0057	1.55
$\alpha = 0.4$	1/256	0.0241	0.42	0.0020	1.55
	1/512	0.0182	0.41	6.63e-04	1.56
	1/1024	0.0138	0.40	2.23e-04	1.57
	1/64	0.0154		0.0114	
	1/128	0.0104	0.57	0.0044	1.37
$\alpha = 0.6$	1/256	0.0069	0.59	0.0017	1.37
	1/512	0.0046	0.60	6.54e-04	1.38
	1/1024	0.0030	0.60	2.50e-04	1.39
	1/64	0.0046		0.0073	
	1/128	0.0028	0.70	0.0031	1.24
$\alpha = 0.8$	1/256	0.0017	0.70	0.0013	1.23
	1/512	0.0010	0.75	5.64e-04	1.23
	1/1024	5.98e-04	0.78	2.42e-04	1.22

TABLE 4. Maximum errors and convergence orders of scheme (23) for Example 5.2.

l_∞ norm	τ	$r=1$	Rate	$r=\frac{2-\alpha}{\alpha}$	Rate
	1/64	0.0311		0.0010	
	1/128	0.0228	0.38	6.65e-04	0.66
$\alpha = 0.4$	1/256	0.0181	0.40	4.33e-04	0.62
	1/512	0.0137	0.40	2.83e-04	0.61
	1/1024	0.0104	0.40	1.86e-04	0.61
	1/64	0.0119		0.0018	
	1/128	0.0078	0.61	0.0012	0.55
$\alpha = 0.6$	1/256	0.0058	0.43	8.28e-04	0.53
	1/512	0.0045	0.35	5.74e-04	0.53
	1/1024	0.0036	0.31	3.90e-04	0.56
	1/64	0.0053		0.0025	
	1/128	0.0040	0.39	0.0018	0.48
$\alpha = 0.8$	1/256	0.0032	0.35	0.0013	0.46
	1/512	0.0025	0.33	9.62e-04	0.45
	1/1024	0.0020	0.32	6.98e-04	0.46

mesh sizes $h_x = \pi/2^{10}$ and $h_y = \pi/2^{10}$ to guarantee that the spatial discretization error is relatively negligible. Table 4 presents the L^∞ errors and convergence rates, which can not achieve good stability and high convergence order. This inspires us to try to design more efficient numerical algorithms for (2) with slightly smoother data in the future.

Declarations

Data availability statement

The data that support the findings of this study are openly available at the following URL/DOI: https://github.com/WangCanLZU/GM.

Conflict of interest statement

We declare that we have no conflict of interest.

Ethical Approval

Not Applicable.

Authors contributions

C.W. wrote the main manuscript text and implement the algorithm; all authors discussed the analysis, the results, and revised the manuscript.

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Appendix A. The equivalent form of (1)

For a real-valued function f defined on $[0, \infty)$, we use $\widehat{f}$ to denote its Laplace transform:

$$\widehat{f}(\lambda) := \mathscr{L}\left\{f(t)\right\}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\,\mathrm{d}t,\ \ \lambda > 0,$$

whenever the integral is absolutely convergent.

Now taking the Laplace transform on both sides of Eq. (1) with respect to the time variable t, one has

$$\lambda \widehat{u}(x, y, \lambda) - u(x, y, 0) = \frac{\lambda}{(\lambda + \mu)^\alpha - \mu^\alpha} \Delta \widehat{u}(x, y, \lambda),$$

which is equivalent to

$$\lambda \frac{(\lambda + \mu)^\alpha - \mu^\alpha}{\lambda} \widehat{u}(x, y, \lambda) - \frac{(\lambda + \mu)^\alpha - \mu^\alpha}{\lambda} u(x, y, 0) = \Delta \widehat{u}(x, y, \lambda).$$

Then taking the inverse Laplace transform on both sides of the above equation results in

$$\partial_t^{\alpha,\mu} u(x, y, t) = \Delta u(x, y, t), \tag{A.1}$$

where

$$\partial_t^{\alpha,\mu} u(x, y, t) = \int_0^t w_\mu^\alpha(t - s) \partial_s u(x, y, s)\,\mathrm{d}s$$

with $\widehat{w}_\mu^\alpha(\lambda) = \frac{(\lambda+\mu)^\alpha - \mu^\alpha}{\lambda}$ being the Laplace transform of w_μ^α with respect to the time variable t.

Appendix B. The proof of (54)

Taking the Laplace transform of the left side of Eq. (54) with respect to the time variable t and using the integral relation $\int_0^{+\infty} e^{-t} E_{\alpha,\beta}(zt^\alpha)t^{\beta-1}\,\mathrm{d}t = \frac{1}{1-z}$ ($\beta > 0, |z| < 1$) (cf. [34, Eq. (4.4.11), p. 62]), we arrive at

$$
\begin{aligned}
&\mathscr{L}\left\{\partial_t^{\alpha,\mu} e^{-\mu t} t^\gamma E_{\alpha,\gamma+1}(\mu^\alpha t^\alpha)\right\}(\lambda) \\
&= [(\lambda+\mu)^\alpha - \mu^\alpha]\,\mathscr{L}\left\{e^{-\mu t} t^\gamma E_{\alpha,\gamma+1}(\mu^\alpha t^\alpha)\right\}(\lambda) \\
&= [(\lambda+\mu)^\alpha - \mu^\alpha]\int_0^{+\infty} e^{-(\mu+\lambda)t} t^\gamma E_{\alpha,\gamma+1}(\mu^\alpha t^\alpha)\,\mathrm{d}t \\
&= \frac{(\lambda+\mu)^\alpha - \mu^\alpha}{(\mu+\lambda)^{\gamma+1}} \\
&\qquad \cdot \int_0^{+\infty} e^{-(\mu+\lambda)t}[(\mu+\lambda)t]^\gamma E_{\alpha,\gamma+1}\left(\mu^\alpha t^\alpha\right)\,\mathrm{d}\left[(\mu+\lambda)t\right] \\
&= \frac{(\lambda+\mu)^\alpha - \mu^\alpha}{(\mu+\lambda)^{\gamma+1}} \cdot \frac{1}{1 - \frac{\mu^\alpha}{(\mu+\lambda)^\alpha}} \\
&= (\mu+\lambda)^{\alpha-\gamma-1};
\end{aligned}
$$

similarly taking the Laplace transform of the right side of Eq. (54) with respect to the time variable t, we obtain

$$
\begin{aligned}
&\mathscr{L}\left\{\frac{e^{-\mu t} t^{\gamma-\alpha}}{\Gamma(1+\gamma-\alpha)}\right\}(\lambda) \\
&= \frac{1}{\Gamma(1+\gamma-\alpha)}\mathscr{L}\left\{e^{-\mu t} t^{\gamma-\alpha}\right\}(\lambda) \\
&= \frac{1}{\Gamma(1+\gamma-\alpha)}\int_0^{+\infty} e^{-(\mu+\lambda)t} t^{\gamma-\alpha}\,\mathrm{d}t \\
&= \frac{(\mu+\lambda)^{\alpha-\gamma-1}}{\Gamma(1+\gamma-\alpha)}\int_0^{+\infty} e^{-(\mu+\lambda)t}[(\mu+\lambda)t]^{\gamma-\alpha}\,\mathrm{d}\left[(\mu+\lambda)t\right] \\
&= \frac{(\mu+\lambda)^{\alpha-\gamma-1}}{\Gamma(1+\gamma-\alpha)}\Gamma(\gamma-\alpha+1) \\
&= (\mu+\lambda)^{\alpha-\gamma-1}.
\end{aligned}
$$

Then the following equality holds

$$
\mathscr{L}\left\{\partial_t^{\alpha,\mu} e^{-\mu t} t^\gamma E_{\alpha,\gamma+1}(\mu^\alpha t^\alpha)\right\}(\lambda) = \mathscr{L}\left\{\frac{e^{-\mu t} t^{\gamma-\alpha}}{\Gamma(1+\gamma-\alpha)}\right\}(\lambda),
$$

the inverse Laplace transform of which results in

$$
\partial_t^{\alpha,\mu} e^{-\mu t} t^\gamma E_{\alpha,\gamma+1}(\mu^\alpha t^\alpha) = \frac{e^{-\mu t} t^{\gamma-\alpha}}{\Gamma(1+\gamma-\alpha)}.
$$

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