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WELL-POSEDNESS AND CONVERGENCE ANALYSIS OF A NONLOCAL MODEL WITH SINGULAR MATRIX KERNEL

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Abstract. In this paper, we consider a two-dimensional linear nonlocal model involving a singular matrix kernel. For the initial value problem, we first give well-posedness results and energy conservation via Fourier transform. Meanwhile, we also discuss the corresponding Dirichlet-type nonlocal boundary value problems in the cases of both positive and semi-positive definite kernels, where the core is the coercivity of bilinear forms. In addition, in the limit of vanishing nonlocality, the solution of the nonlocal model is seen to converge to a solution of its classical elasticity local model provided that $c_t = 0$.

Key words. Nonlocal model, well-posedness, convergence, singular matrix kernel, coercivity.

1. Introduction

In this paper, we consider a two-parameter nonlocal model as follows,

$$\mathbf{u}_{tt}(t,\mathbf{x}) = \mathcal{L}_\delta \mathbf{u}(t,\mathbf{x}) + \boldsymbol{b}(t,\mathbf{x}),\ (t,\mathbf{x}) \in (0,T)\times\mathcal{S}, \tag{1}$$

where the nonlocal integral operator $\mathcal{L}_\delta$ is given by

$$\begin{aligned}\mathcal{L}_\delta \mathbf{u}(t,\mathbf{x}) := \int_{\mathcal{S}\cup\Omega_\delta} \left(\frac{c_n(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} + \frac{c_t\big[(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})\big]^*}{|\mathbf{x}'-\mathbf{x}|^2}\right) \\ \times \big(\mathbf{u}(t,\mathbf{x}') - \mathbf{u}(t,\mathbf{x})\big)\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}',\end{aligned} \tag{2}$$

$\mathcal{S} \subseteq \mathbb{R}^2$ is an open domain ($\mathcal{S} = \Omega$ or $\mathcal{S} = \mathbb{R}^2$), $\Omega_\delta = \{\mathbf{x} \in \mathbb{R}^2 \setminus \Omega : \mathrm{dist}(\mathbf{x}, \partial\Omega) \le \delta\}$ is a collar domain surrounding a bounded open set $\Omega \subseteq \mathbb{R}^2$. $\mathbf{u} : (0,T)\times\mathcal{S}\cup\Omega_\delta$ represents displacement, and $\boldsymbol{b}$ is the external force density. c_n, c_t denote the tensile parameter and shear parameter, their expressions can be derived as

$$c_n = \frac{8E(1+\nu)}{\pi\delta^4(1-\nu^2)},\ c_t = \frac{8E(1-3\nu)}{\pi\delta^4(1-\nu^2)}, \tag{3}$$

here E, ν are the Young's modulus and Poisson's ratio, and we note that $0 \le \nu \le 1/3$. The horizon parameter δ characterizes the effective range of nonlocal interaction between the material point $\mathbf{x}'$ and point $\mathbf{x}$, and $\chi_\delta(\cdot)$ is the standard canonical function, i.e.,

$$\chi_\delta(\mathbf{x}'-\mathbf{x}) = \begin{cases} 1, & |\mathbf{x}'-\mathbf{x}| \le \delta, \\ 0, & \text{otherwise.}\end{cases}$$

In recent years, there have been lots of works done on nonlocal equation of the type (1) and relevant variational problems, including theory analysis [1, 2, 3], numerical methods [4, 5, 6, 23], model development and applications [8, 20, 29]. Regarding the well-posedness theory for equations similar to (1), Emmrich and Weckner [13, 14] proved the well-posedness of the initial problem on bounded domains by using

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semigroup theory of operators. In [17], the well-posedness of a scalar nonlocal evolution problem is obtained by utilizing properties of Neumann series and Volterra integral equations, where the boundary data is proposed on the classical boundary domain $\partial\Omega$. In particular, Du and Zhou [21] established the well-posedness results for a nonlocal initial problem in the Fourier space, which takes into account the non-integrable kernels. In addition, Aksoylu and Parks [15] considered scalar linear stationary nonlocal problems, and gave the well-posedness results, the key step is to utilize domain decomposition methods to prove the coercivity, (see also [16] for a similar discussion). More generally, Mengesha and Du [12] proved the well-posedness for a nonlinear stationary nonlocal problem based on variational methods. We refer to [10, 17, 18] for an exhaustive introduction of well-posedness results.

On the other hand, observe that δ acts as a bridge between nonlocal models and the corresponding local models, so the study of reduction of nonlocal models to local models in the limit of $\delta \to 0$ has attracted much attention. In [10], the authors proved that the nonlocal integral operator applied to smooth functions converges asymptotically to the corresponding classical differential operator by using using Taylor expansion. In particular, based on Fourier transform, Mikata [11] analyzed the limit behaviors of solutions for a kinds of peristatic and peridynamic nonlocal problems, where solutions of these nonlocal equations approach solutions of the corresponding local equations with horizon vanishes. More results can be found in [7, 9, 19, 21] and references therein.

Inspired by the above papers [10, 15, 21], we will prove the well-posedness and convergence results for the initial and stationary cases of equation (1), which are the focus of our paper. For the well-posedness results of stationary nonlocal problems, the coercivity of bilinear forms is ensured by using relative compactness and some key inequalities. In particular, we don't rely on the proof in [10, 21] for convergence results as $\delta \to 0$, but made some modifications, and introduce some other techniques.

This paper is organized as follows. In Section 2, for the initial value problem associated to equation (1), we prove the well-posedness results and energy conservation via Fourier transform. In Section 3, for the corresponding Dirichlet-type nonlocal boundary problems, the well-posedness results of solution are established in the cases of positive definite kernel and semi-positive definite kernel. In Section 4, we shall analyze the limit behaviors of solutions of nonlocal problems as $\delta \to 0$. Finally, we complete the paper with an appendix.

Notation 1.1. Throughout the paper, we will denote various generic positive constants by the same letter C, although the constants may differ from line to line. Moreover, relevant dependencies on parameters will be emphasized using parentheses, i.e., $C \equiv C(T, \delta)$ means that C depends only on T, δ. The notation $\otimes$ denotes the dyadic product and $[(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})]^* = \mathbb{I}-(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})$, here $\mathbb{I}$ is a second order identity matrix. $(\cdot,\cdot)$ is the inner product defined as $(\mathbf{u},\mathbf{v}) = \int_{\mathbb{R}^2} \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})d\mathbf{x}$. M is a finite number.

To achieve our main results, let us first give a brief review of model equation (1). Equation (1) can be deduced from the following two-parameter nonlocal peridynamic model by a series of simplification,

$$\mathbf{u}_{tt}(t,\mathbf{x}) = \int_{B_\delta(\mathbf{x})} (c_n\eta_n\hat{\mathbf{e}}_n + c_t\eta_t\hat{\mathbf{e}}_t)d\mathbf{x}' + \boldsymbol{b}(t,\mathbf{x}), (t,\mathbf{x}) \in (0,T)\times\mathcal{S}. \tag{4}$$

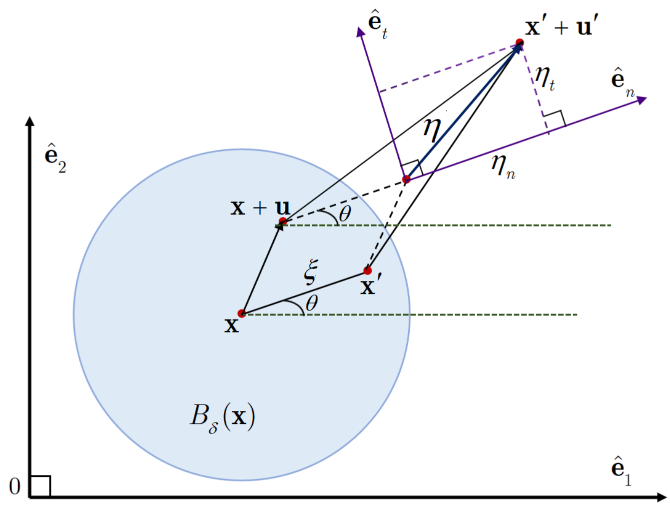

FIGURE 1. The components within the peridynamic horizon.

We write $\boldsymbol{\eta} := \mathbf{u}(\mathbf{x}', t) - \mathbf{u}(\mathbf{x}, t)$ and $\boldsymbol{\xi} := \mathbf{x}' - \mathbf{x}$, which denote the relative displacement and bond. η_n and η_t are the components of $\boldsymbol{\eta}$ along the normal direction $\hat{\mathbf{e}}_n$ and tangential direction $\hat{\mathbf{e}}_t$, respectively. Here $\hat{\mathbf{e}}_n$ is in the bond $\boldsymbol{\xi}$ while $\hat{\mathbf{e}}_t$ is orthogonal to $\hat{\mathbf{e}}_n$. The integrand term $c_n\eta_n\hat{\mathbf{e}}_n + c_t\eta_t\hat{\mathbf{e}}_t$ is the force density function that characterizes the interaction between the material point $\mathbf{x}'$ and point $\mathbf{x}$, and $B_\delta(\mathbf{x})$ is the ball centered at $\mathbf{x} \in \mathcal{S}$ with radius δ. The mentioned components above are illustrated in Figure 1. In fact, equation (4) was first proposed in [22] to expand the range of Poisson's ratios for bond-based nonlocal peridynamic models, and retain the simplicity of the bond-based models in form. More precisely, it is suited to model the materials whose Poisson's ratio belongs to the interval $[0, \frac{1}{3}]$.

Lemma 1.2. *The nonlocal integral operator* $-\mathcal{L}_\delta : L^2(\mathcal{S} \cup \Omega_\delta) \to L^2(\mathcal{S} \cup \Omega_\delta)$ *is self-adjoint, bounded and nonnegative.*

Proof. It's easy to see that the linearity of $\mathcal{L}_\delta$ from (2). Then for simplicity, we write $\mathbf{u}(t, \mathbf{x})$ as $\mathbf{u}(\mathbf{x})$ and rewrite the kernel function $\mathbf{k}(\cdot)$ as

$$
\begin{aligned}
\mathbf{k}(\mathbf{x}' - \mathbf{x}) &= c_n \frac{(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} + c_t \frac{\left[(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x})\right]^*}{|\mathbf{x}' - \mathbf{x}|^2} \\
&=: \begin{pmatrix} c_n \cos^2\theta + c_t \sin^2\theta & (c_n - c_t)\cos\theta\sin\theta \\ (c_n - c_t)\cos\theta\sin\theta & c_n \sin^2\theta + c_t \cos^2\theta \end{pmatrix}, \ \theta \in [0, 2\pi].
\end{aligned} \tag{5}
$$

Then c_n, c_t are two eigenvalues of the symmetric matrix function $\mathbf{k}(\cdot)$ exactly, thus there exists an orthogonal matrix function $\mathbf{O}$, such that

$$
\mathbf{k}(\mathbf{x}' - \mathbf{x}) = \mathbf{O}^{\mathrm{T}} \begin{pmatrix} c_n & 0 \\ 0 & c_t \end{pmatrix} \mathbf{O}. \tag{6}
$$

For any $\mathbf{u}(\mathbf{x}), \mathbf{v}(\mathbf{x}) \in L^2(\mathcal{S} \cup \Omega_\delta)$, inserting (5)-(6) into (2) yields

$$
\big(-\mathcal{L}_\delta \mathbf{u}(\mathbf{x}), \mathbf{v}(\mathbf{x})\big) = \big(\mathbf{u}(\mathbf{x}), -\mathcal{L}_\delta \mathbf{v}(\mathbf{x})\big), \tag{7a}
$$

$$
\big(-\mathcal{L}_\delta \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{x})\big) \geq 0. \tag{7b}
$$

By Hölder inequality, we have

$$
\begin{aligned}
&\big(-\mathcal{L}_\delta \mathbf{u}(\mathbf{x}), \mathbf{v}(\mathbf{x})\big) \\
\le&\frac{1}{2}\bigg(\int_{\mathcal{S}\cup\Omega_\delta}\int_{\mathcal{S}\cup\Omega_\delta} (\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))^{\mathrm{T}}\mathbf{O}^{\mathrm{T}}\begin{pmatrix} c_n & 0 \\ 0 & c_t \end{pmatrix}\mathbf{O}(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}'d\mathbf{x}\bigg)^{\frac{1}{2}} \\
\times&\bigg(\int_{\mathcal{S}\cup\Omega_\delta}\int_{\mathcal{S}\cup\Omega_\delta} (\mathbf{v}(\mathbf{x}')-\mathbf{v}(\mathbf{x}))^{\mathrm{T}}\mathbf{O}^{\mathrm{T}}\begin{pmatrix} c_n & 0 \\ 0 & c_t \end{pmatrix}\mathbf{O}(\mathbf{v}(\mathbf{x}')-\mathbf{v}(\mathbf{x}))\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}'d\mathbf{x}\bigg)^{\frac{1}{2}} \\
\le&\frac{1}{2}\bigg(\int_{\mathcal{S}\cup\Omega_\delta}\int_{\mathcal{S}\cup\Omega_\delta} c_n|\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x})|^2\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}'d\mathbf{x}\bigg)^{1/2} \\
\times&\bigg(\int_{\mathcal{S}\cup\Omega_\delta}\int_{\mathcal{S}\cup\Omega_\delta} c_n|\mathbf{v}(\mathbf{x}')-\mathbf{v}(\mathbf{x})|^2\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}'d\mathbf{x}\bigg)^{1/2} \\
\le&c_n\bigg(\int_{\mathcal{S}\cup\Omega_\delta}\int_{\mathcal{S}\cup\Omega_\delta} (\mathbf{u}^2(\mathbf{x}')+\mathbf{u}^2(\mathbf{x}))\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}'d\mathbf{x}\bigg)^{1/2} \\
\times&\bigg(\int_{\mathcal{S}\cup\Omega_\delta}\int_{\mathcal{S}\cup\Omega_\delta} (\mathbf{v}^2(\mathbf{x}')+\mathbf{v}^2(\mathbf{x}))\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}'d\mathbf{x}\bigg)^{1/2} \\
\le&2\pi c_n\delta^2\|\mathbf{u}\|_{L^2(\mathcal{S}\cup\Omega_\delta)}\|\mathbf{v}\|_{L^2(\mathcal{S}\cup\Omega_\delta)},
\end{aligned}
\tag{8}
$$

where we have used the fact that $\int_{\mathcal{S}\cup\Omega_\delta}\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}' \le \pi\delta^2$, $c_n \ge c_t$ and symmetry. This completes the proof.

□

2. Initial value problem

In this section, we consider the following initial value problem,

$$\mathbf{u}_{tt}(t,\mathbf{x}) = \mathcal{L}_\delta \mathbf{u}(t,\mathbf{x}) + \boldsymbol{b}(t,\mathbf{x}), \qquad (t,\mathbf{x})\in(0,T)\times\mathbb{R}^2, \tag{9a}$$

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}),\ \mathbf{u}_t(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^2. \tag{9b}$$

Observe that functions on the whole space $\mathbb{R}^2$ is suitable to take Fourier transform, then we prove the existence results of problem (9a)-(9b) via Fourier transform.

Theorem 2.1. *Let* $\mathbf{u}_0(\mathbf{x})\in L^2(\mathbb{R}^2), \mathbf{v}_0(\mathbf{x})\in L^2(\mathbb{R}^2), \boldsymbol{b}(t,\mathbf{x})\in L^2(0,T;L^2(\mathbb{R}^2))$*, then for a given finite* $T>0$*, there exists a solution* $\mathbf{u}(t,\mathbf{x})\in\mathcal{C}^2(0,T;L^2(\mathbb{R}^2))$ *to the initial value problem (9a)-(9b).*

Proof. Applying Fourier transform to (9a)-(9b) with respect to space variable $\mathbf{x}$, it follows that

$$\begin{cases} \bar{\mathbf{u}}_{tt}(t,\boldsymbol{\omega}) + \mathbf{A}_\delta(\boldsymbol{\omega})\bar{\mathbf{u}}(t,\boldsymbol{\omega}) = \bar{\boldsymbol{b}}(t,\boldsymbol{\omega}), & (t,\boldsymbol{\omega})\in(0,T)\times\mathbb{R}^2, \\ \bar{\mathbf{u}}(0,\boldsymbol{\omega}) = \bar{\mathbf{u}}_0(\boldsymbol{\omega}),\ \bar{\mathbf{u}}_t(0,\boldsymbol{\omega}) = \bar{\mathbf{v}}_0(\boldsymbol{\omega}), & \boldsymbol{\omega}\in\mathbb{R}^2, \end{cases} \tag{10}$$

where

$$\mathbf{A}_\delta(\boldsymbol{\omega}) := c_n\int_{B_{\delta(\mathbf{0})}}\frac{(1-\cos(\boldsymbol{\xi},\boldsymbol{\omega}))}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi}\otimes\boldsymbol{\xi}d\boldsymbol{\xi} + c_t\int_{B_{\delta(\mathbf{0})}}\frac{(1-\cos(\boldsymbol{\xi},\boldsymbol{\omega}))}{|\boldsymbol{\xi}|^2}\big[\boldsymbol{\xi}\otimes\boldsymbol{\xi}\big]^*d\boldsymbol{\xi}$$

is the Fourier symbol of the nonlocal operator $-\mathcal{L}_\delta$, and

$$\boldsymbol{\omega} = (\omega_1,\omega_2)\in\mathbb{R}^2,\ \boldsymbol{\xi} = (\xi_1,\xi_2)\in\mathbb{R}^2. \tag{11}$$

Directly from Duhamel's principle, we have

$$\begin{aligned}\bar{\mathbf{u}}(t,\boldsymbol{\omega}) &= \cos(\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}t)\bar{\mathbf{u}}_0(\boldsymbol{\omega}) + \frac{\sin(\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}t)}{\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}}\bar{\mathbf{v}}_0(\boldsymbol{\omega}) \\ &\quad + \int_0^t \frac{\sin(\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}\tau)}{\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}}\bar{\boldsymbol{b}}(t-\tau,\boldsymbol{\omega})d\tau.\end{aligned} \tag{12}$$

Using the convolution property and the inverse Fourier transform yields an integral expression of the solution, i.e.,

$$\begin{aligned}\mathbf{u}(t,\mathbf{x}) &= \int_{\mathbb{R}^2} \mathbf{u}_0(\mathbf{x}-\boldsymbol{\xi})g_t(t,\boldsymbol{\xi})d\boldsymbol{\xi} + \int_{\mathbb{R}^2} \mathbf{v}_0(\mathbf{x}-\boldsymbol{\xi})g(t,\boldsymbol{\xi})d\boldsymbol{\xi} \\ &\quad + \int_0^t \int_{\mathbb{R}^2} \boldsymbol{b}(t-\tau,\mathbf{x}-\boldsymbol{\xi})g(t,\boldsymbol{\xi})d\boldsymbol{\xi}d\tau,\end{aligned} \tag{13}$$

here

$$g(t,\boldsymbol{\xi}) := \mathcal{F}^{-1}\left\{\frac{\sin(\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}t)}{\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}}\right\} = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\sin(\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}t)}{\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}} e^{i(\boldsymbol{\xi},\boldsymbol{\omega})}d\boldsymbol{\omega}$$

is a Green's function.

Consequently, $\mathbf{u} \in \mathcal{C}(0,T;L^2(\mathbb{R}^2))$. From (9a), we obtain $\mathbf{u} \in \mathcal{C}^2(0,T;L^2(\mathbb{R}^2))$. □

Remark 2.2. *Eigenvalues of the matrix function $\mathbf{A}_\delta(\boldsymbol{\omega})$ are positive (see the Appendix for details), which guarantees that $\frac{1}{\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}}$ is well-defined. Moreover, the matrix functions*

$$\begin{aligned}\cos(\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}t) &:= \sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}t)^{2n}}{(2n)!}, \\ \sin(\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}t) &:= \sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}t)^{2n+1}}{(2n+1)!}.\end{aligned}$$

Remark 2.3. *Physically, nonlocal peridynamic framework permits the discontinuity of the solution with respect to the space variable* $\mathbf{x}$*, which matches with the existence results of solution in* $L^2(\mathbb{R}^2)$.

Proposition 2.4. Under the conditions of Theorem 2.1, then for any $t \in (0,T)$, the solution obtained above is unique in $\mathcal{C}^2(0,T;L^2(\mathbb{R}^2))$, and satisfies

$$\|\mathbf{u}(t)\|_{L^2(\mathbb{R}^2)} \le C(T,\frac{1}{\delta})\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{R}^2)} + \|\mathbf{v}_0\|_{L^2(\mathbb{R}^2)} + \|\boldsymbol{b}\|_{L^2(0,T;L^2(\mathbb{R}^2))}\Big). \tag{14}$$

Proof. To shorten notation, we suppress $\mathbf{x}$ and write $\mathbf{u}(t)$ instead of $\mathbf{u}(t,\mathbf{x})$. Multiplying both sides of (9a) by $\mathbf{u}_t(t)$ and integrating over $\mathbb{R}^2$ in space variable $\mathbf{x}$, then by Hölder inequality, we have

$$\begin{aligned}&\frac{d}{dt}\Big\{\|\mathbf{u}_t(t)\|^2_{L^2(\mathbb{R}^2)} + \big(-\mathcal{L}_\delta\mathbf{u}(t),\mathbf{u}(t)\big)_{L^2(\mathbb{R}^2)}\Big\} \\ \le&\|\mathbf{u}_t(t)\|^2_{L^2(\mathbb{R}^2)} + \|\boldsymbol{b}(t)\|^2_{L^2(\mathbb{R}^2)} \\ \le&\|\mathbf{u}_t(t)\|^2_{L^2(\mathbb{R}^2)} + \big(-\mathcal{L}_\delta\mathbf{u}(t),\mathbf{u}(t)\big)_{L^2(\mathbb{R}^2)} + \|\boldsymbol{b}(t)\|^2_{L^2(\mathbb{R}^2)},\end{aligned} \tag{15}$$

where we have used the fact that $\big(-\mathcal{L}_\delta \mathbf{u}(t), \mathbf{u}(t)\big)_{L^2(\mathbb{R}^2)} \geq 0$. Let $\varphi(t) := \|\mathbf{u}_t(t)\|^2_{L^2(\mathbb{R}^2)} + \big(-\mathcal{L}_\delta \mathbf{u}(t), \mathbf{u}(t)\big)_{L^2(\mathbb{R}^2)}$, then (15) becomes

$$\frac{d}{dt}\varphi(t) \leq \varphi(t) + \|\boldsymbol{b}(t)\|^2_{L^2(\mathbb{R}^2)},$$

using Gronwall's inequality, we obtain

$$\varphi(t) \leq e^{\int_0^t 1ds}\left[\varphi(0) + \int_0^t \|\boldsymbol{b}\|^2_{L^2(\mathbb{R}^2)} ds\right], \forall t \in [0,T].$$

Thus,

$$\begin{aligned}\max_{0\leq t\leq T}\varphi(t) = &\max_{0\leq t\leq T}\Big\{\|\mathbf{u}_t(t)\|^2_{L^2(\mathbb{R}^2)} + (-\mathcal{L}_\delta\mathbf{u}(t),\mathbf{u}(t))_{L^2(\mathbb{R}^2)}\Big\}\\ \leq & C(T)\Big(\|\boldsymbol{b}\|^2_{L^2(0,T;L^2(\mathbb{R}^2))} + \|\mathbf{v}_0\|^2_{L^2(\mathbb{R}^2)} + (-\mathcal{L}_\delta \mathbf{u}_0, \mathbf{u}_0)_{L^2(\mathbb{R}^2)}\Big).\end{aligned} \tag{16}$$

Assume that $\mathbf{u}, \mathbf{v} \in \mathcal{C}^2(0,T; L^2(\mathbb{R}^2))$ are the solutions of (9a)-(9b) with $\mathbf{u} \neq \mathbf{v}$. Let $\mathbf{w} = \mathbf{u} - \mathbf{v}$, then $\mathbf{w}$ satisfies

$$\begin{cases}\mathbf{w}_{tt}(t,\mathbf{x}) = \mathcal{L}_\delta \mathbf{w}(t,\mathbf{x}), & (t,\mathbf{x}) \in (0,T)\times\mathbb{R}^2,\\ \mathbf{w}(0,\mathbf{x}) = \mathbf{0},\ \mathbf{w}_t(0,\mathbf{x}) = \mathbf{0}, & \mathbf{x}\in\mathbb{R}^2.\end{cases}$$

From (16), we can deduce that

$$\|\mathbf{w}_t(t)\|^2_{L^2(\mathbb{R}^2)} + \big(-\mathcal{L}_\delta \mathbf{w}(t), \mathbf{w}(t)\big)_{L^2(\mathbb{R}^2)} = 0.$$

Observe that $\big(-\mathcal{L}_\delta \mathbf{w}(t), \mathbf{w}(t)\big)_{L^2(\mathbb{R}^2)} \geq 0$, then we have $\|\mathbf{w}_t(t)\|_{L^2(\mathbb{R}^2)} = 0$, and so

$$\|\mathbf{w}(t)\|_{L^2(\mathbb{R}^2)} = \|\mathbf{w}(t) - \mathbf{w}(0)\|_{L^2(\mathbb{R}^2)} \leq \int_0^t \|\mathbf{w}_s(s)\|_{L^2(\mathbb{R}^2)} ds = 0.$$

Thus $\mathbf{w}(t) = \mathbf{0}$ for all $t \in (0,T)$, i.e., $\mathbf{u} = \mathbf{v}$. This implies that the uniqueness of the solution. Due to

$$(-\mathcal{L}_\delta \mathbf{u}_0, \mathbf{u}_0) \leq 2\pi c_n \delta^2 \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^2)} \leq \frac{16E(1+\nu)}{(1-\nu)\delta^2}\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^2)}, \tag{17}$$

together with calculus basic formulas, we can obtain the desired the estimate (14) with the constant $C \equiv C(T, \frac{1}{\delta})$. □

Observe that $\mathbf{u}_t \in \mathcal{C}(0,T; L^2(\mathbb{R}^2))$, then we have an immediate corollary as follows.

Corollary 2.5. The solution $\mathbf{u}(t,\mathbf{x})$ of problem (9a)-(9b) is uniformly Lipschitz in time.

Proof. Directly from (16), for any $t_1, t_2 \in (0,T)$, we can deduce that

$$\|\mathbf{u}(t_1) - \mathbf{u}(t_2)\|_{L^2(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2}\Big|\int_{t_2}^{t_1}\mathbf{u}_t(\tau)d\tau\Big|^2 d\mathbf{x}\right)^{1/2} \leq C(T, \frac{1}{\delta})|t_1 - t_2|. \tag{18}$$

□

Remark 2.6. *For the spatial domain $\mathcal{S} = \mathbb{R}^2$, applying the Fourier transform is a natural choice. But for the spatial domain $\mathcal{S} = \Omega$, extending the solution by* $\mathbf{0}$ *outside Ω, that is,*

$$\tilde{\mathbf{u}}(\mathbf{x}) = \begin{cases}\mathbf{u}(\mathbf{x}), & \mathbf{x}\in\Omega,\\ \mathbf{0}, & \mathbf{x}\in\mathbb{R}^2\setminus\Omega,\end{cases}$$

it is impossible to extend the above developed results to study the problem on a finite domain by the extension method, since we don't know the right-hand side of the equation $-\mathcal{L}_\delta \tilde{\mathbf{u}}(\mathbf{x}) = \tilde{\mathbf{b}}(\mathbf{x})$ *for* $\mathbf{x} \in \mathbb{R}^2 \setminus \Omega$*, then the existence of its solution can't be guaranteed necessarily.*

Energy conservation usually plays a key role in the study of properties of the solution. Next, we give the conditions of energy conservation for initial problem (9a)-(9b).

Theorem 2.7. *Under the conditions of Theorem 2.1, then the total energy* $\Phi(t)$ *is a constant in* $[0, T]$ *provided* $\boldsymbol{b}(t, \mathbf{x}) = \boldsymbol{b}(\mathbf{x})$*, where* $\Phi(t)$ *contains potential energy and kinetic energy, i.e.,*

$$\Phi(t) := \frac{1}{2}\|\mathbf{u}_t(t)\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}\big(-\mathcal{L}_\delta \mathbf{u}(t), \mathbf{u}(t)\big) - \int_{\mathbb{R}^2} \boldsymbol{b}(t, \mathbf{x})\mathbf{u}(t, \mathbf{x})d\mathbf{x}. \tag{19}$$

Proof. Multiplying both sides of (9a) by $\mathbf{u}_t(t)$ and integrating over space $\mathbb{R}^2$ with respect to the variable $\mathbf{x}$ gives

$$\frac{1}{2}\frac{d}{dt}\Big\{\|\mathbf{u}_t(t)\|^2_{L^2(\mathbb{R}^2)} + \big(-\mathcal{L}_\delta \mathbf{u}(t), \mathbf{u}(t)\big)\Big\} = \int_{\mathbb{R}^2} \boldsymbol{b}(t)\mathbf{u}_t(t)d\mathbf{x}. \tag{20}$$

Integrating over time from 0 to t and integrating by parts, we have

$$\begin{aligned} \frac{d\Phi(t)}{dt} &= \int_{\mathbb{R}^2} \mathbf{u}_t(t)\mathbf{u}_{tt}(t)d\mathbf{x} - \int_{\mathbb{R}^2} \mathbf{u}(t)\boldsymbol{b}_t(t)d\mathbf{x} - \int_{\mathbb{R}^2} \mathbf{u}_t(t)\boldsymbol{b}(t)d\mathbf{x} + \big(-\mathcal{L}_\delta \mathbf{u}(t), \mathbf{u}_t(t)\big) \\ &= -\int_{\mathbb{R}^2} \mathbf{u}(t)\boldsymbol{b}_t(t)d\mathbf{x}, \end{aligned} \tag{21}$$

it follows that the total energy $\Phi(t)$ is conserved if the external force does not change with the time, i.e., $\boldsymbol{b}(\mathbf{x}, t) = \boldsymbol{b}(\mathbf{x})$. □

3. Stationary problems

In this section, we assume $\mathcal{S} = \Omega$ and consider nonlocal Dirichlet problems associated to equation (1).

3.1. Positive definite kernel ($c_t > 0$). The nonhomogeneous Dirichlet-type nonlocal boundary value problem is posed as follows,

$$\begin{cases} -\mathcal{L}_\delta \mathbf{u}(\mathbf{x}) = \boldsymbol{b}(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \mathbf{u}(\mathbf{x}) = \mathbf{g}(\mathbf{x}), & \mathbf{x} \in \Omega_\delta. \end{cases} \tag{22}$$

We set

$$\tilde{\mathbf{g}}(\mathbf{x}) := \begin{cases} \mathbf{0}, & \mathbf{x} \in \Omega, \\ \mathbf{g}(\mathbf{x}), & \mathbf{x} \in \Omega_\delta, \end{cases} \quad \text{and} \quad \tilde{\mathbf{u}}(\mathbf{x}) := \mathbf{u}(\mathbf{x}) - \tilde{\mathbf{g}}(\mathbf{x}). \tag{23}$$

Then nonlocal problem (22) can be reformulated as

$$\begin{cases} -\mathcal{L}_\delta \tilde{\mathbf{u}}(\mathbf{x}) = \tilde{\boldsymbol{b}}(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \tilde{\mathbf{u}}(\mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \Omega_\delta, \end{cases} \tag{24}$$

where

$$\tilde{\boldsymbol{b}}(\mathbf{x}) =: \boldsymbol{b}(\mathbf{x}) + \int_{\Omega_\delta} \mathbf{k}(\mathbf{x}' - \mathbf{x})\boldsymbol{g}(\mathbf{x}')\chi_\delta(\mathbf{x}' - \mathbf{x})d\mathbf{x}', \tag{25}$$

and $\mathbf{k}(\mathbf{x}' - \mathbf{x})$ is defined as (5).

The weak formulation of (24) is given by

$$\begin{cases} \text{given } \tilde{\boldsymbol{b}} \in L^2(\Omega), \text{ seek } \tilde{\mathbf{u}} \in L^2(\Omega \cup \Omega_\delta), \\ \text{such that } \tilde{\mathbf{u}} = \mathbf{0} \text{ for } \mathbf{x} \in \Omega_\delta \text{ and } A_\delta(\tilde{\mathbf{u}}, \mathbf{v}) = F(\mathbf{v}), \ \forall \mathbf{v} \in \mathcal{V}_\delta. \end{cases} \tag{26}$$

Here $A_\delta(\cdot,\cdot)$ denotes the bilinear form, i.e.,

(27)

$$A_\delta(\tilde{\mathbf{u}}, \mathbf{v}) := \frac{1}{2}\int_{\Omega\cup\Omega_\delta}\int_{\Omega\cup\Omega_\delta} \big(\tilde{\mathbf{u}}(\mathbf{x}')-\tilde{\mathbf{u}}(\mathbf{x})\big)^{\mathrm{T}}\mathbf{k}(\mathbf{x}'-\mathbf{x})\big(\mathbf{v}(\mathbf{x}')-\mathbf{v}(\mathbf{x})\big)\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}'d\mathbf{x}.$$

$$\mathcal{V}_\delta := \Big\{\mathbf{v} \in L^2(\Omega \cup \Omega_\delta) : \mathbf{v}|_{\Omega_\delta} = \mathbf{0}\Big\} \text{ and } F(\mathbf{v}) := \textstyle\int_\Omega \tilde{\boldsymbol{b}}(\mathbf{x})\mathbf{v}(\mathbf{x})d\mathbf{x}.$$

Theorem 3.1. *Assume that $\boldsymbol{b}(\mathbf{x}) \in L^2(\Omega)$ and $\mathbf{g}(\mathbf{x}) \in L^2(\Omega_\delta)$, then there exists a unique solution $\mathbf{u}(\mathbf{x}) \in L^2(\Omega \cup \Omega_\delta)$ to problem (22), and the estimate*

$$\|\mathbf{u}\|_{L^2(\Omega\cup\Omega_d)} \le C\Big(\|\boldsymbol{b}\|_{L^2(\Omega)} + \|\mathbf{g}\|_{L^2(\Omega_\delta)}\Big) \tag{28}$$

holds for some constant $C = C(l, \delta) > 0$.

Proof. By Hölder inequality, we have

$$|F(\mathbf{v})| \le \|\boldsymbol{b}\|_{L^2(\Omega)}\|\mathbf{v}\|_{L^2(\Omega)} + c_n\pi\delta^2\|\mathbf{g}\|_{L^2(\Omega_\delta)}\|\mathbf{v}\|_{L^2(\Omega_\delta)}. \tag{29}$$

From Lemma 1.2, we can see that $\big(-\mathcal{L}_\delta\tilde{\mathbf{u}}(\mathbf{x}), \mathbf{v}(\mathbf{x})\big) = A_\delta(\tilde{\mathbf{u}}, \mathbf{v})$, and so $A_\delta : L^2(\Omega \cup \Omega_\delta) \times L^2(\Omega \cup \Omega_\delta) \to \mathbb{R}$ is a symmetric, bounded bilinear form.

The remaining proof is an adaptation of the the process of [24] dealing with the nonlocal p-Laplace evolution problem, we present the proof for completeness. Let us construct a finite number of non-empty sets $\mathcal{B}_j$ as follows,

$$\begin{cases} \mathcal{B}_{-1} = \big\{\mathbf{x} \in \Omega_\delta : \frac{\delta}{2} \le d(\mathbf{x}, \partial\Omega) \le \delta\big\}, \\ \mathcal{B}_0 = \big\{\mathbf{x} \in \Omega_\delta : d(\mathbf{x}, \partial\Omega) \le \frac{\delta}{2}\big\}, \\ \mathcal{B}_1 = \big\{\mathbf{x} \in \Omega : d(\mathbf{x}, \mathcal{B}_0) \le \frac{\delta}{2}\big\}, \\ \mathcal{B}_2 = \big\{\mathbf{x} \in \Omega : d(\mathbf{x}, \mathcal{B}_1) \le \frac{\delta}{2}\big\}, \\ \mathcal{B}_j = \big\{\mathbf{x} \in \Omega \setminus \bigcup_{k=1}^{j-1} \mathcal{B}_k : d(\mathbf{x}, \mathcal{B}_{j-1}) \le \frac{\delta}{2}\big\}, j = 2, 3, ...l, \end{cases} \tag{30}$$

where $\mathcal{B}_{j-1} \cap \mathcal{B}_j = \emptyset$ and $\mathcal{B}_{j-1} \cap B_\delta(\mathbf{x}) \ne \emptyset$ for $\mathbf{x} \in \bar{\mathcal{B}}_j (j = 1, 2, ...l.)$, l is a finite integer, see Figure 2 for details. Then we have

(31)

$$\begin{aligned} A_\delta(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) \ge& \frac{c_t}{2}\int_{\Omega\cup\Omega_\delta}\int_{\Omega\cup\Omega_\delta} |\tilde{\mathbf{u}}(\mathbf{x}') - \tilde{\mathbf{u}}(\mathbf{x})|^2\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}'d\mathbf{x} \\ \ge& \frac{c_t}{4}\int_{\mathcal{B}_j}\int_{\mathcal{B}_{j-1}} |\tilde{\mathbf{u}}(\mathbf{x})|^2\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}'d\mathbf{x} - \frac{c_t}{2}\int_{\mathcal{B}_j}\int_{\mathcal{B}_{j-1}} |\tilde{\mathbf{u}}(\mathbf{x}')|^2\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}'d\mathbf{x}. \end{aligned}$$

Define a linear continuous function in $\bar{\mathcal{B}}_j$ by

$$H_j(\mathbf{x}) := \int_{\mathcal{B}_{j-1}} \chi_\delta(\mathbf{x}' - \mathbf{x})d\mathbf{x}' \text{ for } \mathbf{x} \in \bar{\mathcal{B}}_j. \tag{32}$$

Clearly, it attains the minimum in $\bar{\mathcal{B}}_j$, and write as

$$\alpha_j := \min_{\mathbf{x}\in\bar{\mathcal{B}}_j} H_j(\mathbf{x}) > 0. \tag{33}$$

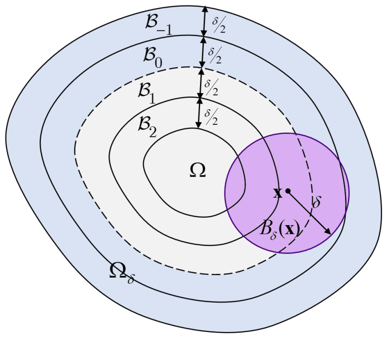

FIGURE 2. Domain Decomposition.

Inserting (33) into (31) yields a key inequality as follows,

$$A_\delta(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) \geq \frac{c_t}{4}\alpha_j \|\tilde{\mathbf{u}}(\mathbf{x})\|^2_{L^2(\mathcal{B}_j)} - \frac{c_t \pi \delta^2}{2}\|\tilde{\mathbf{u}}(\mathbf{x}')\|^2_{L^2(\mathcal{B}_{j-1})}. \tag{34}$$

Further, for $j = 1, 2, ..., l$, we derive that in turn

$$\|\tilde{\mathbf{u}}\|^2_{L^2(\mathcal{B}_1)} \leq \frac{4}{c_t \alpha_1} A_\delta(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) + \frac{2\pi\delta^2}{\alpha_1}\|\tilde{\mathbf{u}}\|^2_{L^2(\mathcal{B}_0)}. \tag{35}$$

$$\|\tilde{\mathbf{u}}\|^2_{L^2(\mathcal{B}_2)} \leq \Big(\frac{4}{c_t \alpha_2} + \frac{2 \times 4\pi\delta^2}{c_t \alpha_1 \alpha_2}\Big) A_\delta(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) + \frac{(2\pi\delta^2)^2}{\alpha_1 \alpha_2}\|\tilde{\mathbf{u}}\|^2_{L^2(\mathcal{B}_0)}. \tag{36}$$

$$\|\tilde{\mathbf{u}}\|^2_{L^2(\mathcal{B}_3)} \leq \Big(\frac{4}{c_t \alpha_3} + \frac{2 \times 4\pi\delta^2}{c_t \alpha_2 \alpha_3} + \frac{(4\pi\delta^2)^2}{c_t \alpha_1 \alpha_2 \alpha_3}\Big) A_\delta(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) + \frac{(2\pi\delta^2)^3}{\alpha_1 \alpha_2 \alpha_3}\|\tilde{\mathbf{u}}\|^2_{L^2(\mathcal{B}_0)}. \tag{37}$$

By the standard method of induction, we can deduce that there exists a constant $\gamma = \gamma(c_t, \alpha_1, \alpha_2, ..., \alpha_j) > 0$, such that for $j = 1, 2, ..., l$, the estimate

$$\|\tilde{\mathbf{u}}\|^2_{L^2(\mathcal{B}_j)} \leq \gamma A_\delta(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) + \frac{(2\pi\delta^2)^j}{\alpha_1 \times \alpha_2 \times \cdots \times \alpha_j}\|\tilde{\mathbf{u}}\|^2_{L^2(\mathcal{B}_0)} \tag{38}$$

is valid. Due to $\tilde{\mathbf{u}}|_{\Omega_\delta} = \mathbf{0}$ and $\mathcal{B}_0 \subseteq \Omega_\delta$, then the second term of the right-hand side in (38) vanishes, thus summing for $j = 1, 2, 3, ..., l$, we have

$$\frac{1}{l\gamma}\|\tilde{\mathbf{u}}\|^2_{L^2(\Omega)} = \frac{1}{l\gamma}\|\tilde{\mathbf{u}}\|^2_{L^2(\Omega \cup \Omega_d)} \leq A_\delta(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}), \tag{39}$$

which follows that the coercivity of $A_\delta(\cdot, \cdot)$ on $L^2(\Omega \cup \Omega_\delta) \times L^2(\Omega \cup \Omega_\delta)$.

Lax-Milgram theorem, we obtain existence results of problem (24) and the estimate

$$\|\tilde{\mathbf{u}}\|_{L^2(\Omega \cup \Omega_\delta)} \leq l\gamma\Big(\|\boldsymbol{b}\|_{L^2(\Omega)} + c_n \pi \delta^2 \|\mathbf{g}\|_{L^2(\Omega_\delta)}\Big). \tag{40}$$

Consequently, according to (23), we can derive that the existence and uniqueness of weak solution $\mathbf{u}(\mathbf{x})$ to the problem (22), as well as the desired estimate (28). $\square$

3.2. Semi-positive definite kernel ($c_t = 0$). In this subsection, we consider the following homogenous Dirichlet-type nonlocal boundary problem,

$$\begin{cases} -\mathcal{L}_\delta \mathbf{u}(\mathbf{x}) = \boldsymbol{b}(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \mathbf{u}(\mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \Omega_\delta, \end{cases} \tag{41}$$

where

$$\mathcal{L}_\delta \mathbf{u}(\mathbf{x}) := c_n \int_{\Omega\cup\Omega_\delta} \frac{(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}'. \tag{42}$$

The bilinear form $A_\delta(\cdot,\cdot)$ and linear functional $F(\cdot)$ associated to (41) are as follows,

$$A_\delta(\mathbf{u},\mathbf{u}) := \frac{3E}{4\pi\delta^4}\int_{\Omega\cup\Omega_\delta}\int_{\Omega\cup\Omega_\delta} \chi_\delta(\mathbf{x}'-\mathbf{x})\left|\frac{(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|}\cdot(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))\right|^2 d\mathbf{x}'d\mathbf{x}, \tag{43a}$$

$$F(\mathbf{u}) := \int_\Omega \boldsymbol{b}(\mathbf{x})\mathbf{u}(\mathbf{x})d\mathbf{x}. \tag{43b}$$

Observe that in (42), the kernel $\mathbf{k}(\mathbf{x}'-\mathbf{x})$ reduces to a semi-positive definite kernel, then the coercivity of $A_\delta(\cdot,\cdot)$ can't be directly deduced from the key inequality (34), which needs other compactness arguments to establish. To verify that the coercivity of $A_\delta(\cdot,\cdot)$ in $L^2(\Omega\cup\Omega_\delta)\times L^2(\Omega\cup\Omega_\delta)$ holds, we first recall a key lemma from Bourgain, Brezis and Mironescu, for more details on this lemma, we refer to [25], [26] and the references therein.

Lemma 3.2. *([25], Lemma 2) Let $g(\tau), h(\tau) : (0,\delta) \to \mathbb{R}^+$. Assume $g(\tau) \le g(\tau/2)$ for $\tau \in (0,\delta)$, and that $h(\tau)$ is nonincreasing. Then there exists a constant $C = C(N) > 0$, such that*

$$\int_0^\delta \tau^{N-1}g(\tau)h(\tau)d\tau \ge C\delta^{-N}\int_0^\delta \tau^{N-1}g(\tau)d\tau\int_0^\delta \tau^{N-1}h(\tau)d\tau, \tag{44}$$

where N denotes a positive constant.

Lemma 3.3. *If there exists a uniformly bounded sequence $\bar{\mathbf{u}}_j = (\bar{u}_{j,1}, \bar{u}_{j,2}) \in L^2(\mathbb{R}^2)$ satisfying*

$$\lim_{j\to\infty}\|\mathbf{Q}^\delta * \bar{\mathbf{u}}_j - \bar{\mathbf{u}}_j\|_{L^2(\mathbb{R}^2)} = 0. \tag{45}$$

Then for any open bounded subset $\mathcal{D}$ of $\mathbb{R}^2$, $\bar{\mathbf{u}}_j\big|_{\mathcal{D}}$ is relatively compact in $L^2(\mathcal{D})$, where

$$\mathbf{Q}^\delta(\mathbf{x}'-\mathbf{x}) := \frac{2}{\pi\delta^2}\frac{(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\chi_\delta(\mathbf{x}'-\mathbf{x}) =: \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22}\end{pmatrix}. \tag{46}$$

Proof. Without loss of generality, we may assume that $\|\bar{\mathbf{u}}_j\|_{L^2(\mathbb{R}^2)} = 1$. Due to $\mathbf{Q}^\delta \in L^1(\mathbb{R}^2)$, then every component element $Q_{ik}(i,k=1,2)$ also belongs to $L^1(\mathbb{R}^2)$. Using compact arguments in [27, p.74, Corollary 4.27], it follows that $Q_{ik} * \bar{u}_{j,k}$ is relatively compact in $L^2(\mathcal{D})$ for any open bounded subset $\mathcal{D} \subset \mathbb{R}^2$, and so $\Big(\mathbf{Q}^\delta * \bar{\mathbf{u}}_j\Big)_i = \sum_{k=1}^{2} Q_{ik} * \bar{u}_{j,k}$ is also relatively compact in $L^2(\mathcal{D})$. Since $L^2(\mathcal{D})$ is a complete Banach space, so $\Big(\mathbf{Q}^\delta * \bar{\mathbf{u}}_j\Big)_i$ is totally bounded, and has a finite ε-cover. Together with (45), we have

$$\|\bar{u}_{j,i}\|_{L^2(\mathcal{D})} \le \|\Big(\mathbf{Q}^\delta * \bar{\mathbf{u}}_j\Big)_i\|_{L^2(\mathcal{D})} + \|\Big(\mathbf{Q}^\delta * \bar{\mathbf{u}}_j\Big)_i - \bar{u}_{j,i}\|_{L^2(\mathcal{D})}. \tag{47}$$

Hence, each component $\bar{u}_{j,i}$ of $\bar{\mathbf{u}}_j$ has a finite ε-cover, and so do $\bar{\mathbf{u}}_j$. This completes the proof. $\square$

The next lemma reveals that conditions of the coercivity of bilinear form $A_\delta(\cdot,\cdot)$ in $L^2(\Omega\cup\Omega_\delta)\times L^2(\Omega\cup\Omega_\delta)$ holds. This lemma is a modified version of the Proposition 1 in [12] for complicated form of $c_n = \frac{12E}{\pi\delta^4}$. But due to its importance, we present the proof here.

Lemma 3.4. *There exists a positive constant ϑ independent of δ, such that*

$$A_\delta(\mathbf{u},\mathbf{u}) \geq \vartheta \|\mathbf{u}\|^2_{L^2(\Omega\cup\Omega_\delta)}. \tag{48}$$

Proof. In fact, it suffices to show $\vartheta > 0$, since (48) is equivalent to

$$\inf_{\|\mathbf{u}\|_{L^2(\Omega\cup\Omega_\delta)}=1} A_\delta(\mathbf{u},\mathbf{u}) \geq \vartheta. \tag{49}$$

Clearly, $\vartheta \geq 0$. Let us prove $\vartheta > 0$ by contradiction. Assume that $\vartheta = 0$, then there exists a sequence $\mathbf{u}_j \in L^2(\Omega\cup\Omega_\delta)$, such that

$$\|\mathbf{u}_j\|_{L^2(\Omega\cup\Omega_\delta)} = 1, \tag{50}$$

and

$$\lim_{j\to\infty} \frac{3E}{4\pi\delta^4}\int_{\Omega\cup\Omega_\delta}\int_{\Omega\cup\Omega_\delta} \chi_\delta(\mathbf{x}'-\mathbf{x})\left|\frac{(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|}\cdot\big(\mathbf{u}_j(\mathbf{x}')-\mathbf{u}_j(\mathbf{x})\big)\right|^2 d\mathbf{x}'d\mathbf{x} = 0. \tag{51}$$

We define a radial function sequence ρ_j as

$$\rho_j(|\boldsymbol{\xi}|) := \frac{4j^4}{2\pi}|\boldsymbol{\xi}|^2\chi_{[0,1]}(j|\boldsymbol{\xi}|),\ \boldsymbol{\xi}\in\mathbb{R}^2. \tag{52}$$

Then it is easy to check that $\rho_j(|\boldsymbol{\xi}|)$ satisfies

$$\begin{cases} \rho_j(|\boldsymbol{\xi}|) \geq 0 \text{ a.e in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2}\rho_j(|\boldsymbol{\xi}|)d\boldsymbol{\xi} = 1 \text{ for all } j\in\mathbb{N}, \\ \displaystyle\lim_{j\to\infty}\int_{|\boldsymbol{\xi}|>\kappa}\rho_j(|\boldsymbol{\xi}|)d\boldsymbol{\xi} = 0 \text{ for every } \kappa>0. \end{cases} \tag{53}$$

Inserting (52) into (51) and taking $\delta = \frac{1}{j}$ leads to

$$\begin{aligned} &\lim_{j\to\infty} A_\delta(\mathbf{u}_j,\mathbf{u}_j) \\ =&\frac{3E}{8}\int_{\Omega\cup\Omega_{\frac{1}{j}}}\int_{\Omega\cup\Omega_{\frac{1}{j}}}\left|\frac{(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|}\cdot(\mathbf{u}_j(\mathbf{x}')-\mathbf{u}_j(\mathbf{x}))\right|^2\frac{\rho_j(|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-!\mathbf{x}|^2}d\mathbf{x}'d\mathbf{x} = 0. \end{aligned} \tag{54}$$

On the other hand, let

$$\bar{\mathbf{u}}_j(\mathbf{x}) := \begin{cases} \mathbf{u}_j(\mathbf{x}), & \mathbf{x}\in\Omega, \\ \mathbf{0}, & \mathbf{x}\notin\Omega. \end{cases} \tag{55}$$

By virtue of (50) and (46), we have $\|\bar{\mathbf{u}}_j\|_{L^2(\mathbb{R}^2)} = 1$ and

$$\begin{aligned} &\int_{\mathbb{R}^2}\left|\Big(\mathbf{Q}^\delta * \bar{\mathbf{u}}_j\Big)(\mathbf{x}) - \bar{\mathbf{u}}_j(\mathbf{x})\right|^2 d\mathbf{x} \\ \leq&\frac{2}{\pi\delta^2}\int_0^\delta\Bigg(\underbrace{\int_{S^1}\int_{\mathbb{R}^2}\left|s\big(\bar{\mathbf{u}}_j(\mathbf{x}+\tau s)-\bar{\mathbf{u}}_j(\mathbf{x})\big)\right|^2 d\mathbf{x}d\sigma(s)}_{:=F_j(\tau)}\Bigg)\tau d\tau. \end{aligned} \tag{56}$$

Clearly, $F_j(2\tau) \leq 2^2 F_j(\tau)$. By Lemma 3.2 with $g(\tau) = \frac{F_j(\tau)}{\tau^2}$, $h(\tau) = \rho_j(\tau)$ and $N = 2$, we obtain

$$\int_0^\delta \tau \frac{F_j(\tau)}{\tau^2} d\tau \leq C\delta^2 \int_0^\delta \tau \frac{F_j(\tau)}{\tau^2} \rho_j(\tau) d\tau = C\delta^2 \frac{8}{3E} A_\delta(\bar{\mathbf{u}}_j, \bar{\mathbf{u}}_j). \tag{57}$$

Observe that $\delta > 0$ and $\delta \to 0$, so we can take $\delta > 0$ small enough, such that

$$\int_{\mathbb{R}^2} \left| \left(\mathbf{Q}^\delta * \bar{\mathbf{u}}_j \right)(\mathbf{x}) - \bar{\mathbf{u}}_j(\mathbf{x}) \right|^2 d\mathbf{x} \leq \frac{2}{\pi\delta^2} \int_0^\delta F_j(\tau)\tau d\tau \leq \frac{16C\delta^2}{3\pi E} A_\delta(\bar{\mathbf{u}}_j, \bar{\mathbf{u}}_j) \to 0 \tag{58}$$

as $j \to \infty$. By Lemma 3.3, we have the relative compactness of $\bar{\mathbf{u}}_j\big|_{\mathcal{D}}$ in $L^2(\mathcal{D})$ with $\mathcal{D} = \Omega$. Thus there exists a function $\mathbf{u}(\mathbf{x})$, such that

$$\|\mathbf{u}\|_{L^2(\mathcal{D})} = \|\mathbf{u}\|_{L^2(\Omega \cup \Omega_\delta)} = 1. \tag{59}$$

Using Lebesgue's dominated convergence theorem and passing to the limit yields that

$$\begin{aligned} 0 = &\lim_{j\to\infty} A_\delta(\bar{\mathbf{u}}_j, \bar{\mathbf{u}}_j) = \lim_{j\to\infty} A_\delta(\mathbf{u}_j, \mathbf{u}_j) \\ = &\lim_{j\to\infty} \frac{3E}{4\pi\delta^4} \int_{\Omega\cup\Omega_\delta} \int_{\Omega\cup\Omega_\delta} \chi_\delta(\mathbf{x}' - \mathbf{x}) \left| \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}_j(\mathbf{x}') - \mathbf{u}_j(\mathbf{x})) \right|^2 d\mathbf{x}' d\mathbf{x} \\ = &\frac{3Ej^4}{4\pi} \int_\Omega \int_\Omega \chi_{\frac{1}{j}}(\mathbf{x}' - \mathbf{x}) \left| \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \right|^2 d\mathbf{x}' d\mathbf{x} \geq 0, \end{aligned} \tag{60}$$

it follows that

$$\frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) = 0 \text{ a.e. in } \Omega \times \Omega. \tag{61}$$

By Proposition 1.2 in [28], (61) holds if and only if there exists a vector $\mathbf{r} \in \mathbb{R}^2$ and a antisymmetric matrix $\mathbb{B}$, such that

$$\mathbf{u}(\mathbf{x}) = \mathbb{B}\mathbf{x} + \mathbf{r}, \forall \mathbf{x} \in \Omega, \tag{62}$$

then $\mathbf{u}$ is displacement of a rigid body. Since we don't consider such kinds of solutions in this paper, so by the uniqueness of limits,

$$\mathbf{u} \equiv \mathbf{0} \text{ a.e. in } \Omega, \tag{63}$$

which contradicts with (59). Hence, $\vartheta > 0$. □

We now come to nonlocal problem (41), and establish the corresponding well-posedness results.

Theorem 3.5. *Assume that $\boldsymbol{b} \in L^2(\Omega)$ and the conditions in Lemma 3.4 are fulfilled, then there exists a unique solution $\mathbf{u} \in L^2(\Omega)$ to problem (41), and the estimate*

$$\|\mathbf{u}\|_{L^2(\Omega)} \leq \frac{1}{\vartheta} \|\boldsymbol{b}\|_{L^2(\Omega)} \tag{64}$$

is valid.

Proof. This statement immediately follows from Lemma 3.4 and Lax-Milgram theorem. □

Remark 3.6. *The above Theorem 3.5 only holds for $\delta >$ small enough, which implies that $\Omega \cup \Omega_\delta$ is towards to Ω. Since the $\Omega \cup \Omega_\delta$ is a closed domain, and the compact result in Lemma 3.3 only holds for any open subset $\mathcal{D} \subset \mathbb{R}^2$. Together with $\mathbf{u}\big|_{\Omega_\delta} = \mathbf{0}$, so here we take $\mathcal{D} = \Omega$. Notice that the statement of Lemma 3.4 becomes invalid in the general case $\mathbf{u}\big|_{\Omega_\delta} = \mathbf{g} \not\equiv \mathbf{0}$, this is because the new right-hand term $c_n \pi \delta^2 \|\mathbf{g}\|_{L^2(\Omega_\delta)} \to \infty$ as $\delta \to 0$.*

4. Limit behaviour with horizon vanishes

In the previous sections, we have gained the well-posedness of problems (9a)-(9b) and (41). Then it is desirable to consider the limit behaviour of their solutions as $\delta \to 0$. We first recall a two-dimensional classical (local) elasticity model associated with (1) as follows,

$$\mathbf{u}_{tt}(t,\mathbf{x}) - \mathcal{L}\mathbf{u}(t,\mathbf{x}) = \boldsymbol{b}(t,\mathbf{x}), \quad (t,\mathbf{x}) \in (0,T) \times \mathcal{S}, \tag{65}$$

where $\mathcal{L}\mathbf{u}(t,\mathbf{x}) = G\Delta\mathbf{u}(t,\mathbf{x}) + G\frac{1+\nu}{1-\nu}\nabla\nabla\cdot\mathbf{u}(t,\mathbf{x})$, $G = \frac{E}{2(1+\nu)}$ is the shear modulus. Once the asymptotic convergence of $\mathcal{L}_\delta$ to $\mathcal{L}$ is established, then the convergence proof of solutions between such two kinds of models is standard, thus the core here lies in the proof of the former. For the initial case, we mainly follow the idea in [21], the main difference of this proof is that we do not rely on commutativity of the matrix function's product for diagonalizing matrix functions but direct calculation. In addition, the convergence result of solutions is deduced from continuous dependence estimate instead of the difference in the Fourier expression of the solution. Meanwhile, for the stationary case, the main novelty our proof is that we use integration by parts repeatedly, rather than Taylor expansion. This is the only part of our proof of convergence which departs from that in [13].

4.1. Initial value case. Observe that Fourier symbol matrix functions of these two operators $\mathcal{L}$ and $\mathcal{L}_\delta$ are given by

$$\mathbf{A}(\boldsymbol{\omega}) := G|\boldsymbol{\omega}|^2\mathbb{I} + G\frac{1+\nu}{1-\nu}\boldsymbol{\omega}\otimes\boldsymbol{\omega}, \tag{66}$$

(67)
$$\begin{aligned}\mathbf{A}_\delta(\boldsymbol{\omega}) :=& c_n\int_{B_\delta(\mathbf{0})}\frac{(\boldsymbol{\xi},\boldsymbol{\omega})^2}{2}\frac{1}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi}\otimes\boldsymbol{\xi}d\boldsymbol{\xi} + c_t\int_{B_\delta(\mathbf{0})}\frac{(\boldsymbol{\xi},\boldsymbol{\omega})^2}{2}\frac{1}{|\boldsymbol{\xi}|^2}[\boldsymbol{\xi}\otimes\boldsymbol{\xi}]^*d\boldsymbol{\xi}\\ &+c_n\int_{B_\delta(\mathbf{0})}\frac{\cos(\kappa)(\boldsymbol{\xi},\boldsymbol{\omega})^4}{4!}\frac{1}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi}\otimes\boldsymbol{\xi}d\boldsymbol{\xi} + c_t\int_{B_\delta(\mathbf{0})}\frac{\cos(\kappa)(\boldsymbol{\xi},\boldsymbol{\omega})^4}{4!}\frac{1}{|\boldsymbol{\xi}|^2}[\boldsymbol{\xi}\otimes\boldsymbol{\xi}]^*d\boldsymbol{\xi}\\ =&: I_1 + I_2 + I_3 + I_4\end{aligned}$$

for some κ. From symmetry, we have
(68)
$$I_1 + I_2 = c_n\frac{\pi\delta^4}{32}\begin{pmatrix} 3\omega_1^2+\omega_2^2 & 2\omega_1\omega_2\\ 2\omega_1\omega_2 & 3\omega_2^2+\omega_1^2\end{pmatrix} + c_t\frac{\pi\delta^4}{32}\begin{pmatrix} 3\omega_1^2+\omega_2^2 & -2\omega_1\omega_2\\ -2\omega_1\omega_2 & 3\omega_2^2+\omega_1^2\end{pmatrix}.$$

On the other hand, we set $r_\delta(\boldsymbol{\omega}) := I_3 + I_4$, then for any $\boldsymbol{\omega} \in \mathbb{R}^2$,

$$\begin{aligned} r_\delta(\boldsymbol{\omega}) =& \frac{\cos(\kappa)\delta^6 c_n}{144}\begin{pmatrix} 5\omega_1^4+6\omega_1^2\omega_2^2+\omega_2^4 & 4\omega_1\omega_2^3+4\omega_1^3\omega_2\\ 4\omega_1\omega_2^3+4\omega_1^3\omega_2 & 5\omega_2^4+6\omega_1^2\omega_2^2+\omega_1^4\end{pmatrix}\\ &+\frac{\cos(\kappa)\delta^6 c_t}{144}\begin{pmatrix} 5\omega_1^4+6\omega_1^2\omega_2^2+\omega_2^4 & -4\omega_1\omega_2^3-4\omega_1^3\omega_2\\ -4\omega_1\omega_2^3-4\omega_1^3\omega_2 & 5\omega_2^4+6\omega_1^2\omega_2^2+\omega_1^4\end{pmatrix}.\end{aligned} \tag{69}$$

Together with (3), we can deduce that

$$r_\delta(\boldsymbol{\omega}) \to 0 \text{ as } \delta \to 0. \tag{70}$$

Combining with (66)-(70), then the following relations hold for $\nu = \frac{1}{3}$(that is $c_t = 0$),

$$\mathbf{A}_\delta(\boldsymbol{\omega}) = \mathbf{A}(\boldsymbol{\omega}) + \mathbf{r}_\delta(\boldsymbol{\omega}), \tag{71a}$$

$$\lim_{\delta\to 0} \mathbf{A}_\delta(\boldsymbol{\omega}) = \mathbf{A}(\boldsymbol{\omega}) \text{ a.e. for } \boldsymbol{\omega} \in \mathbb{R}^2. \tag{71b}$$

Let $\{\lambda_{\delta 1}, \lambda_{\delta 2}\}$ and $\{\lambda_1, \lambda_2\}$ be the eigenvalues of the matrix functions $\mathbf{A}_\delta(\boldsymbol{\omega})$ and $\mathbf{A}(\boldsymbol{\omega})$, respectively. Directly from (71b), we can see that

$$\lambda_{\delta i} \to \lambda_i \text{ as } \delta \to 0 \ (i = 1, 2). \tag{72}$$

Proposition 4.1. There exists an orthogonal matrix function $\mathbf{P}$, such that matrix functions $\mathbf{A}(\boldsymbol{\omega})$ and $\mathbf{A}_\delta(\boldsymbol{\omega})$ diagonalize simultaneously.

Proof. Observe that matrix functions $\mathbf{A}_\delta(\boldsymbol{\omega})$ and $\mathbf{A}(\boldsymbol{\omega})$ are symmetric, so they can diagonalize and we can find an orthogonal matrix $\mathbf{P}$, such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}(\boldsymbol{\omega})\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{73}$$

Now we claim that $\mathbf{P}$ also can diagonalize the $\mathbf{A}_\delta(\boldsymbol{\omega})$. In light of (71a) and (73), it suffices to prove that $\mathbf{P}$ can diagonalize the matrix $r_\delta(\boldsymbol{\omega})$. We assume the matrix

$$\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then it is not difficult to see that the subdiagonal elements of left-hand side in (73) satisfy

$$3ab\omega_1^2 + ab\omega_2^2 + 2bc\omega_1\omega_2 + 2ad\omega_1\omega_2 + 3cd\omega_2^2 + cd\omega_1^2 = 0. \tag{74}$$

Multiplying (74) by $2\omega_1^2$ and $2\omega_2^2$, respectively, which follow that

$$6ab\omega_1^4 + 2ab\omega_2^2\omega_1^2 + 4bc\omega_1^3\omega_2 + 4ad\omega_1^3\omega_2 + 6cd\omega_2^2\omega_1^2 + 2cd\omega_1^4 = 0, \tag{75a}$$

$$6ab\omega_1^2\omega_1^2 + 2ab\omega_2^4 + 4bc\omega_1\omega_2^3 + 4ad\omega_1\omega_2^3 + 6cd\omega_2^4 + 2cd\omega_1^2\omega_2^2 = 0. \tag{75b}$$

On the other hand, the subdiagonal elements of the matrix $\mathbf{P}^{\mathrm{T}}r_\delta(\boldsymbol{\omega})\mathbf{P}$ are the same, that is,

$$\begin{aligned} &\left(\mathbf{P}^{\mathrm{T}}r_\delta(\boldsymbol{\omega})\mathbf{P}\right)_{12} = \left(\mathbf{P}^{\mathrm{T}}r_\delta(\boldsymbol{\omega})\mathbf{P}\right)_{21} \\ =&5ab\omega_1^4 + 6ab\omega_1^2\omega_2^2 + ab\omega_2^4 + 4bc\omega_1^3\omega_2^2 + 4bc\omega_1\omega_2^3 \\ &+4ad\omega_1^3\omega_2 + 4ad\omega_1\omega_2^3 + cd\omega_1^4 + 6cd\omega_2^2\omega_1^2 + 5cd\omega_2^4. \end{aligned} \tag{76}$$

Using (75a)-(76) and properties of the orthogonal matrix, we have

$$-(76) = (75a) + (75b) - (76) = (ab + cd)(\omega_1^2 + \omega_2^2)^2 = 0, \tag{77}$$

which implies that $\mathbf{P}^{\mathrm{T}}r_\delta(\boldsymbol{\omega})\mathbf{P}$ is an orthogonal matrix. The proof is completed. □

Theorem 4.2. *Let $\mathbf{u}^\delta$ and $\mathbf{u}^0$ be the solutions of problems (9a) and (65) under the same initial value conditions, respectively. Then for every $t \in (0, T)$,*

$$\lim_{\delta\to 0} \|\mathbf{u}^\delta(t, \cdot) - \mathbf{u}^0(t, \cdot)\|_{L^2(\mathbb{R}^2)} = 0. \tag{78}$$

Proof. By Plancherel formula, for the uniformly bounded and smooth function $\mathbf{u}(t,\mathbf{x})$, we have

$$\begin{aligned}\|\mathcal{L}_\delta\mathbf{u}-\mathcal{L}\mathbf{u}\|_{L^2(\mathbb{R}^2)} &= \|\mathbf{A}_\delta(\boldsymbol{\omega})\bar{\mathbf{u}}(t)-\mathbf{A}(\boldsymbol{\omega})\bar{\mathbf{u}}(t)\|_{L^2(\mathbb{R}^2)}\\ &=\int_{\mathbb{R}^2}\bar{\mathbf{u}}^{\mathrm{T}}(t)[\mathbf{A}_\delta(\boldsymbol{\omega})-\mathbf{A}(\boldsymbol{\omega})]^2\bar{\mathbf{u}}(t)d\boldsymbol{\omega}\\ &=\int_{\mathbb{R}^2}\bar{\mathbf{u}}^{\mathrm{T}}(t)\mathbf{P}^{\mathrm{T}}\begin{pmatrix}(\lambda_{\delta1}-\lambda_1)^2 & 0\\ 0 & (\lambda_{\delta2}-\lambda_2)^2\end{pmatrix}\mathbf{P}\bar{\mathbf{u}}(t)d\boldsymbol{\omega}\\ &\le \max\left\{(\lambda_{\delta1}-\lambda_1)^2,(\lambda_{\delta2}-\lambda_2)^2\right\}\|\bar{\mathbf{u}}(t)\|^2_{L^2(\mathbb{R}^2)}\to 0\end{aligned}\tag{79}$$

as $\delta\to 0$.

Set $\mathbf{v}(t,\mathbf{x}):=\mathbf{u}^\delta(t,\mathbf{x})-\mathbf{u}^0(t,\mathbf{x})$, then it is clear to see that $\mathbf{v}(t,\mathbf{x})$ satisfies

$$\begin{cases}\mathbf{v}_{tt}(t,\mathbf{x})=\mathcal{L}_\delta\mathbf{v}(t,\mathbf{x})+(\mathcal{L}_\delta-\mathcal{L})\mathbf{u}^0, & (t,\mathbf{x})\in(0,T)\times\mathbb{R}^2,\\ \mathbf{v}(0,\mathbf{x})=\mathbf{0},\ \mathbf{v}_t(0,\mathbf{x})=\mathbf{0}, & \mathbf{x}\in\mathbb{R}^2.\end{cases}\tag{80}$$

Using the estimate (16), (17) and calculus basic formulas, we have

$$\|\mathbf{v}(t)\|_{L^2(\mathbb{R}^2)}\le C(T)\|\mathcal{L}_\delta\mathbf{u}^0-\mathcal{L}\mathbf{u}^0\|_{L^2(\mathbb{R}^2)}.\tag{81}$$

Replacing $\mathbf{u}$ by $\mathbf{u}^0$ in (79) and passing to limit in δ completes the proof. □

4.2. Stationary case. Let $\zeta:\mathbb{R}^2\to\mathbb{R}$ be a function with

$$\nabla_{\boldsymbol{z}}\zeta(|\boldsymbol{z}|)=\frac{\boldsymbol{z}}{|\mathbf{z}|^2}\boldsymbol{k}(\boldsymbol{z})=c_n\frac{1}{|\boldsymbol{z}|^2}(z_1,z_2)^T=c_n\frac{\boldsymbol{z}^T}{|\boldsymbol{z}|^2},$$

where

$$\zeta(|\mathbf{z}|):=\int\frac{\mathbf{z}}{|\mathbf{z}|^2}\boldsymbol{k}(\mathbf{z})d\mathbf{z}=c_n\int_\delta^{|\mathbf{z}|}\frac{1}{r^2}rdr,\quad \boldsymbol{k}(\mathbf{z}):=c_n\frac{\mathbf{z}\otimes\mathbf{z}}{|\mathbf{z}|^2}.$$

Theorem 4.3. *Let* $\mathbf{u}\in\mathcal{C}^4(\Omega\cup\Omega_\delta)$ *with* $\sup_{\mathbf{x}\in\Omega}|\mathbf{u}^{(4)}(\mathbf{x})|\le M<\infty$. *If* $c_t=0$, *then*

$$\lim_{\delta\to0}\mathcal{L}_\delta\mathbf{u}(\mathbf{x})=\mathcal{L}\mathbf{u}(\mathbf{x})\ \text{in}\ L^2(\Omega).\tag{82}$$

Proof. By the fundamental theorem of calculus, for any $\mathbf{x}\in\Omega$, we have

$$\begin{aligned}\mathcal{L}_\delta\mathbf{u}(\mathbf{x})&=-\int_{B_\delta(\mathbf{0})}\boldsymbol{k}(\mathbf{z})(\mathbf{u}(\mathbf{x}+\mathbf{z})-\mathbf{u}(\mathbf{x}))d\mathbf{z}\\ &=-c_n\int_{B_\delta(\mathbf{0})}\frac{\mathbf{z}\otimes\mathbf{z}}{|\mathbf{z}|^2}\int_0^1\nabla\mathbf{u}(\mathbf{x}+s\mathbf{z})\mathbf{z}dsd\mathbf{z}\\ &=c_n\int_{B_\delta(\mathbf{0})}\int_0^1\mathrm{div}_{\mathbf{z}}\Big(\mathbf{z}\otimes\mathbf{z}\nabla\mathbf{u}(\mathbf{x}+s\mathbf{z})\Big)\zeta(|\mathbf{z}|)dsd\mathbf{z}.\end{aligned}\tag{83}$$

Notice that

$$\begin{aligned}\mathrm{div}_{\boldsymbol{z}}\Big(\mathbf{z}\otimes\mathbf{z}\nabla\mathbf{u}(\mathbf{x}+s\mathbf{z})\Big)&=\begin{pmatrix}z_1^2\Delta u_1(\mathbf{x}+s\mathbf{z})s\\ z_2^2\Delta u_2(\mathbf{x}+s\mathbf{z})s\end{pmatrix}+\begin{pmatrix}z_1z_2\Delta u_2(\mathbf{x}+s\mathbf{z})s\\ z_2z_2\Delta u_1(\mathbf{x}+s\mathbf{z})s\end{pmatrix}\\ &+\begin{pmatrix}2z_1\frac{\partial u_1}{\partial z_1}(\mathbf{x}+s\mathbf{z})+z_2\frac{\partial u_2}{\partial z_1}(\mathbf{x}+s\mathbf{z})+z_1\frac{\partial u_2}{\partial z_2}(\mathbf{x}+s\mathbf{z})\\ z_2\frac{\partial u_1}{\partial z_1}(\mathbf{x}+s\mathbf{z})+z_1\frac{\partial u_1}{\partial z_2}(\mathbf{x}+s\mathbf{z})+2z_1\frac{\partial u_2}{\partial z_2}(\mathbf{x}+s\mathbf{z})\end{pmatrix}\\ &:=\mathbf{B}_1(\mathbf{z})+\mathbf{B}_2(\mathbf{z})+\mathbf{B}_3(\mathbf{z}).\end{aligned}\tag{84}$$

Concerning the first term $\mathbf{B}_1(\mathbf{z})$, using the fact that $\zeta(|\mathbf{z}|) = 0$ on $\partial B_\delta(\mathbf{0})$, together with symmetry and integration by parts, we derive that

$$
\begin{aligned}
& c_n \int_{\mathcal{B}_\delta(\mathbf{0})} \int_0^1 \mathbf{B}_1(\mathbf{z})\zeta(|\mathbf{z}|) ds d\mathbf{z} \\
=& - c_n \int_{\mathcal{B}_\delta(\mathbf{0})} \int_0^1 \begin{pmatrix} z_1^2\zeta(|\mathbf{z}|)\left(\frac{1-s^2}{2}\right)\Delta\nabla u_1(\mathbf{x}+s\mathbf{z})\mathbf{z} \\ z_2^2\zeta(|\mathbf{z}|)\left(\frac{1-s^2}{2}\right)\Delta\nabla u_2(\mathbf{x}+s\mathbf{z})\mathbf{z} \end{pmatrix} ds d\mathbf{z} \\
& - \frac{c_n\pi}{8}\int_0^\delta r^3 dr \Delta\mathbf{u}(\mathbf{x}) \\
=& c_n \int_{\mathcal{B}_\delta(\mathbf{0})} \int_0^1 \begin{pmatrix} z_1^2\zeta(|\mathbf{z}|)\left(\frac{1}{3} - \frac{s^2}{2} + \frac{s^3}{6}\right)\Delta\nabla^2 u_1(\mathbf{x}+s\mathbf{z})\mathbf{z}\cdot\mathbf{z} \\ z_2^2\zeta(|\mathbf{z}|)\left(\frac{1}{3} - \frac{s^2}{2} + \frac{s^3}{6}\right)\Delta\nabla^2 u_2(\mathbf{x}+s\mathbf{z})\mathbf{z}\cdot\mathbf{z} \end{pmatrix} ds d\mathbf{z} \\
& - \frac{c_n\pi}{8}\int_0^\delta r^3 dr \Delta\mathbf{u}(\mathbf{x}),
\end{aligned} \tag{85}
$$

it follows that

$$
\begin{aligned}
& \left| c_n \int_{\mathcal{B}_\delta(\mathbf{0})} \int_0^1 \mathbf{B}_1(\mathbf{z})\zeta(|\mathbf{z}|) ds d\mathbf{z} + \frac{c_n\pi}{8}\int_0^\delta r^3 dr \Delta\mathbf{u}(\mathbf{x}) \right| \\
\leq & \frac{5c_n\pi\delta^2 M}{24}\int_0^\delta r^3 dr.
\end{aligned} \tag{86}
$$

Arguing as the previous $\mathbf{B}_1(\mathbf{z})$ for $\mathbf{B}_2(\mathbf{z})$ and $\mathbf{B}_3(\mathbf{z})$, we can get

$$
c_n \int_{\mathcal{B}_\delta(\mathbf{0})} \int_0^1 \mathbf{B}_2(\mathbf{z})\zeta(|\mathbf{z}|) ds d\mathbf{z} = \mathbf{0}, \tag{87}
$$

and

$$
\begin{aligned}
& \left| c_n \int_{\mathcal{B}_\delta(\mathbf{0})} \int_0^1 \mathbf{B}_3(\mathbf{z})\zeta(|\mathbf{z}|) d\mathbf{z} ds + \frac{c_n\pi}{8}\int_0^\delta r^3 dr 2\nabla(\nabla\cdot\mathbf{u})(\mathbf{x}) \right| \\
\leq & c_n M\delta^2\pi \int_0^\delta r^3 dr.
\end{aligned} \tag{88}
$$

Inserting (3) into (86)-(88) and taking $\nu = \frac{1}{3}$ yields to

$$
\mathcal{L}_\delta\mathbf{u}(\mathbf{x}) + \underbrace{\frac{E(1+\nu)}{4(1-\nu^2)}[\Delta\mathbf{u} + 2\nabla(\nabla\cdot\mathbf{u})(\mathbf{x})]}_{=:-\mathcal{L}\mathbf{u}(\mathbf{x})} \leq 2M\delta^2 c_n\pi \int_0^\delta r^3 dr = O(\delta^2), \tag{89}
$$

which implies that the nonlocal operator $\mathcal{L}_\delta$ applied to smooth $\mathbf{u}$ approaches the classical elastic operator $\mathcal{L}$ applied to $\mathbf{u}$ at a rate of δ^2 provided that $c_t = 0$.

Consequently, using Lebesgue's dominated convergence theorem, we can conclude that

$$
\lim_{\delta\to 0} \mathcal{L}_\delta\mathbf{u}(\mathbf{x}) = \mathcal{L}\mathbf{u}(\mathbf{x}) \text{ in } L^2(\Omega). \tag{90}
$$

□

Remark 4.4. *The condition $c_t = 0$ is need, which meets the results for the initial case.*

More precisely, the following convergence result holds.

Theorem 4.5. *Let* $\mathbf{u}^\delta$ *be the solution of problem (41), then there exists a limit function* $\mathbf{u}^0 \in H_0^1(\Omega)$*, which is a weak solution of the following classical elastic boundary problem exactly,*

$$\begin{cases} -\mathcal{L}\mathbf{u}(\mathbf{x}) = \boldsymbol{b}(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \mathbf{u}(\mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \partial\Omega. \end{cases} \tag{91}$$

Proof. Observe that

$$\lim_{\delta \to 0} \|\mathbf{Q}^\delta * \bar{\mathbf{u}}^\delta - \bar{\mathbf{u}}^\delta\|_{L^2(\mathbb{R}^2)} = 0, \tag{92}$$

where $\bar{\mathbf{u}}^\delta$ is obtained by extending $\mathbf{u}^\delta$ outside Ω by $\mathbf{0}$.

In light of (64) and Lemma 3.3, then $\mathbf{u}^\delta$ is relative compact in $L^2(\Omega)$ and $\mathbf{u}^\delta = \mathbf{0}$ on Ω_δ. Further, there exists a limit function $\mathbf{u}^0(\mathbf{x}) \in H_0^1(\Omega)$, such that for $\forall \boldsymbol{\varphi} \in C_c^\infty(\Omega \cup \Omega_\delta)$ with $\Omega \subset \text{supp}(\boldsymbol{\varphi})$,

$$\lim_{\delta \to 0}(-\mathcal{L}_\delta \mathbf{u}_\delta, \boldsymbol{\varphi}) = \lim_{\delta \to 0}(\mathbf{u}_\delta, -\mathcal{L}_\delta \boldsymbol{\varphi}) = (\mathbf{u}^0, -\mathcal{L}\boldsymbol{\varphi}) = (-\mathcal{L}\mathbf{u}^0, \boldsymbol{\varphi}) = (\boldsymbol{b}, \boldsymbol{\varphi}), \tag{93}$$

which follows that $\mathbf{u}^0(\mathbf{x})$ satisfies the problem (91) in the sense of distributions. The proof is completed. □

Appendix

Finally, we analyze eigenvalues of the matrix function $\mathbf{A}_\delta(\boldsymbol{\omega})$. Note that $\mathbf{A}_\delta(\boldsymbol{\omega})$ can be rewritten as the following form,

$$\mathbf{A}_\delta(\boldsymbol{\omega}) = \int_{B_{\delta(\mathbf{0})}} (1 - \cos(\boldsymbol{\xi}, \boldsymbol{\omega})) \begin{pmatrix} c_n \cos^2\theta + c_t \sin^2\theta & (c_n - c_t)\cos\theta\sin\theta \\ (c_n - c_t)\cos\theta\sin\theta & c_n \sin^2\theta + c_t\cos^2\theta \end{pmatrix} d\boldsymbol{\xi},$$

where the sign of its eigenvalues $\{\lambda_{\delta i}\}_{i=1}^2$ can be determined by the following quadratic form.

For any $\bar{\mathbf{u}}(\boldsymbol{\omega}) = (\bar{u}_1(\boldsymbol{\omega}), \bar{u}_2(\boldsymbol{\omega})) \not\equiv \mathbf{0}$, we have

$$\begin{aligned} \bar{\mathbf{u}}(\boldsymbol{\omega})\mathbf{A}_\delta(\boldsymbol{\omega})\bar{\mathbf{u}}(\boldsymbol{\omega})^{\mathrm{T}} =& (\bar{u}_1(\boldsymbol{\omega}), \bar{u}_2(\boldsymbol{\omega})) \int_{B_\delta(\mathbf{0})} \frac{1 - \cos(\boldsymbol{\xi}, \boldsymbol{\omega})}{|\boldsymbol{\xi}|^2} \boldsymbol{\xi} \otimes \boldsymbol{\xi} d\boldsymbol{\xi} (\bar{u}_1(\boldsymbol{\omega}), \bar{u}_2(\boldsymbol{\omega}))^{\mathrm{T}} \\ =& \int_{B_\delta(\mathbf{0})} \left(\xi_1 \bar{u}_1(\boldsymbol{\omega}) + \xi_2 \bar{u}_2(\boldsymbol{\omega})\right)^2 \frac{1 - \cos(\boldsymbol{\xi}, \boldsymbol{\omega})}{|\boldsymbol{\xi}|^2} d\boldsymbol{\xi} > 0, \end{aligned}$$

it follows that $\lambda_{\delta 1} > 0, \lambda_{\delta 2} > 0$. Hence, the matrix function $\mathbf{A}_\delta(\boldsymbol{\omega})$ is positive definite, and so $\frac{1}{\sqrt{\mathbf{A}_\delta(\boldsymbol{\omega})}}$ is well-defined.

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