INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 4, Pages 557–576

doi: 10.4208/ijnam2023-1024

A PENALTY FINITE ELEMENT METHOD FOR THE STATIONARY CLOSED-LOOP GEOTHERMAL MODEL

HAOCHEN LIU, PENGZHAN HUANG*, AND YINNIAN HE

Abstract. In this article, we give a penalty finite element method for the steady-state closed-loop geothermal model. Firstly, we construct the stationary penalty closed-loop geothermal equations. Secondly, we propose a finite element method for the penalty system and deduce error estimates. Finally, some numerical experiments are used to illustrate the theoretical results of the presented method.

Key words. Closed-loop geothermal model, penalty finite element method, error estimate, numerical test.

1. Introduction

In this paper, we propose and study a penalty finite element method for a steady-state closed-loop geothermal model. The governing equations of this model include the Navier-Stokes/Darcy equations and heat equations [20, 21].

Let $\Omega \subset \mathbb{R}^2$ consist of two subdomains Ω_f and Ω_p with Lipschitz continuous boundaries $\partial\Omega_f$ and $\partial\Omega_p$, separated by the interface Γ. The Navier-Stokes equations coupled with the heat equation (velocity vector u_f, pressure p_f and temperature θ_f) describe fluid flow in the fluid domain Ω_f:

$$-\nu\Delta u_f + (u_f \cdot \nabla)\, u_f + \nabla p_f = G_r \nu^2 \theta_f \xi \qquad \text{in } \Omega_f, \tag{1}$$

$$\nabla \cdot u_f = 0 \qquad \text{in } \Omega_f, \tag{2}$$

$$-\alpha_f \Delta\theta_f + u_f \cdot \nabla\theta_f = g_f \qquad \text{in } \Omega_f. \tag{3}$$

Besides, the Darcy equations coupled with the heat equation (velocity vector u_p, pressure p_p and temperature θ_p) describe Darcy flow in the porous media domain Ω_p:

$$\frac{\nu}{D_a} u_p + \nabla p_p = G_r \nu^2 \theta_p \xi \qquad \text{in } \Omega_p, \tag{4}$$

$$\nabla \cdot u_p = 0 \qquad \text{in } \Omega_p, \tag{5}$$

$$-\alpha_p \Delta\theta_p + u_p \cdot \nabla\theta_p = g_p \qquad \text{in } \Omega_p, \tag{6}$$

where ν is the kinetic viscosity and G_r is the Grashof number, and $\xi = (0, -1)^T$ is the unit vector in the direction of the gravitational acceleration. In addition, D_a is the Darcy number, assuming the porous media is isotropic and homogeneous. α_f and α_p refer to the thermal diffusivity in the fluid and porous media domains, respectively. g_f and g_p are heat sources in the fluid and porous media domains.

Received by the editors on October 5, 2022 and, accepted on April 14, 2023.
2000 *Mathematics Subject Classification.* 65N30, 76M10, 76N10.
*Corresponding author.

Additionally, the problem (1)-(6) is considered in conjunction with the following boundary conditions on $\partial\Omega_f$ and $\partial\Omega_p$

$$u_f = 0 \text{ on } \partial\Omega_f\backslash\Gamma, \quad \theta_f = 0 \text{ on } \Gamma_{fD}, \quad \frac{\partial\theta_f}{\partial n_f} = 0 \text{ on } \Gamma_{fN},$$
$$u_p\cdot n_p = 0 \text{ on } \partial\Omega_p\backslash\Gamma, \quad \theta_p = 0 \text{ on } \Gamma_{pD}, \quad \frac{\partial\theta_p}{\partial n_p} = 0 \text{ on } \Gamma_{pN},$$

where Γ_{fD} and Γ_{fN} are the pipe region boundaries with $\partial\Omega_f\backslash\Gamma = \Gamma_{fN}\cup\Gamma_{fD}$ and denote the Dirichlet and Neumann boundary conditions, respectively. Γ_{pD} and Γ_{pN} are the porous media region boundaries with $\partial\Omega_p\backslash\Gamma = \Gamma_{pN}\cup\Gamma_{pD}$. The unit outward normal vectors satisfy the condition of $n_p = -n_f$ on the interface Γ. Furthermore, for the closed-loop geothermal model, in order to describe heat exchanging and no-fluid communication on the interface Γ, we utilize several critical interface conditions as follows [18]:

$$\theta_f = \theta_p, \qquad \text{(Continuity of temperature)}, \tag{7}$$
$$\alpha_f\frac{\partial\theta_f}{\partial n_f} + \alpha_p\frac{\partial\theta_p}{\partial n_p} = 0, \qquad \text{(Continuity of heat flux)}, \tag{8}$$
$$u_p\cdot n_p = 0, \quad u_f\cdot n_f = 0, \qquad \text{(No-communication condition)}, \tag{9}$$
$$u_f\cdot\tau = 0, \qquad \text{(No-slip condition)}, \tag{10}$$

where τ is the unit tangential vector along Γ.

Numerically solving the governing problem remains challenging because it has multiple physical quantities and domain couplings. In [24], Valencia-López et al. propose some finite element methods to study Navier-Stoke/Darcy equations coupled with heat equations, where the Navier-Stokes/Darcy equations are coupled with the Beavers-Joseph interface conditions. In [26], Zhang et al. consider the well-posedness and numerical scheme for the natural convection in a composite fluid layer overlying a porous media layer with internal heat generation. In addition, Mahbub et al. [18] use an unsteady-state model with no-communication conditions on the interface for the closed-loop geothermal system and design a decoupled stabilized finite element approach. A decoupled iterative finite element method and a two-grid finite element method [15, 14] are proposed and analyzed for the steady-state case.

Since the Darcy velocity u_p in (4) has low regularity ($u_p \in L^2(\Omega_p)^2$), it is not easy to prove the existence of a solution to the problem (1)-(10) and have a challenge in the numerical computations. As is known, the penalty method [4, 22, 23, 19, 9, 11, 12, 13, 17] is a practical algorithm for fluid flow problems. In this article, we propose and study a penalty finite element method for the steady-state closed-loop geothermal model. Firstly, we construct the stationary penalty closed-loop geothermal equations and prove the existence of the weak solution to the penalty system, which is easier to obtain than that of the original system. Then we get error estimates of the weak solutions to the penalty and the original system. Secondly, we propose a finite element method for the penalty system and deduce the convergence of the finite element discretization. Finally, since the finite element system is nonlinear, we linearize the nonlinear problem and deduce the iterative error.

Now, we use the penalty method for the original equations of the closed-loop geothermal problem. The penalty method applied to (1)-(6) is to approximate the solution ($\boldsymbol{u} = (u_f, u_p), \boldsymbol{p} = (p_f, p_p), \boldsymbol{\theta} = (\theta_f, \theta_p)$) by ($\boldsymbol{u}^\varepsilon = (u_f^\varepsilon, u_p^\varepsilon), \boldsymbol{p}^\varepsilon =$

$(p_f^\varepsilon, p_p^\varepsilon), \boldsymbol{\theta}^\varepsilon = (\theta_f^\varepsilon, \theta_p^\varepsilon))$ satisfying the following penalty closed-loop geothermal equations:

$$-\nu\Delta u_f^\varepsilon + \left(u_f^\varepsilon \cdot \nabla\right) u_f^\varepsilon + \nabla p_f^\varepsilon = G_r \nu^2 \theta_f^\varepsilon \xi \quad \text{in } \Omega_f, \tag{11}$$

$$\nabla \cdot u_f^\varepsilon = 0 \quad \text{in } \Omega_f, \tag{12}$$

$$-\alpha_f \Delta\theta_f^\varepsilon + \left(u_f^\varepsilon \cdot \nabla\right) \theta_f^\varepsilon = g_f \quad \text{in } \Omega_f, \tag{13}$$

$$-\varepsilon\Delta u_p^\varepsilon + \frac{\nu}{D_a} u_p^\varepsilon + \nabla p_p^\varepsilon = G_r \nu^2 \theta_p^\varepsilon \xi \quad \text{in } \Omega_p, \tag{14}$$

$$\nabla \cdot u_p^\varepsilon = 0 \quad \text{in } \Omega_p, \tag{15}$$

$$-\alpha_p \Delta\theta_p^\varepsilon + (u_p^\varepsilon \cdot \nabla)\theta_p^\varepsilon = g_p \quad \text{in } \Omega_p, \tag{16}$$

where the term $-\varepsilon\Delta u_p^\varepsilon$ in (14) is the penalty term and $1 > \varepsilon > 0$ is the penalty parameter.

In addition, we need the following boundary conditions:

$$u_f^\varepsilon = 0 \quad \text{on } \partial\Omega_f \backslash \Gamma, \quad \theta_f^\varepsilon = 0 \quad \text{on } \Gamma_{fD}, \quad \frac{\partial \theta_f^\varepsilon}{\partial n_f} = 0 \quad \text{on } \Gamma_{fN},$$

$$u_p^\varepsilon \cdot n_p = 0 \quad \text{on } \partial\Omega_p \backslash \Gamma, \quad \theta_p^\varepsilon = 0 \quad \text{on } \Gamma_{pD}, \quad \frac{\partial \theta_p^\varepsilon}{\partial n_p} = 0 \quad \text{on } \Gamma_{pN},$$

and the following conditions on the interface Γ:

$$\theta_f^\varepsilon = \theta_p^\varepsilon, \quad \alpha_f \frac{\partial \theta_f^\varepsilon}{\partial n_f} + \alpha_p \frac{\partial \theta_p^\varepsilon}{\partial n_p} = 0, \tag{17}$$

$$u_f^\varepsilon \cdot n_f = 0, \quad u_f^\varepsilon \cdot \tau = 0, \quad u_p^\varepsilon \cdot n_p = 0. \tag{18}$$

We notice that the Darcy velocity u_p^ε in the penalty closed-loop geothermal equations has high regularity ($u_p^\varepsilon \in H^2(\Omega_p)^2$), which makes it much easier to obtain the existence of solution than that of the solution to the original problem (1)-(10).

2. Preliminaries

In this section, we introduce some notations, function spaces, and results used in this paper. For $1 \leq q \leq \infty$ and $k \in \mathbb{N}^+$, we denote the Lebesgue space by $L^q(D)$ and a special Sobolev space by $H^k(D)$ [1], where D may be Ω_f, Ω_p. We denote the inner product and the norm on $L^2(D)$ or $L^2(D)^2$ by $(\cdot,\cdot)_D$ and $\|\cdot\|_{L^2(D)}$, respectively. Then, we also denote the norms of the spaces $L^q(D)$ and $H^k(D)$ by $\|\cdot\|_{L^q(D)}$ and $\|\cdot\|_{H^k(D)}$, respectively. To denote briefly:

$$\|\cdot\|_k = \|\cdot\|_{H^k(D)}, \quad \|\cdot\|_0 = \|\cdot\|_{L^2(D)}, \quad \|\cdot\|_\Gamma = \|\cdot\|_{L^2(\Gamma)}.$$

Moreover, we define the following function spaces:

$$X_f := \left\{ v_f \in H^1(\Omega_f)^2 : v_f = 0 \text{ on } \partial\Omega_f \backslash \Gamma \right\},$$

$$X_p := \left\{ v_p \in L^2(\Omega_p)^2, \nabla \cdot v_p \in L^2(\Omega_p) : v_p \cdot n_p = 0 \text{ on } \partial\Omega_p \backslash \Gamma \right\},$$

$$Y_f := \left\{ q_f \in L^2(\Omega_f) : (q_f, 1)_{\Omega_f} = 0 \right\}, \quad Y_p := \left\{ q_p \in L^2(\Omega_p) : (q_p, 1)_{\Omega_p} = 0 \right\},$$

$$W_f := \left\{ \omega_f \in H^1(\Omega_f) : \omega_f = 0 \text{ on } \Gamma_{fD} \right\}, \; W_p := \left\{ \omega_p \in H^1(\Omega_p) : \omega_p = 0 \text{ on } \Gamma_{pD} \right\}.$$

Besides, we define some product spaces:

$$X := X_f \times X_p, \quad Y := Y_f \times Y_p, \quad W := W_f \times W_p,$$

and

$$W_\Gamma := \left\{ \boldsymbol{\omega} = (\omega_f, \omega_p) \in W_f \times W_p : \omega_f|_\Gamma = \omega_p|_\Gamma \right\}.$$

Furthermore, we define $H^{-1}(D)$ as the dual space of $H_0^1(D)$ and its norm is defined by

$$\|g\|_{-1} = \sup_{\omega \in H_0^1(D)} \frac{|(g,\omega)_D|}{\|\nabla \omega\|_0}.$$

Additionally, we recall the Poincaré inequality [1] as follows: for $u_f \in X_f$,

$$\|u_f\|_0 \le C_p \|\nabla u_f\|_0, \tag{19}$$

with a constant C_p that only depends on Ω.

Now, we introduce the continuous trilinear form

$$\begin{aligned} c_f(u_f, v_f, w_f)_{\Omega_f} &= ((u_f \cdot \nabla) v_f, w_f)_{\Omega_f} + 0.5((\nabla \cdot u_f) v_f, w_f)_{\Omega_f} \\ &= 0.5((u \cdot \nabla) v_f, w_f)_{\Omega_f} - 0.5((u_f \cdot \nabla) w_f, v_f)_{\Omega_f} \quad \forall u_f, v_f, w_f \in X_f. \end{aligned}$$

Similarly, we define another two trilinear forms for any $\boldsymbol{\omega} \in W_\Gamma$ and $\boldsymbol{\theta} \in W_\Gamma$:

$$\tilde{c}_f \left(u_f, \theta_f, \omega_f\right)_{\Omega_f} = 0.5 \left(u_f \cdot \nabla \theta_f, \omega_f\right)_{\Omega_f} - 0.5 \left(u_f \cdot \nabla \omega_f, \theta_f\right)_{\Omega_f} \quad \forall u_f \in X_f, \tag{20}$$

$$\tilde{c}_p \left(u_p, \theta_p, \omega_p\right)_{\Omega_p} = 0.5 \left(u_p \cdot \nabla \theta_p, \omega_p\right)_{\Omega_p} - 0.5 \left(u_p \cdot \nabla \omega_p, \theta_p\right)_{\Omega_p} \quad \forall u_p \in X_p. \tag{21}$$

Next, denote $\boldsymbol{v} = (v_f, v_p)$ and $\boldsymbol{q} = (q_f, q_p)$. With the above notations, we obtain the variational formulation of (1)-(10): find $(\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\theta}) \in X \times Y \times W_\Gamma$ such that for any $(\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\omega}) \in X \times Y \times W_\Gamma$

(22)

$$\nu \left(\nabla u_f, \nabla v_f\right)_{\Omega_f} + c_f(u_f, u_f, v_f)_{\Omega_f} - (p_f, \nabla \cdot v_f)_{\Omega_f} + (\nabla \cdot u_f, q_f)_{\Omega_f} = G_r \nu^2 \left(\theta_f \xi, v_f\right)_{\Omega_f},$$

$$\frac{\nu}{D_a} \left(u_p, v_p\right)_{\Omega_p} - (p_p, \nabla \cdot v_p)_{\Omega_p} + (\nabla \cdot u_p, q_p)_{\Omega_p} = G_r \nu^2 \left(\theta_p \xi, v_p\right)_{\Omega_p}, \tag{23}$$

(24)

$$\begin{aligned} &\alpha_f \left(\nabla \theta_f, \nabla \omega_f\right)_{\Omega_f} + \alpha_p \left(\nabla \theta_p, \nabla \omega_p\right)_{\Omega_p} + \tilde{c}_f(u_f, \theta_f, \omega_f)_{\Omega_f} + \tilde{c}_p \left(u_p, \theta_p, \omega_p\right)_{\Omega_p} \\ &- \alpha_f \int_\Gamma n_f \cdot \nabla \theta_f (\omega_f - \omega_p) + \frac{\alpha_f \gamma}{h} \int_\Gamma (\theta_f - \theta_p)(\omega_f - \omega_p) = \left(g_f, \omega_f\right)_{\Omega_f} + \left(g_p, \omega_p\right)_{\Omega_p}, \end{aligned}$$

where $\gamma > 0$ is a stabilization parameter which is independent of h, and h is mesh size defined in the next section. Due to the low regularity ($u_p \in L^2(\Omega_p)^2$) of the Darcy velocity, it is not easier to obtain the existence of the solution.

3. Penalty problem of the closed-loop geothermal system

In this section, we show the variational formulation of the penalty closed-loop geothermal problem (11)-(18), prove the existence of the solution to the penalty system and show error bounds between the weak solutions $(\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\theta})$ and $(\boldsymbol{u}^\varepsilon, \boldsymbol{p}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$.

Next, we need the function space for the penalty system:

$$X_p^\varepsilon := \left\{ v_p^\varepsilon \in H^1\left(\Omega_p\right)^2 : v_p^\varepsilon \cdot n_p = 0 \text{ on } \Gamma_p \right\}, \quad X^\varepsilon := X_f \times X_p^\varepsilon.$$

Then we obtain the variational formulation of the penalty system (11)-(18), find $(\boldsymbol{u}^\varepsilon, \boldsymbol{p}^\varepsilon, \boldsymbol{\theta}^\varepsilon) \in X^\varepsilon \times Y \times W_\Gamma$ satisfying:

(25)

$$\nu \left(\nabla u_f^\varepsilon, \nabla v_f\right)_{\Omega_f} + c_f \left(u_f^\varepsilon, u_f^\varepsilon, v_f\right)_{\Omega_f} - (p_f^\varepsilon, \nabla \cdot v_f)_{\Omega_f} + (\nabla \cdot u_f^\varepsilon, q_f)_{\Omega_f} = G_r \nu^2 \left(\theta_f^\varepsilon \xi, v_f\right)_{\Omega_f},$$

(26)

$$\varepsilon \left(\nabla u_p^\varepsilon, \nabla v_p\right)_{\Omega_p} + \frac{\nu}{D_a} \left(u_p^\varepsilon, v_p\right)_{\Omega_p} - (p_p^\varepsilon, \nabla \cdot v_p)_{\Omega_p} + (\nabla \cdot u_p^\varepsilon, q_p)_{\Omega_p} = G_r \nu^2 \left(\theta_p^\varepsilon \xi, v_p\right)_{\Omega_p},$$

(27)
$$\alpha_f\left(\nabla\theta_f^\varepsilon,\nabla\omega_f\right)_{\Omega_f}+\alpha_p\left(\nabla\theta_p^\varepsilon,\nabla\omega_p\right)_{\Omega_p}+\tilde c_f\left(u_f^\varepsilon,\theta_f^\varepsilon,\omega_f\right)_{\Omega_f}+\tilde c_p\left(u_p^\varepsilon,\theta_p^\varepsilon,\omega_p\right)_{\Omega_p}$$
$$-\alpha_f\int_\Gamma n_f\cdot\nabla\theta_f^\varepsilon(\omega_f-\omega_p)+\frac{\alpha_f\gamma}{h}\int_\Gamma(\theta_f^\varepsilon-\theta_p^\varepsilon)(\omega_f-\omega_p)=(g_f,\omega_f)_{\Omega_f}+(g_p,\omega_p)_{\Omega_p},$$

for all $(\boldsymbol{v},\boldsymbol{q},\boldsymbol{\omega})\in X^\varepsilon\times Y\times W_\Gamma$.

Besides, it is easy to verify that these trilinear forms satisfy the following property.

Lemma 3.1. [6, 27] *The trilinear forms $c_f(\cdot,\cdot,\cdot)_{\Omega_f}$ and $\tilde c_i(\cdot,\cdot,\cdot)_{\Omega_i}$, $i=f$ or p satisfy:*

$$\begin{aligned}&|c_f(u,v,w)_{\Omega_f}|\le N\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0,\quad\forall u,v,w\in X^\varepsilon,\\&|\tilde c_i(u_i,\theta_i,\omega_i)_{\Omega_i}|\le\tilde N\|\nabla u_i\|_0\|\nabla\theta_i\|_0\|\nabla\omega_i\|_0,\quad\forall u_i\in X^\varepsilon,\theta_i,w_i\in W,\end{aligned}\tag{28}$$

where N and $\tilde N$ denote the positive constants depending only on the domain.

The existence and uniqueness of the solution to (25)-(27) are shown in the following theorem.

Theorem 3.1. *If the following conditions hold*

$$\begin{aligned}&0<2NG_rC_p^2\mathcal{S}_\theta+G_r^2\nu^2C_p^2\max\{C_p^2\alpha_f^{-1},D_a\alpha_p^{-1}\}\tilde N^2\alpha_f^{-1}\mathcal{S}_\theta^2<1,\\&0<D_aG_r^2\nu^2C_p^2\max\{C_p^2\alpha_f^{-1},D_a\alpha_p^{-1}\}\tilde N^2\alpha_p^{-1}\|\nabla\theta_p^\varepsilon\|^2_{L^3(\Omega_p)}<1,\end{aligned}\tag{29}$$

then the problem (25)-(27) admits a unique solution $(\boldsymbol{u}^\varepsilon,\boldsymbol{p}^\varepsilon,\boldsymbol{\theta}^\varepsilon)\in X^\varepsilon\times Y\times W_\Gamma$ such that

$$\begin{aligned}&\|\nabla u_f^\varepsilon\|_0\le G_rC_p{}^2\nu\mathcal{S}_\theta,\quad\varepsilon\left\|\nabla u_p^\varepsilon\right\|_0\le G_rC_p^2\nu^2\mathcal{S}_\theta,\quad\|u_p^\varepsilon\|_0\le D_aG_r\nu C_p\mathcal{S}_\theta,\\&\|\nabla\theta_f^\varepsilon\|_0^2+\left\|\nabla\theta_p^\varepsilon\right\|_0^2\le\mathcal{S}_\theta^2,\end{aligned}\tag{30}$$

where $\mathcal{S}_\theta=\frac{1}{\min\{\alpha_f,\alpha_p\}}\|g(x)\|_{-1}$ with $g(x)|_{\Omega_f}=g_f(x)$ and $g(x)|_{\Omega_p}=g_p(x)$.

Proof. As [26], applying Brouwer's fixed point theorem, the existence of the solution can be easily obtained due to the high regularity ($u_p^\varepsilon\in H^2(\Omega_p)^2$) of the Darcy velocity.

Now, we mainly prove the uniqueness of the solution. Assume that there exist two solutions $(\tilde{\boldsymbol{u}}^\varepsilon,\tilde{\boldsymbol{p}}^\varepsilon,\tilde{\boldsymbol{\theta}}^\varepsilon)$ and $(\hat{\boldsymbol{u}}^\varepsilon,\hat{\boldsymbol{p}}^\varepsilon,\hat{\boldsymbol{\theta}}^\varepsilon)$ to the problem (25)-(27). We define $e_{\boldsymbol{u}}=\tilde{\boldsymbol{u}}^\varepsilon-\hat{\boldsymbol{u}}^\varepsilon$, $e_{\boldsymbol{p}}=\tilde{\boldsymbol{p}}^\varepsilon-\hat{\boldsymbol{p}}^\varepsilon$ and $e_{\boldsymbol{\theta}}=\tilde{\boldsymbol{\theta}}^\varepsilon-\hat{\boldsymbol{\theta}}^\varepsilon$, and choose test functions $\boldsymbol{v}=e_{\boldsymbol{u}},\boldsymbol{q}=e_{\boldsymbol{p}}$ and $\boldsymbol{\omega}=e_{\boldsymbol{\theta}}$.

(31)
$$\nu\|\nabla e_{u_f}\|_0^2+c_f\left(e_{u_f},\tilde u_f^\varepsilon,e_{u_f}\right)_{\Omega_f}+c_f\left(u_f^\varepsilon,e_{u_f},e_{u_f}\right)_{\Omega_f}=G_r\nu^2\left(e_{\theta_f}\xi,e_{u_f}\right)_{\Omega_f},$$

$$\varepsilon\|\nabla e_{u_p}\|_0^2+\frac{\nu}{D_a}\|e_{u_p}\|_0^2=G_r\nu^2\left(e_{\theta_p}\xi,e_{u_p}\right)_{\Omega_p},\tag{32}$$

$$\begin{aligned}&\alpha_f\|\nabla e_{\theta_f}\|_0^2+\alpha_p\|\nabla e_{\theta_p}\|_0^2+\tilde c_f\left(e_{u_f},\tilde\theta_f^\varepsilon,e_{\theta_f}\right)_{\Omega_f}+\tilde c_f\left(\hat u_f^\varepsilon,e_{\theta_f},e_{\theta_f}\right)_{\Omega_f}\\&+\tilde c_p\left(e_{u_p},\tilde\theta_p^\varepsilon,e_{\theta_p}\right)_{\Omega_p}+\tilde c_p\left(\hat u_p^\varepsilon,e_{\theta_p},e_{\theta_p}\right)_{\Omega_p}=0.\end{aligned}\tag{33}$$

Note that the interface term $-\alpha_f\int_\Gamma n_f\cdot\nabla\theta_f^\varepsilon(\omega_f-\omega_p)$ and stabilized term $\frac{\alpha_f\gamma}{h}\int_\Gamma(\theta_f^\varepsilon-\theta_p^\varepsilon)(\omega_f-\omega_p)$ vanish when the coupled space W_Γ is applied. Besides, the nonlinear terms $c_f\left(u_f^\varepsilon,e_{u_f},e_{u_f}\right)_{\Omega_f}$, $\tilde c_f\left(\hat u_f^\varepsilon,e_{\theta_f},e_{\theta_f}\right)_{\Omega_f}$, and $\tilde c_p\left(\hat u_p^\varepsilon,e_{\theta_p},e_{\theta_p}\right)_{\Omega_p}$ vanish due to the definition of these trilinear terms.

Next, applying Lemma 3.1, the Hölder inequality and the Young's inequality, we rewrite (33) as

$$\begin{aligned}\alpha_f\|\nabla e_{\theta_f}\|_0^2+\alpha_p\|\nabla e_{\theta_p}\|_0^2\le&\frac{\alpha_f}{2}\|\nabla e_{\theta_f}\|_0^2+\frac{\tilde{N}^2}{2\alpha_f}\|\nabla\tilde{\theta}_f^\varepsilon\|_0^2\|\nabla e_{u_f}\|_0^2\\&+\frac{\alpha_p}{2}\|\nabla e_{\theta_p}\|_0^2+\frac{\tilde{N}^2}{2\alpha_p}\|\nabla\tilde{\theta}_p^\varepsilon\|_{L^3(\Omega_p)}^2\|e_{u_p}\|_0^2.\end{aligned}\tag{34}$$

Rearranging (34), one has

$$\alpha_f\|\nabla e_{\theta_f}\|_0^2+\alpha_p\|\nabla e_{\theta_p}\|_0^2\le\frac{\tilde{N}^2}{\alpha_f}\mathcal{S}_\theta^2\|\nabla e_{u_f}\|_0^2+\frac{\tilde{N}^2}{\alpha_p}\|\nabla\tilde{\theta}_p^\varepsilon\|_{L^3(\Omega_p)}^2\|e_{u_p}\|_0^2.\tag{35}$$

Similarly, from (31) and (32) we obtain

$$\begin{aligned}&\nu\|\nabla e_{u_f}\|_0^2+\varepsilon\|\nabla e_{u_p}\|_0^2+\frac{\nu}{D_a}\|e_{u_p}\|_0^2\\&\le N\|\nabla\tilde{u}_f^\varepsilon\|_0\|\nabla e_{u_f}\|_0^2+G_r\nu^2C_p^2\|\nabla e_{\theta_f}\|_0\|\nabla e_{u_f}\|_0+G_r\nu^2C_p\|\nabla e_{\theta_p}\|_0\|e_{u_p}\|_0\\&\le N\|\nabla\tilde{u}_f^\varepsilon\|_0\|\nabla e_{u_f}\|_0^2+\frac{\nu}{2}\|\nabla e_{u_f}\|_0^2+\frac{{G_r}^2\nu^3C_p^4}{2\alpha_f}\alpha_f\|\nabla e_{\theta_f}\|_0^2\\&\quad+\frac{\nu}{2D_a}\|e_{u_p}\|_0^2+\frac{G_r^2\nu^3C_p^2D_a}{2\alpha_p}\alpha_p\|\nabla e_{\theta_p}\|_0^2.\end{aligned}\tag{36}$$

Moreover, by combining (35) with (36), we have

$$\begin{aligned}&\nu\|\nabla e_{u_f}\|_0^2+\varepsilon\|\nabla e_{u_p}\|_0^2+\frac{\nu}{D_a}\|e_{u_p}\|_0^2\\&\le{G_r}^2C_p^2\nu^3\max\{C_p^2\alpha_f^{-1},D_a\alpha_p^{-1}\}(\frac{\tilde{N}^2}{\alpha_f}\mathcal{S}_\theta^2\|\nabla e_{u_f}\|_0^2+\frac{\tilde{N}^2}{\alpha_p}\|\nabla\tilde{\theta}_p^\varepsilon\|_{L^3(\Omega_p)}^2\|e_{u_p}\|_0^2)\\&\quad+2N\|\nabla\tilde{u}_f^\varepsilon\|_0\|\nabla e_{u_f}\|_0^2.\end{aligned}\tag{37}$$

Based on the conditions (29), we arrive at

$$\begin{aligned}&2N\|\nabla\tilde{u}_f^\varepsilon\|_0+{G_r}^2C_p^2\nu^3\max\{C_p^2\alpha_f^{-1},D_a\alpha_p^{-1}\}\frac{\tilde{N}^2}{\alpha_f}\mathcal{S}_\theta^2\\&\le 2NG_r{C_p}^2\nu\mathcal{S}_\theta+{G_r}^2C_p^2\nu^3\max\{C_p^2\alpha_f^{-1},D_a\alpha_p^{-1}\}\frac{\tilde{N}^2}{\alpha_f}\mathcal{S}_\theta^2<\nu,\end{aligned}\tag{38}$$

and

$${G_r}^2C_p^2\nu^3\max\{C_p^2\alpha_f^{-1},D_a\alpha_p^{-1}\}\frac{\tilde{N}^2}{\alpha_p}\|\nabla\tilde{\theta}_p^\varepsilon\|_{L^3(\Omega_p)}^2<\frac{\nu}{D_a}.\tag{39}$$

Since $\varepsilon>0$, we deduce that $\|\nabla e_{u_f}\|_0\le 0$, $\|\nabla e_{u_p}\|_0\le 0$ and $\|e_{u_p}\|_0\le 0$. Considering (35), we know $e_{\theta_f}=e_{\theta_p}=0$.

Finally, based on the *inf-sup* condition (there exists a positive constant β such that for all $\boldsymbol{q}\in Y$, $\sup_{\boldsymbol{v}\in X}\frac{|(\boldsymbol{q},\nabla\cdot\boldsymbol{v})_\Omega|}{\|\nabla\boldsymbol{v}\|_0\|\boldsymbol{q}\|_0}\ge\beta$), the uniqueness of the pressure $\boldsymbol{p}$ can be proved directly.

Now, let us prove (30). Firstly, set $\boldsymbol{\omega}=\boldsymbol{\theta}^\varepsilon$ in (27) to get

$$\alpha_f\|\nabla\theta_f^\varepsilon\|_0^2+\alpha_p\|\nabla\theta_p^\varepsilon\|_0^2=(g_f,\theta_f^\varepsilon)_{\Omega_f}+\left(g_p,\theta_p^\varepsilon\right)_{\Omega_p}.\tag{40}$$

Note that $\tilde{c}_f\left(u_f^\varepsilon,\theta_f^\varepsilon,\theta_f^\varepsilon\right)_{\Omega_f}$ and $\tilde{c}_p\left(u_p^\varepsilon,\theta_p^\varepsilon,\theta_p^\varepsilon\right)_{\Omega_p}$ are equal to zero. The interface and stabilized terms disappear due to the application of the space W_Γ. By using the Cauchy-Schwarz inequality and the Young's inequality, we arrive at

$$\alpha_f \left\|\nabla\theta_f^\varepsilon\right\|_0^2 + \alpha_p \left\|\nabla\theta_p^\varepsilon\right\|_0^2 \leq \frac{1}{\min\{\alpha_f, \alpha_p\}}\|g(x)\|_{-1}^2. \tag{41}$$

Secondly, taking $(v_f, q_f) = \left(u_f^\varepsilon, p_f^\varepsilon\right)$ in (25) and using Lemma 3.1, the Poincaré inequality, one can easily obtain

$$\nu \left\|\nabla u_f^\varepsilon\right\|_0^2 \leq G_r\nu^2 C_p^2 \|\nabla\theta_f^\varepsilon\|_0 \|\nabla u_f^\varepsilon\|_0, \tag{42}$$

which leads to

$$\|\nabla u_f^\varepsilon\|_0 \leq G_r {C_p}^2 \nu \mathcal{S}_\theta. \tag{43}$$

Next, by taking $(v_p, q_p) = \left(u_p^\varepsilon, p_p^\varepsilon\right)$ in (26), we arrive at

$$\varepsilon\|\nabla u_p^\varepsilon\|_0^2 + \frac{\nu}{D_a}\|u_p^\varepsilon\|_0^2 = G_r\nu^2\left(\theta_p^\varepsilon\xi, u_p^\varepsilon\right)_{\Omega_p}. \tag{44}$$

Moreover, applying the Cauchy-Schwarz inequality and Young's inequality on the right-hand side of (44), we have

$$\varepsilon\|\nabla u_p^\varepsilon\|_0 \leq G_r\nu^2 C_p^2 \|\nabla\theta_p^\varepsilon\|_0, \quad \|u_p^\varepsilon\|_0 \leq D_a G_r \nu C_p \|\nabla\theta_p^\varepsilon\|_0, \tag{45}$$

which combines with (41) to finish the proof. □

Theorem 3.2. *If the following conditions hold*

$$\begin{aligned} &0 < 2NG_rC_p^2\mathcal{S}_\theta + G_r^2\nu^2C_p^2\max\{C_p^2\alpha_f^{-1}, D_a\alpha_p^{-1}\}\tilde{N}^2\alpha_f^{-1}\mathcal{S}_\theta^2 < 1,\\ &0 < D_aG_r^2\nu^2C_p^2\max\{C_p^2\alpha_f^{-1}, D_a\alpha_p^{-1}\}\tilde{N}^2\alpha_p^{-1}\|\nabla\theta_p\|_{L^3(\Omega_p)}^2 < 1, \end{aligned} \tag{46}$$

then one has

$$\|\nabla u_f\|_0 \leq G_r\nu C_p^2\mathcal{S}_\theta, \quad \|u_p\|_0 \leq D_aG_r\nu C_p\mathcal{S}_\theta, \quad \|\nabla\theta_f\|_0^2 + \|\nabla\theta_p\|_0^2 \leq \mathcal{S}_\theta^2. \tag{47}$$

Proof. One can prove (47) by a similar argument as (30), so we omit it. □

Now, we derive the bound of error concerning $(\boldsymbol{u}^\varepsilon, \boldsymbol{p}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ and $(\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\theta})$.

Theorem 3.3. *Under the assumptions of Theorem 3.2 and 3.1, if $u_p^\varepsilon \in X_p^\varepsilon \cap H^2(\Omega_p)^2$, then one has the following bound of error:*

$$\left\|\nabla\left(u_f - u_f^\varepsilon\right)\right\|_0 + \left\|u_p - u_p^\varepsilon\right\|_0 + \|\boldsymbol{p} - \boldsymbol{p}^\varepsilon\|_0 + \|\nabla(\boldsymbol{\theta} - \boldsymbol{\theta}^\varepsilon)\|_0 \leq C\varepsilon, \tag{48}$$

where the constant $C > 0$ is independent of ε.

Proof. Subtracting (25)-(27) from (22)-(24) gives

$$\begin{aligned} &\nu\left(\nabla(u_f - u_f^\varepsilon), \nabla v_f\right)_{\Omega_f} + c_f\left(u_f - u_f^\varepsilon, u_f, v_f\right)_{\Omega_f} + c_f\left(u_f^\varepsilon, u_f - u_f^\varepsilon, v_f\right)_{\Omega_f}\\ &- (p_f - p_f^\varepsilon, \nabla\cdot v_f)_{\Omega_f} + (\nabla\cdot(u_f - u_f^\varepsilon), q_f)_{\Omega_f} = G_r\nu^2\left((\theta_f - \theta_f^\varepsilon)\xi, v_f\right)_{\Omega_f}, \end{aligned} \tag{49}$$

and

$$\begin{aligned} &-\varepsilon\left(\nabla u_p^\varepsilon, \nabla v_p\right)_{\Omega_p} + \frac{\nu}{D_a}\left(u_p - u_p^\varepsilon, v_p\right)_{\Omega_p} - (p_p - p_p^\varepsilon, \nabla\cdot v_p)_{\Omega_p} + (\nabla\cdot(u_p - u_p^\varepsilon), q_p)_{\Omega_p}\\ &= G_r\nu^2\left((\theta_p - \theta_p^\varepsilon)\xi, v_p\right)_{\Omega_p}, \end{aligned} \tag{50}$$

as well as

(51)

$$\alpha_f\left(\nabla(\theta_f-\theta_f^\varepsilon),\nabla\omega_f\right)_{\Omega_f}+\alpha_p\left(\nabla(\theta_p-\theta_p^\varepsilon),\nabla\omega_p\right)_{\Omega_p}+\tilde c_f\left(u_f-u_f^\varepsilon,\theta_f,\omega_f\right)_{\Omega_f}$$
$$+\tilde c_f\left(u_f^\varepsilon,\theta_f-\theta_f^\varepsilon,\omega_f\right)_{\Omega_f}+\tilde c_p\left(u_p-u_p^\varepsilon,\theta_p,\omega_p\right)_{\Omega_p}+\tilde c_p\left(u_p^\varepsilon,\theta_p-\theta_p^\varepsilon,\omega_p\right)_{\Omega_p}$$
$$-\alpha_f\int_\Gamma n_f\cdot\nabla(\theta_f-\theta_f^\varepsilon)(\omega_f-\omega_p)+\frac{\alpha_f\gamma}{h}\int_\Gamma((\theta_f-\theta_f^\varepsilon)-(\theta_p-\theta_p^\varepsilon))(\omega_f-\omega_p)=0.$$

Note that the interface term $-\alpha_f\int_\Gamma n_f\cdot\nabla(\theta_f-\theta_f^\varepsilon)(\omega_f-\omega_p)$ and the stabilization term $\frac{\alpha_f\gamma}{h}\int_\Gamma((\theta_f-\theta_f^\varepsilon)-(\theta_p-\theta_p^\varepsilon))(\omega_f-\omega_p)$ in (51) vanish when the space W_Γ is used.

Next, by taking $\boldsymbol{\omega}=\boldsymbol{\theta}-\boldsymbol{\theta}^\varepsilon$ and rearranging (51), we have

$$\begin{aligned}&\alpha_f\|\nabla(\theta_f-\theta_f^\varepsilon)\|_0^2+\alpha_p\|\nabla(\theta_p-\theta_p^\varepsilon)\|_0^2\\ \le&\frac{1}{2\alpha_f}\tilde N^2\|\nabla\theta_f\|_0^2\|\nabla(u_f-u_f^\varepsilon)\|_0^2+\frac{\alpha_f}{2}\|\nabla(\theta_f-\theta_f^\varepsilon)\|_0^2\\ &+\frac{1}{2\alpha_p}\tilde N^2\|\nabla\theta_p\|_{L^3(\Omega_p)}^2\|u_p-u_p^\varepsilon\|_0^2+\frac{\alpha_p}{2}\|\nabla(\theta_p-\theta_p^\varepsilon)\|_0^2.\end{aligned}\tag{52}$$

Taking $(\boldsymbol{v},\boldsymbol{q})=(\boldsymbol{u}-\boldsymbol{u}^\varepsilon,\boldsymbol{p}-\boldsymbol{p}^\varepsilon)$ in (49)-(50), we get

$$\begin{aligned}&\nu\|\nabla(u_f-u_f^\varepsilon)\|_0^2+\frac{\nu}{D_a}\|u_p-u_p^\varepsilon\|_0^2\\ \le&N\|\nabla u_f\|_0\|\nabla(u_f-u_f^\varepsilon)\|_0^2+G_r\nu^2C_p^2\|\nabla(\theta_f-\theta_f^\varepsilon)\|_0\|\nabla(u_f-u_f^\varepsilon)\|_0\\ &+\varepsilon\|\Delta u_p^\varepsilon\|_0\|u_p-u_p^\varepsilon\|_0+G_r\nu^2C_p\|\nabla(\theta_p-\theta_p^\varepsilon)\|_0\|u_p-u_p^\varepsilon\|_0.\end{aligned}\tag{53}$$

In addition, by using (52), Young's inequality and (47), we arrive at

(54)

$$\begin{aligned}&\nu\|\nabla(u_f-u_f^\varepsilon)\|_0^2+\frac{\nu}{D_a}\|u_p-u_p^\varepsilon\|_0^2\\ \le&N\|\nabla u_f\|_0\|\nabla(u_f-u_f^\varepsilon)\|_0^2+\frac{\nu}{2}\|\nabla(u_f-u_f^\varepsilon)\|_0^2+\frac{G_r^2\nu^3C_p^4}{2}\|\nabla(\theta_f-\theta_f^\varepsilon)\|_0^2\\ &+\frac{\varepsilon^2D_a}{\nu}\|\Delta u_p^\varepsilon\|_0^2+\frac{\nu}{2D_a}\|u_p-u_p^\varepsilon\|_0^2+G_r^2\nu^3C_p^2D_a\|\nabla(\theta_p-\theta_p^\varepsilon)\|_0^2\\ \le&NG_r\nu C_p^2\mathcal{S}_\theta\|\nabla(u_f-u_f^\varepsilon)\|_0^2+\frac{\nu}{2}\|\nabla(u_f-u_f^\varepsilon)\|_0^2+\frac{\varepsilon^2D_a}{\nu}\|\Delta u_p^\varepsilon\|_0^2+\frac{\nu}{2D_a}\|u_p-u_p^\varepsilon\|_0^2\\ &+{G_r}^2C_p^2\nu^3\max\{\frac{C_p^2}{2\alpha_f},\frac{D_a}{\alpha_p}\}(\frac{\tilde N^2}{\alpha_f}\mathcal{S}_\theta^2\|\nabla(u_f-u_f^\varepsilon)\|_0^2+\frac{\tilde N^2}{\alpha_p}\|\nabla\tilde\theta_p^\varepsilon\|_{L^3(\Omega_p)}^2\|(u_p-u_p^\varepsilon)\|_0^2).\end{aligned}$$

Based on (47) and the regularity assumption of u_p^ε, i.e., $u_p^\varepsilon\in H^2(\Omega_p)^2$, we rearrange (54) as

(55)

$$\begin{aligned}&(\nu-2NG_r{C_p}^2\nu\mathcal{S}_\theta-{G_r}^2C_p^2\nu^3\max\{C_p^2\alpha_f^{-1},2D_a\alpha_p^{-1}\}\frac{\tilde N^2}{\alpha_f}\mathcal{S}_\theta^2)\|\nabla(u_f-u_f^\varepsilon)\|_0^2\\ &+(\frac{\nu}{D_a}-{G_r}^2C_p^2\nu^3\max\{C_p^2\alpha_f^{-1},2D_a\alpha_p^{-1}\}\frac{\tilde N^2}{\alpha_p}\|\nabla\tilde\theta_p^\varepsilon\|_{L^3(\Omega_p)}^2)\|u_p-u_p^\varepsilon\|_0^2\le C\varepsilon^2.\end{aligned}$$

Therefore, it is easy to get the result from (52) for the temperature

$$\begin{aligned}&\alpha_f\|\nabla(\theta_f-\theta_f^\varepsilon)\|_0^2+\alpha_p\|\nabla(\theta_p-\theta_p^\varepsilon)\|_0^2\\ \le&\frac{\tilde N^2}{\alpha_f}\mathcal{S}_\theta^2\|\nabla(u_f-u_f^\varepsilon)\|_0^2+\frac{\tilde N^2}{\alpha_p}\|\nabla\theta_p\|_{L^3(\Omega_p)}^2\|u_p-u_p^\varepsilon\|_0^2\le C\varepsilon^2.\end{aligned}\tag{56}$$

Finally, thanks to the *inf-sup* condition, by taking $\boldsymbol{q} = \mathbf{0}$ in (49) and (50), we have the error bound of the pressures, which finishes the proof. □

4. Penalty finite element discretization

Let h be a real positive parameter. Finite element spaces $X_h^\varepsilon \times Y_h \times W_h \subset X^\varepsilon \times Y \times W$ are characterized by τ_h, a partitioning of Ω into triangles, which is assumed to be uniformly regular as $h \to 0$.

We consider X_h^ε and W_h to be spaces of continuous piecewise polynomials of degree l, and Y_h is the space of continuous piecewise polynomials of degree $l-1$ ($l \geq 1$). We also assume that the finite element spaces X_h^ε and Y_h satisfy the discrete *inf-sup* condition. Let mappings $R_h : X^\varepsilon \to X_h^\varepsilon$, $Q_h : Y \to Y_h$ and $P_h : W \to W_h$. Assume that $R_h v \in X_h^\varepsilon$, $Q_h q \in Y_h$ and $P_h \omega \in W_h$ satisfy the following approximation properties,

(57)
$$\|v - R_h v\|_0 + \|\omega - P_h\omega\|_0 \leq Ch^{k+1}(\|v\|_{k+1} + \|\omega\|_{k+1}),$$
$$\|q - Q_h q\|_0 + \|\nabla(v - R_h v)\|_0 + \|\nabla(\omega - P_h\omega)\|_0 \leq Ch^{k}(\|q\|_k + \|v\|_{k+1} + \|\omega\|_{k+1}),$$

where $0 \leq k \leq l$.

We also need the local inverse inequality [8, 18] for W_h: there exists a constant $C_{in} > 0$ which depends only on the minimum angles of τ_h, such that

$$\|\nabla\theta_h\|_\Gamma \leq C_{in}^{\frac{1}{2}} h^{-\frac{1}{2}} \|\nabla\theta_h\|_0 \quad \forall\theta_h \in W_h. \tag{58}$$

Next, we approximate the variational formulation (25)-(27) of the penalty system (11)-(18): find $(\boldsymbol{u}_h^\varepsilon, \boldsymbol{p}_h^\varepsilon, \boldsymbol{\theta}_h^\varepsilon) \in X_h^\varepsilon \times Y_h \times W_h$ satisfying

$$\begin{aligned}\nu\left(\nabla u_{f,h}^\varepsilon, \nabla v_f\right)_{\Omega_f} &+ c_f\left(u_{f,h}^\varepsilon, u_{f,h}^\varepsilon, v_f\right)_{\Omega_f} - (p_{f,h}^\varepsilon, \nabla\cdot v_f) + (\nabla\cdot u_{f,h}^\varepsilon, q_f)\\ &= G_r\nu^2\left(\theta_{f,h}^\varepsilon \xi, v_f\right)_{\Omega_f},\end{aligned} \tag{59}$$

$$\begin{aligned}\varepsilon\left(\nabla u_{p,h}^\varepsilon, \nabla v_p\right)_{\Omega_p} &+ \frac{\nu}{D_a}\left(u_{p,h}^\varepsilon, v_p\right)_{\Omega_p} - (p_{p,h}^\varepsilon, \nabla\cdot v_p) + (\nabla\cdot u_{p,h}^\varepsilon, q_p)\\ &= G_r\nu^2\left(\theta_{p,h}^\varepsilon \xi, v_p\right)_{\Omega_p},\end{aligned} \tag{60}$$

$$\begin{aligned}&\alpha_f\left(\nabla\theta_{f,h}^\varepsilon, \nabla\omega_f\right)_{\Omega_f} + \alpha_p\left(\nabla\theta_{p,h}^\varepsilon, \nabla\omega_p\right)_{\Omega_p} + \tilde{c}_f\left(u_{f,h}^\varepsilon, \theta_{f,h}^\varepsilon, \omega_f\right)_{\Omega_f}\\ &+ \tilde{c}_p\left(u_{p,h}^\varepsilon, \theta_{p,h}^\varepsilon, \omega_p\right)_{\Omega_p} - \alpha_f\int_\Gamma n_f\cdot\nabla\theta_{f,h}^\varepsilon(\omega_f - \omega_p)\\ &+ \frac{\alpha_f\gamma}{h}\int_\Gamma(\theta_{f,h}^\varepsilon - \theta_{p,h}^\varepsilon)(\omega_f - \omega_p) = (g_f, \omega_f)_{\Omega_f} + (g_p, \omega_p)_{\Omega_p},\end{aligned} \tag{61}$$

for any $(\boldsymbol{v}^\varepsilon, \boldsymbol{q}^\varepsilon, \boldsymbol{\omega}^\varepsilon) \in X_h^\varepsilon \times Y_h \times W_h$.

We call the above finite element method as the penalty finite element method. Moreover, we have the following stability for the finite element approximation problem (59)-(61).

Theorem 4.1. *Under the assumption of Theorem 3.3, if $\gamma \gg C_{in}$, then one gets*

$$\begin{aligned}&\|\nabla u_{f,h}^\varepsilon\|_0 \leq G_r\nu C_p^2\mathcal{S}_\theta, \quad \left\|\nabla u_{p,h}^\varepsilon\right\|_0 \leq \varepsilon^{-1}G_r\nu^2C_p^2\mathcal{S}_\theta,\\ &\left\|\nabla\theta_{f,h}^\varepsilon\right\|_0^2 + \left\|\nabla\theta_{p,h}^\varepsilon\right\|_0^2 \leq \mathcal{S}_\theta^2.\end{aligned} \tag{62}$$

Proof. One can easily prove it by using a similar manner to (30) in Theorem 3.1. So we omit it. □

Furthermore, we derive the bound of the errors between $(\boldsymbol{u}_h^\varepsilon, \boldsymbol{p}_h^\varepsilon, \boldsymbol{\theta}_h^\varepsilon)$ and $(\boldsymbol{u}^\varepsilon, \boldsymbol{p}^\varepsilon, \boldsymbol{\theta}^\varepsilon)$ in the following theorem. For convenience, we separate the errors into two parts,

$$u_i - u_{i,h} = u_i - R_h u_i + R_h u_i - u_{i,h} := \eta_i + \psi_i,\ i = f \text{ or } p.$$
$$p_i - p_{i,h} = p_i - Q_h p_i + Q_h p_i - p_{i,h} := \rho_i + \pi_i,\ i = f \text{ or } p.$$
$$\theta_i - \theta_{i,h} = \theta_i - P_h \theta_i + P_h \theta_i - \theta_{i,h} := \phi_i + \varphi_i,\ i = f \text{ or } p.$$

Theorem 4.2. *Under the assumption of Theorem 4.1, if $\gamma \gg C_{in}$ and the following conditions hold*

$$\begin{aligned} &\frac{\nu}{2} - NG_r\nu C_p^2 \mathcal{S}_\theta - G_r^2 C_p^2 \nu^3 \max\{\frac{3C_p^2}{2\alpha_f}, \frac{D_a}{\alpha_p}\}\kappa_f > 0,\\ &\frac{\nu}{2D_a} - G_r^2 C_p^2 \nu^3 \max\{\frac{3C_p^2}{2\alpha_f}, \frac{D_a}{\alpha_p}\}\kappa_p > 0, \end{aligned} \tag{63}$$

where $\kappa_f = 4\tilde{N}^2\alpha_f^{-1}\|\nabla\theta_f^\varepsilon\|_0^2$ and $\kappa_p = 4\tilde{N}^2\alpha_p^{-1}\|\theta_p^\varepsilon\|_2^2$, then one has

$$\left\|u_f^\varepsilon - u_{f,h}^\varepsilon\right\|_1 + \varepsilon^{\frac{1}{2}}\left\|u_p^\varepsilon - u_{p,h}^\varepsilon\right\|_1 + h^{-1}\left\|u_p^\varepsilon - u_{p,h}^\varepsilon\right\|_0 + \|\boldsymbol{p}^\varepsilon - \boldsymbol{p}_h^\varepsilon\|_0 + \|\boldsymbol{\theta}^\varepsilon - \boldsymbol{\theta}_h^\varepsilon\|_1 \le C\varepsilon^{-1}h^k.$$

Proof. Subtracting (61) from (27) we have the error equation of temperatures:
(64)

$$\begin{aligned} &\alpha_f\left(\nabla(\theta_f^\varepsilon - \theta_{f,h}^\varepsilon), \nabla\omega_f\right)_{\Omega_f} + \alpha_p\left(\nabla(\theta_p^\varepsilon - \theta_{p,h}^\varepsilon), \nabla\omega_p\right)_{\Omega_p} + \tilde{c}_f\left(u_f^\varepsilon, \theta_f^\varepsilon, \omega_f\right)_{\Omega_f}\\ &- \tilde{c}_f\left(u_{f,h}^\varepsilon, \theta_{f,h}^\varepsilon, \omega_f\right)_{\Omega_f} + \tilde{c}_p\left(u_p^\varepsilon, \theta_p^\varepsilon, \omega_p\right)_{\Omega_p} - \tilde{c}_p\left(u_{p,h}^\varepsilon, \theta_{p,h}^\varepsilon, \omega_p\right)_{\Omega_p}\\ &- \alpha_f\int_\Gamma n_f\cdot\nabla(\theta_f^\varepsilon - \theta_{f,h}^\varepsilon)(\omega_f - \omega_p) + \frac{\alpha_f\gamma}{h}\int_\Gamma((\theta_f^\varepsilon - \theta_{f,h}^\varepsilon) - (\theta_p^\varepsilon - \theta_p^\varepsilon))(\omega_f - \omega_{p,h}) = 0. \end{aligned}$$

Setting $\boldsymbol{\omega} = \boldsymbol{\varphi}$ in (64), we obtain

$$\begin{aligned} &\alpha_f\|\nabla\varphi_f\|_0^2 + \alpha_p\|\nabla\varphi_p\|_0^2 + \alpha_f\gamma(h^{-\frac{1}{2}}\|\varphi_f - \varphi_p\|_\Gamma)^2\\ &= -\alpha_f(\nabla\phi_f, \nabla\varphi_f)_{\Omega_f} - \alpha_p(\nabla\phi_p, \nabla\varphi_p)_{\Omega_p} - \tilde{c}_f\left(u_f^\varepsilon - u_{f,h}^\varepsilon, \theta_f^\varepsilon, \varphi_f\right)_{\Omega_f}\\ &\quad - \tilde{c}_f\left(u_{f,h}^\varepsilon, \phi_f, \varphi_f\right)_{\Omega_f} - \tilde{c}_p\left(u_p^\varepsilon - u_{p,h}^\varepsilon, \theta_p^\varepsilon, \varphi_p\right)_{\Omega_p} - \tilde{c}_p\left(u_{p,h}^\varepsilon, \phi_p, \varphi_p\right)_{\Omega_p}\\ &\quad + \alpha_f\int_\Gamma n_f\cdot\nabla(\theta_f^\varepsilon - \theta_{f,h}^\varepsilon)(\varphi_f - \varphi_p) - \frac{\alpha_f\gamma}{h}\int_\Gamma(\phi_f - \phi_f)(\varphi_f - \varphi_p). \end{aligned} \tag{65}$$

Now, let us estimate each term on the right-hand side of (65) with the help of the Cauchy inequality, Young's inequality, (57) and Theorem 3.2.

$$\begin{aligned} &|-\alpha_f(\nabla\phi_f, \nabla\varphi_f)_{\Omega_f} - \alpha_p(\nabla\phi_p, \nabla\varphi_p)_{\Omega_p}|\\ &\quad\le \alpha_f\|\nabla\phi_f\|_0^2 + \frac{\alpha_f}{4}\|\nabla\varphi_f\|_0^2 + \alpha_p\|\nabla\phi_p\|_0^2 + \frac{\alpha_p}{4}\|\nabla\varphi_p\|_0^2\\ &\quad\le Ch^{2k} + \frac{\alpha_f}{4}\|\nabla\varphi_f\|_0^2 + \frac{\alpha_p}{4}\|\nabla\varphi_p\|_0^2,\\ &|-\tilde{c}_f\left(u_f^\varepsilon - u_{f,h}^\varepsilon, \theta_f^\varepsilon, \varphi_f\right)_{\Omega_f} - \tilde{c}_f\left(u_{f,h}^\varepsilon, \phi_f, \varphi_f\right)_{\Omega_f}|\\ &\quad\le \tilde{N}^2\alpha_f^{-1}\|\nabla(u_f^\varepsilon - u_{f,h}^\varepsilon)\|_0^2\|\nabla\theta_f^\varepsilon\|_0^2 + \tilde{N}^2\alpha_f^{-1}\|\nabla u_{f,h}^\varepsilon\|_0^2\|\nabla\phi_f\|_0^2 + \frac{\alpha_f}{2}\|\nabla\varphi_f\|_0^2\\ &\quad\le Ch^{2k} + \tilde{N}^2\alpha_f^{-1}\|\nabla\theta_f^\varepsilon\|_0^2\|\nabla\psi_f\|_0^2 + \frac{\alpha_f}{2}\|\nabla\varphi_f\|_0^2,\\ &|-\tilde{c}_p\left(u_p^\varepsilon - u_{p,h}^\varepsilon, \theta_p^\varepsilon, \varphi_p\right)_{\Omega_p} - \tilde{c}_p\left(u_{p,h}^\varepsilon, \phi_p, \varphi_p\right)_{\Omega_p}|\\ &\quad\le \tilde{N}\|u_p^\varepsilon - u_{p,h}^\varepsilon\|_0\|\theta_p^\varepsilon\|_2\|\nabla\varphi_p\|_0 + \tilde{N}\|\nabla u_{p,h}^\varepsilon\|_0\|\nabla\phi_p\|_0\|\nabla\varphi_p\|_0\\ &\quad\le \tilde{N}^2\alpha_p^{-1}\|u_p^\varepsilon - u_{p,h}^\varepsilon\|_0^2\|\theta_p^\varepsilon\|_2^2 + \tilde{N}^2\alpha_p^{-1}\|\nabla u_{p,h}^\varepsilon\|_0^2\|\nabla\phi_p\|_0^2 + \frac{\alpha_p}{2}\|\nabla\varphi_p\|_0^2\\ &\quad\le Ch^{2(k+1)} + C\varepsilon^{-2}h^{2k} + \tilde{N}^2\alpha_p^{-1}\|\theta_p^\varepsilon\|_2^2\|\psi_p\|_0^2 + \frac{\alpha_p}{2}\|\nabla\varphi_p\|_0^2, \end{aligned}$$

as well as

$$
\begin{aligned}
|\alpha_f \int_\Gamma n_f \cdot \nabla(\theta_f^\varepsilon - \theta_{f,h}^\varepsilon)(\varphi_f - \varphi_p)| &\le \alpha_f C_{in}^{\frac{1}{2}} h^{-\frac{1}{2}} \|\nabla(\theta_f^\varepsilon - \theta_{f,h}^\varepsilon)\|_0 \|\varphi_f - \varphi_p\|_\Gamma \\
&\le \frac{\alpha_f C_{in}}{2\gamma} \|\nabla(\theta_f^\varepsilon - \theta_{f,h}^\varepsilon)\|_0^2 + \frac{\alpha_f \gamma}{2} (h^{-\frac{1}{2}} \|\varphi_f - \varphi_p\|_\Gamma)^2 \\
&\le Ch^{2k} + \frac{\alpha_f C_{in}}{2\gamma} \|\nabla \varphi_f\|_0^2 + \frac{\alpha_f \gamma}{2} (h^{-\frac{1}{2}} \|\varphi_f - \varphi_p\|_\Gamma)^2, \\
|-\frac{\alpha_f \gamma}{h} \int_\Gamma (\phi_f - \phi_p)(\varphi_f - \varphi_p)| &\le \alpha_f \gamma C_{in}^{-\frac{1}{2}} h^{-\frac{3}{2}} \|\phi_f - \phi_p\|_0 \|\varphi_f - \varphi_p\|_\Gamma \\
&\le \frac{\alpha_f \gamma C_{in}}{2} (h^{-1} \|\phi_f - \phi_p\|_0)^2 + \frac{\alpha_f \gamma}{2} (h^{-\frac{1}{2}} \|\varphi_f - \varphi_p\|_\Gamma)^2 \\
&\le Ch^{2k} + \frac{\alpha_f \gamma}{2} (h^{-\frac{1}{2}} \|\varphi_f - \varphi_p\|_\Gamma)^2.
\end{aligned}
$$

If $\gamma \gg C_{in}$, then the term $\frac{\alpha_f C_{in}}{2\gamma} \|\nabla \varphi_f\|_0^2$ is pretty close to zero. Now, with the above estimation of (65), we get the following results:

$$
\alpha_f \|\nabla \varphi_f\|_0^2 + \alpha_p \|\nabla \varphi_p\|_0^2 \le Ch^{2k} + C\varepsilon^{-2} h^{2k} + \kappa_f \|\nabla \psi_f\|_0^2 + \kappa_p \|\psi_p\|_0^2, \tag{66}
$$

where $\kappa_f = 4\tilde{N}^2 \alpha_f^{-1} \|\nabla \theta_f^\varepsilon\|_0^2$, $\kappa_p = 4\tilde{N}^2 \alpha_p^{-1} \|\theta_p^\varepsilon\|_2^2$.

Moreover, we prove the error bound of the velocity. Similarly, by subtracting (25)-(26) from (59)-(60), we arrive at

(67)

$$
\begin{aligned}
&\nu(\nabla(u_f^\varepsilon - u_{f,h}^\varepsilon), \nabla v_f)_{\Omega_f} + \varepsilon \left(\nabla(u_p^\varepsilon - u_{p,h}^\varepsilon), \nabla v_p\right)_{\Omega_p} + \frac{\nu}{D_a} \left((u_p^\varepsilon - u_{p,h}^\varepsilon), v_p\right)_{\Omega_p} \\
=& c_f \left(u_f^\varepsilon, u_f^\varepsilon, v_f\right)_{\Omega_f} - c_f \left(u_{f,h}^\varepsilon, u_{f,h}^\varepsilon, v_f\right)_{\Omega_f} + G_r \nu^2 \left((\theta_f^\varepsilon - \theta_{f,h}^\varepsilon)\xi, v_f\right)_{\Omega_f} \\
&+ G_r \nu^2 \left((\theta_p^\varepsilon - \theta_{p,h}^\varepsilon)\xi, v_p\right)_{\Omega_p}.
\end{aligned}
$$

Setting $v_f = \psi_f$, $v_p = \psi_p$ in (67), we obtain

$$
\begin{aligned}
&\nu \|\nabla \psi_f\|_0^2 + \varepsilon \|\nabla \psi_p\|_0^2 + \frac{\nu}{D_a} \|\psi_p\|_0^2 \\
=& -\nu(\nabla \eta_f, \nabla \psi_f)_{\Omega_f} - \varepsilon(\nabla \eta_p, \nabla \psi_p)_{\Omega_p} - \frac{\nu}{D_a} (\eta_p, \psi_p)_{\Omega_p} \\
&- c_f \left(u_f^\varepsilon, \eta_f, \psi_f\right)_{\Omega_f} - c_f \left(\eta_f, u_{f,h}^\varepsilon, \psi_f\right)_{\Omega_f} - c_f \left(\psi_f, u_{f,h}^\varepsilon, \psi_f\right)_{\Omega_f} \\
&+ G_r \nu^2 \left((\theta_f^\varepsilon - \theta_{f,h}^\varepsilon)\xi, \psi_f\right)_{\Omega_f} + G_r \nu^2 \left((\theta_p^\varepsilon - \theta_{p,h}^\varepsilon)\xi, \psi_p\right)_{\Omega_p}.
\end{aligned} \tag{68}
$$

Now we restrict each term at the right-hand side of (68). Firstly, we consider the nonlinear terms. Applying the Cauchy-Schwarz inequality, Lemma 3.1 and Young's inequality, we obtain the following estimate:

$$
\begin{aligned}
&|-c_f \left(u_f^\varepsilon, \eta_f, \psi_f\right)_{\Omega_f} - c_f \left(\eta_f, u_{f,h}^\varepsilon, \psi_f\right)_{\Omega_f} - c_f \left(\psi_f, u_{f,h}^\varepsilon, \psi_f\right)_{\Omega_f}| \\
\le& C\nu^{-1} \|\nabla u_f^\varepsilon\|_0^2 \|\nabla \eta_f\|_0^2 + C\nu^{-1} \|\nabla \eta_f\|_0^2 \|\nabla u_{f,h}^\varepsilon\|_0^2 \\
&+ \frac{\nu}{6} \|\nabla \psi_f\|_0^2 + N \|\nabla u_{f,h}^\varepsilon\|_0 \|\nabla \psi_f\|_0^2 \\
\le& Ch^{2k} + \frac{\nu}{6} \|\nabla \psi_f\|_0^2 + N \|\nabla u_{f,h}^\varepsilon\|_0 \|\nabla \psi_f\|_0^2.
\end{aligned} \tag{69}
$$

And then, we further get

(70)
$$
\begin{aligned}
|-\nu(\nabla\eta_f,\nabla\psi_f)_{\Omega_f}| &\le Ch^{2k}+\frac{\nu}{6}\|\nabla\psi_f\|_0^2,\\
|-\varepsilon(\nabla\eta_p,\nabla\psi_p)_{\Omega_p}-\frac{\nu}{D_a}(\eta_p,\psi_p)_{\Omega_p}| &\le C\varepsilon h^{2k}+Ch^{2(k+1)}+\frac{\varepsilon}{2}\|\nabla\psi_p\|_0^2+\frac{\nu}{4D_a}\|\psi_p\|_0^2,\\
G_r\nu^2\left((\theta_f^\varepsilon-\theta_{f,h}^\varepsilon)\xi,\psi_f\right)_{\Omega_f} &\le \frac{3G_r^2C_p^4\nu^3}{2}\|\nabla(\theta_f^\varepsilon-\theta_{f,h}^\varepsilon)\|_0^2+\frac{\nu}{6}\|\nabla\psi_f\|_0^2\\
&\le Ch^{2k}+\frac{3G_r^2C_p^4\nu^3}{2}\|\nabla\varphi_f\|_0^2+\frac{\nu}{6}\|\nabla\psi_f\|_0^2,\\
G_r\nu^2\left((\theta_p^\varepsilon-\theta_{p,h}^\varepsilon)\xi,\psi_p\right)_{\Omega_p} &\le G_r^2C_p^2\nu^3D_a\|\nabla(\theta_p^\varepsilon-\theta_{p,h}^\varepsilon)\|_0^2+\frac{\nu}{4D_a}\|\psi_p\|_0^2\\
&\le Ch^{2k}+G_r^2C_p^2\nu^3D_a\|\nabla\varphi_p\|_0^2+\frac{\nu}{4D_a}\|\psi_p\|_0^2.
\end{aligned}
$$

Combining all terms above and according to (62), (66), we arrive at

$$
\begin{aligned}
&\frac{\nu}{2}\|\nabla\psi_f\|_0^2+\frac{\varepsilon}{2}\|\nabla\psi_p\|_0^2+\frac{\nu}{2D_a}\|\psi_p\|_0^2\\
\le& Ch^{2k}+C\varepsilon h^{2k}+NG_r\nu C_p^2\mathcal{S}_\theta\|\nabla\psi_f\|_0^2\\
&+\frac{3G_r^2C_p^4\nu^3}{2}\|\nabla\varphi_f\|_0^2+G_r^2C_p^2\nu^3D_a\|\nabla\varphi_p\|_0^2\\
\le& Ch^{2k}+C\varepsilon h^{2k}+NG_r\nu C_p^2\mathcal{S}_\theta\|\nabla\psi_f\|_0^2 \qquad (71)\\
&+G_r^2C_p^2\nu^3\max\{\frac{3C_p^2}{2\alpha_f},\frac{D_a}{\alpha_p}\}(\alpha_f\|\nabla\varphi_f\|_0^2+\alpha_p\|\nabla\varphi_p\|_0^2)\\
\le& Ch^{2k}+C\varepsilon h^{2k}+C\varepsilon^{-2}h^{2k}+NG_r\nu C_p^2\mathcal{S}_\theta\|\nabla\psi_f\|_0^2\\
&+G_r^2C_p^2\nu^3\max\{\frac{3C_p^2}{2\alpha_f},\frac{D_a}{\alpha_p}\}\kappa_f\|\nabla\psi_f\|_0^2+G_r^2C_p^2\nu^3\max\{\frac{3C_p^2}{2\alpha_f},\frac{D_a}{\alpha_p}\}\kappa_p\|\psi_p\|_0^2.
\end{aligned}
$$

Therefore, if the condition (63) hold, then we have the following inequality

$$
\begin{aligned}
&(\frac{\nu}{2}-NG_r\nu C_p^2\mathcal{S}_\theta-G_r^2C_p^2\nu^3\max\{\frac{3C_p^2}{2\alpha_f},\frac{D_a}{\alpha_p}\}\kappa_f)\|\nabla\psi_f\|_0^2+\frac{\varepsilon}{2}\|\nabla\psi_p\|_0^2\\
&+(\frac{\nu}{2D_a}-G_r^2C_p^2\nu^3\max\{\frac{3C_p^2}{2\alpha_f},\frac{D_a}{\alpha_p}\}\kappa_p)\|\psi_p\|_0^2\le C\varepsilon^{-2}h^{2k}.
\end{aligned}
\tag{72}
$$

Finally, by using the discrete *inf-sup* condition and (66), (72), we obtain

$$
\|\pi_i\|_0\le Ch^k+C\|\nabla\psi_i\|_0+C\|\nabla\varphi_i\|_0. \tag{73}
$$

□

5. Iterative algorithm

In this section, we design a decoupled, nonlinear iterative algorithm for the finite element approximation problem (59)-(61). Then, we derive stability and error estimates.

Algorithm 5.1. *For given* $(u_{f,h}^{\varepsilon,n},u_{p,h}^{\varepsilon,n})\in X_h^\varepsilon$*, find* $(\boldsymbol{u}_h^{\varepsilon,n+1},\boldsymbol{p}_h^{\varepsilon,n+1},\boldsymbol{\theta}_h^{\varepsilon,n+1})\in X_h^\varepsilon\times Y_h\times W_h$ *such that for all* $(\boldsymbol{v}^\varepsilon,\boldsymbol{q}^\varepsilon,\boldsymbol{\omega}^\varepsilon)\in X_h^\varepsilon\times Y_h\times W_h$

Step 1:

$$\begin{aligned}
&\alpha_f\left(\nabla\theta_{f,h}^{\varepsilon,n+1},\nabla\omega_f\right)_{\Omega_f}+\alpha_p\left(\nabla\theta_{p,h}^{\varepsilon,n+1},\nabla\omega_p\right)_{\Omega_p}+\tilde{c}_f\left(u_{f,h}^{\varepsilon,n},\theta_{f,h}^{\varepsilon,n+1},\omega_f\right)_{\Omega_f}\\
&+\tilde{c}_p\left(u_{p,h}^{\varepsilon,n},\theta_{p,h}^{\varepsilon,n+1},\omega_p\right)_{\Omega_p}-\alpha_f\int_\Gamma n_f\cdot\nabla\theta_{f,h}^{\varepsilon,n+1}(\omega_f-\omega_p)\\
&+\frac{\alpha_f\gamma}{h}\int_\Gamma\left(\theta_{f,h}^{\varepsilon,n+1}-\theta_{p,h}^{\varepsilon,n+1}\right)(\omega_f-\omega_p)=(g_f,\omega_f)_{\Omega_f}+(g_p,\omega_p)_{\Omega_p}.
\end{aligned}\tag{74}$$

Step 2:

$$\begin{aligned}
&\nu\left(\nabla u_{f,h}^{\varepsilon,n+1},\nabla v_f\right)_{\Omega_f}+c_f\left(u_{f,h}^{\varepsilon,n},u_{f,h}^{\varepsilon,n+1},v_f\right)_{\Omega_f}\\
&-\left(p_{f,h}^{\varepsilon,n+1},\nabla\cdot v_f\right)_{\Omega_f}+(\nabla\cdot u_{f,h}^{\varepsilon,n+1},q_f)_{\Omega_f}\\
=&G_r\nu^2\left(\theta_{f,h}^{\varepsilon,n+1}\xi,v_f\right)_{\Omega_f}.
\end{aligned}\tag{75}$$

Step 3:

$$\begin{aligned}
&\varepsilon\left(\nabla u_{p,h}^{\varepsilon,n+1},\nabla v_p\right)_{\Omega_p}+\frac{\nu}{D_a}(u_{p,h}^{\varepsilon,n+1},v_p)_{\Omega_p}\\
&-\left(p_{p,h}^{\varepsilon,n+1},\nabla\cdot v_p\right)_{\Omega_p}+\left(\nabla\cdot u_{p,h}^{\varepsilon,n+1},q_p\right)_{\Omega_p}\\
=&G_r\nu^2\left(\theta_{p,h}^{\varepsilon,n+1}\xi,v_p\right)_{\Omega_p}.
\end{aligned}\tag{76}$$

From Algorithm 5.1, we notice that the closed-loop geothermal model can be solved separately when the initial iterative values $(u_f^{h,0},u_p^{h,0})$ are given. One of the main benefits of the decoupled algorithm is that it solves several smaller subproblems instead of the coupled problem. Then, we can save much computational time.

Next, we consider the stability and error estimation of the decoupled iterative algorithm.

Theorem 5.1. *Algorithm 5.1 is stable if the stabilization parameter γ satisfies the condition: $\gamma\gg C_{in}$. Besides, one has*

$$\nu\left\|\nabla u_{f,h}^{\varepsilon,n+1}\right\|_0^2+\varepsilon\left\|\nabla u_{p,h}^{\varepsilon,n+1}\right\|_0^2+\frac{\nu}{D_a}\left\|u_{p,h}^{\varepsilon,n+1}\right\|_0^2+\alpha_f\left\|\nabla\theta_{f,h}^{\varepsilon,n+1}\right\|_0^2+\alpha_p\left\|\nabla\theta_{p,h}^{\varepsilon,n+1}\right\|_0^2\le C,$$

where $C>0$ is a constant and independent of h.

Proof. One can prove the theorem by a similar argument as Theorem 4.1, so we omit it. Reader also can see [25, 18]. □

Now, we analyze the iterative error of the decoupled iterative finite element method. For convenience, we set $E_{u_f}^{n+1}=u_{f,h}^{\varepsilon}-u_{f,h}^{\varepsilon,n+1}$, $E_{u_p}^{n+1}=u_{p,h}^{\varepsilon}-u_{p,h}^{\varepsilon,n+1}$, $E_{\theta_f}^{n+1}=\theta_{f,h}^{\varepsilon}-\theta_{f,h}^{\varepsilon,n+1}$, and $E_{\theta_p}^{n+1}=\theta_{p,h}^{\varepsilon}-\theta_{p,h}^{\varepsilon,n+1}$.

Theorem 5.2. *Under the conditions of Theorem 4.1 and Theorem 5.1, assume that $(u_{f,h}^{\varepsilon,n+1},u_{p,h}^{\varepsilon,n+1},p_{f,h}^{\varepsilon,n+1},p_{p,h}^{\varepsilon,n+1},\theta_{f,h}^{\varepsilon,n+1},\theta_{p,h}^{\varepsilon,n+1})$ is the function sequence of Algorithm 5.1, and let the iterative factor λ satisfy*

$$\begin{aligned}
0<\lambda:=2G_r^2C_p^2\mathcal{S}_\theta^2\max\left\{N^2C_p^2+\nu^2\max\{\frac{C_p^2}{\alpha_f},\frac{D_a}{2\alpha_p}\}\frac{\tilde{N}^2}{\alpha_f},\nu^2\max\{\frac{C_p^2}{\alpha_f},\frac{D_a}{2\alpha_p}\}\frac{\tilde{N}^2}{\alpha_p}\right\}&\\
<1.&
\end{aligned}\tag{77}$$

Then we have

$$\nu\|\nabla E_{u_f}^{n+1}\|_0^2 + 2\varepsilon\|\nabla E_{u_p}^{n+1}\|_0^2 + \frac{\nu}{D_a}\|E_{u_p}^{n+1}\|_0^2 \le \lambda(\nu\|\nabla E_{u_f}^{n}\|_0^2 + \frac{\nu}{D_a}\|E_{u_p}^{n}\|_0^2). \tag{78}$$

Proof. Subtracting (59) from (74) and taking $\omega_f = E_{\theta_f}^{n+1}$, $\omega_p = E_{\theta_p}^{n+1}$, we obtain an error equation on the temperature.

(79)

$$\begin{aligned}&\alpha_f\left(\nabla E_{\theta_f}^{n+1}, \nabla E_{\theta_f}^{n+1}\right)_{\Omega_f} + \alpha_p\left(\nabla E_{\theta_p}^{n+1}, \nabla E_{\theta_p}^{n+1}\right)_{\Omega_p} + \tilde{c}_f\left(E_{u_f}^{n}, \theta_{f,h}^{\varepsilon}, E_{\theta_f}^{n+1}\right)_{\Omega_f}\\ &+ \tilde{c}_p\left(E_{u_p}^{n}, \theta_{p,h}^{\varepsilon}, E_{\theta_p}^{n+1}\right)_{\Omega_p} - \alpha_f\int_\Gamma n_f\cdot\nabla E_{\theta_f}^{n+1}(E_{\theta_f}^{n+1} - E_{\theta_p}^{n+1})\\ &+ \frac{\alpha_f\gamma}{h}\int_\Gamma (E_{\theta_f}^{n+1} - E_{\theta_p}^{n+1})^2 = 0.\end{aligned}$$

By using the Cauchy-Schwarz inequality and Young's inequality, we show

$$\begin{aligned}&\alpha_f\left\|\nabla E_{\theta_f}^{n+1}\right\|_0^2 + \alpha_p\left\|\nabla E_{\theta_p}^{n+1}\right\|_0^2 + \frac{\alpha_f\gamma}{h}\left\|E_{\theta_f}^{n+1} - E_{\theta_p}^{n+1}\right\|_\Gamma\\ &\le \tilde{N}\|\nabla E_{u_f}^{n}\|_0\|\nabla\theta_{f,h}^{\varepsilon}\|_0\|\nabla E_{\theta_f}^{n+1}\|_0 + \tilde{N}\|\nabla E_{u_p}^{n}\|_0\|\nabla\theta_{p,h}^{\varepsilon}\|_0\|\nabla E_{\theta_p}^{n+1}\|_0\\ &\quad + \frac{\alpha_f C_{in}}{2\gamma}\left\|\nabla E_{\theta_f}^{n+1}\right\|_0^2 + \frac{\alpha_f\gamma}{2h}\left\|E_{\theta_f}^{n+1} - E_{\theta_p}^{n+1}\right\|_\Gamma\\ &\le \frac{\alpha_f}{2}\left\|\nabla E_{\theta_f}^{n+1}\right\|_0^2 + \frac{\tilde{N}^2}{2\alpha_f}\|\nabla E_{u_f}^{n}\|_0^2\|\nabla\theta_{f,h}^{\varepsilon}\|_0^2 + \frac{\alpha_p}{2}\left\|\nabla E_{\theta_p}^{n+1}\right\|_0^2\\ &\quad + \frac{\tilde{N}^2}{2\alpha_p}\|\nabla E_{u_p}^{n}\|_0^2\|\nabla\theta_{p,h}^{\varepsilon}\|_0^2 + \frac{\alpha_f C_{in}}{2\gamma}\left\|\nabla E_{\theta_f}^{n+1}\right\|_0^2 + \frac{\alpha_f\gamma}{2h}\left\|E_{\theta_f}^{n+1} - E_{\theta_p}^{n+1}\right\|_\Gamma.\end{aligned} \tag{80}$$

Thanks to (62), we rearrange (80).

$$\begin{aligned}&\alpha_f(1 - \frac{C_{in}}{\gamma})\left\|\nabla E_{\theta_f}^{n+1}\right\|_0^2 + \alpha_p\left\|\nabla E_{\theta_p}^{n+1}\right\|_0^2 + \frac{\alpha_f\gamma}{h}\left\|E_{\theta_f}^{n+1} - E_{\theta_p}^{n+1}\right\|_\Gamma\\ &\le \frac{\tilde{N}^2}{\alpha_f}\mathcal{S}_\theta^2\|\nabla E_{u_f}^{n}\|_0^2 + \frac{\tilde{N}^2}{\alpha_p}\mathcal{S}_\theta^2\|\nabla E_{u_p}^{n}\|_0^2.\end{aligned} \tag{81}$$

Specially, when the stabilization parameter $\gamma \gg C_{in}$, we have

$$\alpha_f\left\|\nabla E_{\theta_f}^{n+1}\right\|_0^2 + \alpha_p\left\|\nabla E_{\theta_p}^{n+1}\right\|_0^2 \le \frac{\tilde{N}^2}{\alpha_f}\mathcal{S}_\theta^2\|\nabla E_{u_f}^{n}\|_0^2 + \frac{\tilde{N}^2}{\alpha_p}\mathcal{S}_\theta^2\|\nabla E_{u_p}^{n}\|_0^2. \tag{82}$$

For error equation on the velocity, we get by taking $v_f = E_{u_f}^{n+1}$ and $v_p = E_{u_p}^{n+1}$.

(83)

$$\begin{aligned}&\nu\left(\nabla E_{u_f}^{n+1}, \nabla E_{u_f}^{n+1}\right)_{\Omega_f} + c_f(E_{u_f}^{n}, u_{f,h}^{\varepsilon}, E_{u_f}^{n+1})_{\Omega_f} + \varepsilon\left(\nabla E_{u_p}^{n+1}, \nabla E_{u_p}^{n+1}\right)_{\Omega_p}\\ &+ \frac{\nu}{D_a}\left(E_{u_p}^{n+1}, E_{u_p}^{n+1}\right)_{\Omega_p} = G_r\nu^2\left(E_{\theta_f}^{n+1}\xi, E_{u_f}^{n+1}\right)_{\Omega_f} + G_r\nu^2\left(E_{\theta_p}^{n+1}\xi, E_{u_p}^{n+1}\right)_{\Omega_p}.\end{aligned}$$

Then, using the Cauchy-Schwarz inequality, Poincaré inequality, and Young's inequality, we have:

(84)

$$\begin{aligned}&\nu\|\nabla E_{u_f}^{n+1}\|_0^2 + \varepsilon\|\nabla E_{u_p}^{n+1}\|_0^2 + \frac{\nu}{D_a}\|E_{u_p}^{n+1}\|_0^2\\ &\le \frac{N^2}{\nu}\|\nabla E_{u_f}^{n}\|_0^2\|\nabla u_{f,h}^{\varepsilon}\|_0^2 + \frac{\nu}{4}\|\nabla E_{u_f}^{n+1}\|_0^2 + C_p^4G_r^2\nu^3\left\|\nabla E_{\theta_f}^{n+1}\right\|_0^2 + \frac{\nu}{4}\left\|\nabla E_{u_f}^{n+1}\right\|_0^2\\ &\quad + \frac{C_p^2G_r^2\nu^3 D_a}{2}\left\|\nabla E_{\theta_p}^{n+1}\right\|_0^2 + \frac{\nu}{2D_a}\left\|E_{u_p}^{n+1}\right\|_0^2.\end{aligned}$$

Then, based on (62) and (82), we arrive at

(85)

$$
\begin{aligned}
&\frac{\nu}{2}\|\nabla E_{u_f}^{n+1}\|_0^2+\varepsilon\|\nabla E_{u_p}^{n+1}\|_0^2+\frac{\nu}{2D_a}\|E_{u_p}^{n+1}\|_0^2\\
\le& N^2G_r^2\nu C_p^4\mathcal{S}_\theta^2\|\nabla E_{u_f}^n\|_0^2+G_r^2C_p^2\nu^3\max\{\frac{C_p^2}{\alpha_f},\frac{D_a}{2\alpha_p}\}(\alpha_f\left\|\nabla E_{\theta_f}^{n+1}\right\|_0^2+\alpha_p\left\|\nabla E_{\theta_p}^{n+1}\right\|_0^2)\\
\le&(2N^2G_r^2C_p^4\mathcal{S}_\theta^2+2G_r^2C_p^2\nu^2\max\{\frac{C_p^2}{\alpha_f},\frac{D_a}{2\alpha_p}\}\frac{\tilde{N}^2}{\alpha_f}\mathcal{S}_\theta^2)\times\frac{\nu}{2}\|\nabla E_{u_f}^n\|_0^2\\
&+2G_r^2C_p^2\nu^2\max\{\frac{C_p^2}{\alpha_f},\frac{D_a}{2\alpha_p}\}\frac{\tilde{N}^2}{\alpha_p}\mathcal{S}_\theta^2\times\frac{\nu}{2D_a}\|E_{u_p}^n\|_0^2.
\end{aligned}
$$

Therefore, if the iterative factor λ satisfies the condition in Theorem 5.2, then we obtain the desired result. □

6. Numerical experiments

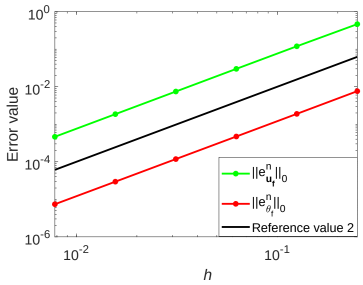
This section will present some numerical examples to demonstrate the accuracy of the decoupled iterative algorithm proposed in this paper. In the following test, the finite element spaces are chosen as the MINI element [2, 3] for the Navier-Stokes equations in the domain Ω_f. As [18], we choose the Brezzi-Douglas-Marini element for the Darcy velocity u_p and the piecewise constant element for the Darcy pressure p_p in the domain Ω_p. We use the linear Lagrangian element for the temperature in the whole domain Ω. Besides, we use FreeFem++ [10] to perform all the numerical experiments.

6.1. Example 1. The first example with an exact solution will show the convergence and error of the presented algorithm in Section 5. Consider the closed-loop geothermal model on the domain $\Omega=[0,1]\times[0,2]$, where $\Omega_p=[0,1]\times[0,1]$ and $\Omega_f=[0,1]\times[1,2]$. Choose $\alpha_f=\alpha_p=1$, $\nu=1$, $D_a=1$, $G_r=1$. The source terms are chosen such that the exact solution is

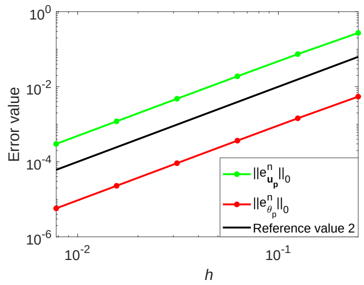
$$
\begin{aligned}
u_f&=\begin{pmatrix}10x^2(x-1)^2y(y-1)(2y-1)\\-10x(x-1)(2x-1)y^2(y-1)^2\end{pmatrix},\quad p_f=10(2x-1)(2y-1),\\
u_p&=\begin{pmatrix}2\pi\sin^2(\pi x)\sin(\pi y)\cos(\pi y)\\-2\pi\sin(\pi x)\sin^2(\pi y)\cos(\pi x)\end{pmatrix},\quad p_p=\cos(\pi x)\cos(\pi y),\\
\theta_f&=x(1-x)(1-y),\quad \theta_p=x(1-x)\left(y-y^2\right).
\end{aligned}
$$

We denote the errors $e_{u_\zeta}^n=u_{\zeta,h}^{\varepsilon,n}-u_\zeta$, $e_{\theta_\zeta}^n=\theta_{\zeta,h}^{\varepsilon,n}-\theta_\zeta$ and $e_{p_\zeta}^n=p_{\zeta,h}^{\varepsilon,n}-p_\zeta$, where $\zeta=f$ or p. In Figure 1, we plot errors and convergence rates of the velocities, pressures, and temperatures concerning fluid flow and Darcy flow. Here, we set the stabilization parameter $\gamma=1.0\mathrm{e}4$ and the penalty parameter $\varepsilon=1.0\mathrm{e}{-5}$. Besides, some mesh sizes are taken as $h=1/4,\ 1/8,\ 1/16,\ 1/32,\ 1/64$ and $1/128$. From this figure, we observe that Algorithm 5.1 provides the second-order accuracy for the velocities and temperatures in L^2-norm, and the first-order accuracy for the Darcy pressure in L^2-norm, for the velocities and temperatures in H^1-norm as expected. Meanwhile, we have an interesting observation that a half-an-order higher accuracy for the pressure on the fluid subdomain in L^2-norm is obtained, which is unsurprising since the MINI element has supercloseness [7].

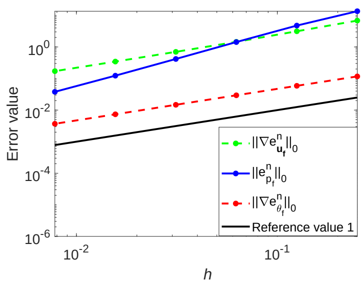
Next, we compare the penalty finite element method (Algorithm 5.1) with the Galerkin finite element method. In Table 1, the absolute errors of the Darcy velocity with different mesh sizes are listed. From this table, we can see that the presented method has better accuracy than the Galerkin finite element method.



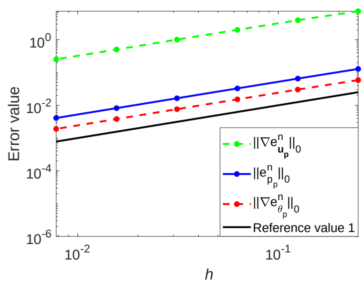
(a) L^2-norm of velocity and temperature on Ω_f.



(b) L^2-norm of velocity and temperature on Ω_p.



(c) H^1-norm of velocity and temperature and L^2-norm of pressure on Ω_f.



(d) H^1-norm of velocity and temperature and L^2-norm of pressure on Ω_p.

FIGURE 1. Convergence orders and errors of Algorithm 5.1.

TABLE 1. Comparison of errors obtained by the presented method and Galerkin finite element method.

h^{-1}	$\|u^{\varepsilon,n}_{p,h} - u_p\|_0$	$\|u^n_{p,h} - u_p\|_0$
4	2.71e–1	1.01
8	7.39e–1	5.47e–1
16	1.90e–2	2.79e–1
32	4.78e–3	1.41e–1
64	1.20e–3	7.04e–2
128	2.99e–4	3.52e–2

Further, we consider the effect of the stabilized parameter γ concerning the convergence performance. In Table 2, we list the errors of the temperatures with different values of the stabilized parameter. From this table, we find that a larger stabilized parameter can obtain better convergence performance. In fact, when $\gamma = 0$ (no stabilization term), the obtained numerical results yield bad convergence rate. For the increasing value of the stabilized parameter, Algorithm 5.1 works well and keeps good convergence rate.

Finally, we test Algorithm 5.1 with a small value of the kinetic viscosity. We consider $\nu = 5.0\text{e–}3$ and $D_a = 1.0\text{e–}3$. The results are shown in Table 3. The table

TABLE 2. The effect of the stabilization parameter γ on the convergence order.

h^{-1}	$\gamma = 0$	$\gamma = 1.0e{-}3$	$\gamma = 1.0e{-}2$	$\gamma = 1.0e{-}1$	$\gamma = 1.0$	$\gamma = 1.0e4$
			$\|e^n_{\theta_f}\|_0$			
16	8.61e–3	6.54e–3	3.92e–3	9.74e–4	4.52e–4	4.71e–4
32	6.41e–3	4.85e–3	1.69e–3	2.95e–4	1.14e–4	1.18e–4
64	6.10e–3	2.76e–3	6.36e–4	8.02e–5	2.92e–5	2.95e–5
128	5.96e–3	1.23e–3	1.91e–4	2.09e–5	7.44e–6	7.39e–6
order	0.03	1.16	1.73	1.94	1.97	1.99
			$\|e^n_{\theta_p}\|_0$			
16	6.94e–3	6.42e–3	3.88e–3	1.00e–3	4.07e–4	3.65e–4
32	6.40e–3	4.88e–3	1.74e–3	3.17e–4	1.02e–4	9.17e–5
64	6.13e–3	2.83e–3	6.49e–4	8.70e–5	2.49e–5	2.29e–5
128	5.99e–3	1.26e–3	1.93e–4	2.24e–5	5.97e–5	5.74e–6
order	0.03	1.17	1.74	1.95	2.05	1.99

shows that Algorithm 5.1 works well with small viscosity and keeps the convergence rates just like the theoretical results.

TABLE 3. The convergence performance of Algorithm 5.1 with $\nu = 5.0e{-}3$.

h^{-1}	$\|e^n_{u_f}\|_0$	$\|\nabla e^n_{u_f}\|_0$	$\|e^n_{\theta_f}\|_0$	$\|\nabla e^n_{\theta_f}\|_0$	$\|e^n_{p_f}\|_0$
16	3.88e–2	2.79	4.86e–4	2.94e–2	6.71e–2
32	9.50e–3	9.25e–1	1.21e–4	1.47e–2	1.73e–2
64	2.34e–3	3.76e–1	3.03e–5	7.36e–3	4.37e–3
128	5.83e-4	1.75e–1	7.57e–6	3.68e–3	1.09e–3
order	2.00	1.10	2.00	0.99	1.99
h^{-1}	$\|e^n_{u_p}\|_0$	$\|\nabla e^n_{u_p}\|_0$	$\|e^n_{\theta_p}\|_0$	$\|\nabla e^n_{\theta_p}\|_0$	$\|e^n_{p_p}\|_0$
16	1.89e–2	2.00	3.65e–4	1.51e–2	3.26e–2
32	4.78e–3	1.01	9.16e–5	7.60e–3	1.63e–2
64	1.21e–3	5.04e–1	2.29e–5	3.80e–3	8.18e–3
128	3.63e–4	2.52e–1	5.73e–6	1.90e–3	4.09e–3
order	1.74	0.99	2.00	0.99	0.99

6.2. Example 2. In this example, we will test the example employed in [16]. The example aims to test the numerical performance of Algorithm 5.1. As shown in Figure 2, the computational domain Ω is a unit square divided into the free flow domain $\Omega_f = \overline{ABCDEFGHIJ}$ and the porous media domain $\Omega_p = \Omega/\Omega_f$, the inflow boundaries $\partial\Omega_{in} = \overline{DE}$ and $\overline{HG}$, the outflow boundaries $\partial\Omega_{out} = \overline{AB}$ and $\overline{AJ}$ and the interface $\Gamma = \bar{\Omega}_f \cap \bar{\Omega}_p$. Besides, the model's parameters are chosen as $\nu = 1$, $D_a = 1.0e{-}5$, $G_r = 1$, $\alpha_f = 0.6$, and $\alpha_p = 0.9$. The heat sources $g_f = 0$ and $g_p = 0$. We will show temperature distribution obtained by Algorithm 5.1 with mesh size $h = 1/64$.

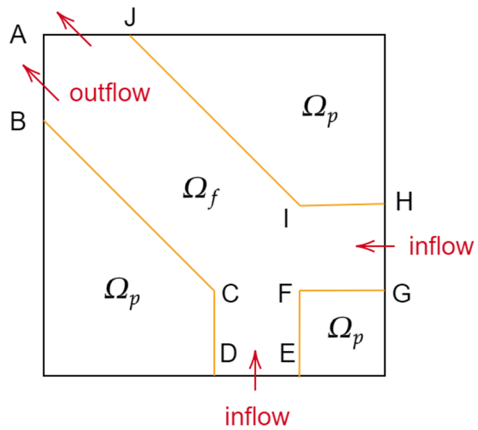
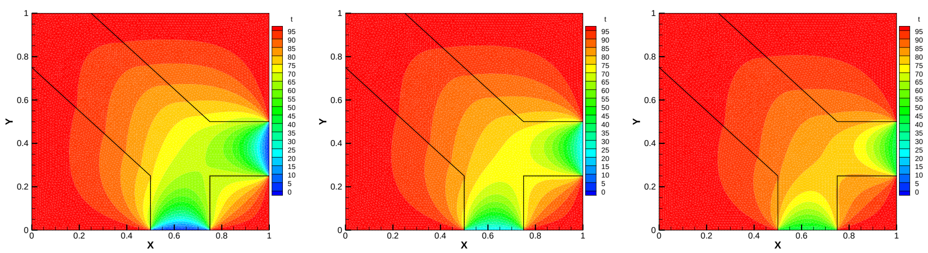


FIGURE 2. An illustration of the domain and interface. Also see [16].

Choose $u_f = 0$ on $\partial\Omega_f \backslash \Gamma$, $u_p \cdot n_p = 0$ on $\partial\Omega_p \backslash \Gamma$ and the velocity Dirichlet boundary conditions as follows:

$$u_f = \begin{cases} (-s_0, s_0) & \text{on } \overline{AB} \text{ and } \overline{JA}, \\ (-s_1, 0) & \text{on } \overline{HG}, \\ (0, s_1) & \text{on } \overline{DE}, \end{cases}$$

where s_1 and s_0 are two constants representing the total inflow and outflow rates, respectively. Here, we take $s_0 = 1$ and $s_1 = 1$. The boundary conditions for the temperatures $\theta_p = 100$ on $\partial\Omega_p \backslash \Gamma$, and $\theta_f = 0, 20, 40$ on $\partial\Omega_{in}$. In addition, we use the conditions (7)-(10) on the interface Γ.



(a) $\theta_f = 0$ (b) $\theta_f = 20$ (c) $\theta_f = 40$

FIGURE 3. Temperature distribution at different injection temperatures.

We change the temperature on $\partial\Omega_{in}$ to test the effect of different injection temperatures. In Figure 3, we see that the higher injection provides better production results in the closed-loop geothermal system.

Acknowledgement

We would like to thank the anonymous reviewers for their helpful comments and suggestions which lead to a considerably improved presentation. This work is supported by the Natural Science Foundation of Xinjiang Province (grant number 2021D01E11) and the Natural Science Foundation of China (grant number 11861067).

References

[1] R.A. Adams and J.J.F. Fournier, Sobolev Spaces, Academic Press, New York, 2003.
[2] D.N. Arnold, F. Brezzi, and M. Fortin, A stable finite element for the Stokes equations, Calcolo, 21:337–344, 1984.
[3] D. Boffi, F. Brezzi, and M. Fortin, Mixed Finite Element Methods and Applications, Springer, Berlin, 2013.
[4] B. Brefort, Attractors for the penalized Navier-Stokes equations, SIAM Journal on Mathematical Analysis, 19:1–21, 1988.
[5] M. Discacciati, Domain decomposition methods for the coupling of surface and groundwater flows, Ph.D. thesis, École Polytechnique Fédérale de Lausanne, Switzerland, 2004.
[6] B. Du, H. Su, and X. Feng, Two-level variational multiscale method based on the decoupling approach for the natural convection problem, International Communications in Heat and Mass Transfer, 61:128–139, 2015.
[7] H. Eichel, L. Tobiska, and H. Xie, Supercloseness and superconvergence of stabilized low-order finite element discretizations of the Stokes problem, Mathematics of computation, 80(274):697–722, 2011.
[8] M.A. Fernández, J.F. Gerbeau, and S. Smaldone, Explicit coupling schemes for a fluid-fluid interaction problem arising in hemodynamics, SIAM Journal on Scientific Computing, 36:A2557–A2583, 2014.
[9] Y. He, J. Li, and X. Yang, Two-level penalized finite element methods for the stationary Navier-Stoke equations, International Journal of Information and Systems Sciences, 2:131–143, 2006.
[10] F. Hecht, New development in FreeFem++, Journal of Numerical Mathematics, 20(3-4):251–266, 2012.
[11] P. Huang, Iterative methods in penalty finite element discretizations for the steady Navier-Stokes equations, Numerical Methods for Partial Differential Equations, 30:74–94, 2014.
[12] P. Huang, Y. He, and X. Feng, Convergence and stability of two-level penalty mixed finite element method for stationary Navier-Stokes equations, Frontiers of Mathematics in China, 8(4):837–854, 2013.
[13] P. Huang and X. Feng, Error estimates for two-level penalty finite volume method for the stationary Navier-Stokes equations, Mathematical Methods in the Applied Sciences, 36:1918–1928, 2013.
[14] H. Liu and P. Huang, A two-grid decoupled finite element method for the stationary closed-loop geothermal system, Journal of Applied Analysis and Computation, doi: 10.11948/20220258, 2023.
[15] H. Liu, P. Huang, and Y. He, Well-posedness and finite element approximation for the steady-state closed-loop geothermal system, Discrete and Continuous Dynamical Systems Series B, doi: 10.3934/dcdsb.2023064, 2023.
[16] X. Liu, R. Li, and Z. Chen, A virtual element method for the coupled Stokes–Darcy problem with the Beaver–Joseph–Saffman interface condition, Calcolo, 56(4):1–28, 2019.
[17] H. Ma and P. Huang, A vector penalty-projection approach for the time-dependent incompressible magnetohydrodynamics flows, Computers and Mathematics with Applications, 120: 28–44, 2022.
[18] M.A.A. Mahbub, X. He, N.J. Nasu, C. Qiu, Y. Wang, and H. Zheng, A coupled multiphysics model and a decoupled stabilized finite element method for the closed-loop geothermal system, SIAM Journal on Scientific Computing, 42(4):B951–B982, 2020.
[19] J.T. Oden and N. Kikuchi, Finite element methods for constrained problems in elasticity, International Journal for Numerical Methods in Engineering, 18(5):701–725, 1982.
[20] C.M. Oldenburg, L. Pan, M.P. Muir, A.D. Eastman, and B.S. Higgins, Numerical simulation of critical factors controlling heat extraction from geothermal systems using a closed-loop heat exchange method, Proceedings of the 41st Workshop on Geothermal Reservoir Engineering, Stanford University, Stanford, 2016.
[21] A. Riahi, P. Moncarz, W. Kolbe, and B. Damjanac, Innovative closed-loop geothermal well designs using water and super critical carbon dioxide as working fluids, Proceedings of the 42nd Workshop on Geothermal Reservoir Engineering, Stanford University, Stanford, 2017.

[22] J. Shen, On error estimates of some higher order projection and penalty-projection methods for Navier-Stokes equations, Numerische Mathematik, 62:49–73, 1992.

[23] J. Shen, On error estimates of the penalty method for unsteady Navier-Stokes equations, SIAM Journal on Numerical Analysis, 32:386–403, 1995.

[24] J.J. Valencia-López and J.A. Ochoa-Tapia, A study of buoyancy-driven flow in a confined fluid overlying a porous layer, International Journal of Heat and Mass Transfer, 44(24):4725–4736, 2001.

[25] J. Yu and Y. Zhang, Nitsche's type stabilization for the fully mixed Navier-Stokes/Darcy problem, Journal of Applied Analysis and Computation, 11(3):1481–1493, 2021.

[26] Y. Zhang, L. Shan, and Y. Hou, Well-posedness and finite element approximation for the convection model in superposed fluid and porous layers, SIAM Journal on Numerical Analysis, 58(1):541–564, 2020.

[27] T. Zhang, X. Zhao, and P. Huang, Decoupled two level finite element methods for the steady natural convection problem, Numerical Algorithms, 68(4):837–866, 2015.

College of Mathematics and System Sciences, Xinjiang University, Urumqi 830017, China
E-mail: liuhaochen@stu.xju.edu.cn and hpzh@xju.edu.cn

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China
E-mail: heyn@mail.xjtu.edu.cn