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THE DISCRETE RAVIART-THOMAS MIXED FINITE ELEMENT METHOD FOR THE p-LAPLACE EQUATION

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Abstract. We consider the discrete Raviart-Thomas mixed finite element method (dRT-MFEM) for the p-Laplace equation in the new sense of measurement. The new measurement of p-Laplace equation for $2 \leq p < \infty$ was studied by D. J. Liu (APPL. NUMER. MATH., 152: 323-337, 2020), where the reliable error analysis for conforming and nonconforming FEM were obtained. This paper provide the reliable and efficient error analysis of dRT-MFEM for p-Laplace equation $(1 < p < 2)$. The numerical investigation for benchmark problem demonstrates the accuracy and robustness of the proposed dRT-MFEM.

Key words. Adaptive finite element methods, discrete Raviart-Thomas mixed finite element method, p-Laplace equation.

1. Introduction

We discuss the following nonlinear p-Laplace equation $(1 < p < 2)$ in the bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ with the given $f \in L^q(\Omega)$ (q conjugate of p),

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The p-Laplace equation (1) admits a unique weak solution [4] satisfying

$$u = \arg\min E(v) \quad \text{for } v \in W_0^{1,p}(\Omega) := \{v \in W^{1,p}(\Omega) : v|_{\partial\Omega} = 0\}. \tag{2}$$

where

$$E(v) := \int_\Omega W(\nabla v)dx - \widetilde{F}(v). \tag{3}$$

The energy density function $W : \mathbb{R}^2 \to \mathbb{R}$ reads $W(a) := |a|^p/p$ with the derivative $\sigma(a) := DW(a) = |a|^{p-2}a$ for all $a \in \mathbb{R}^2 \setminus \{0\}$ which is recorded as σ for the convenience of subsequent discussion and $\widetilde{F}(v) := \int_\Omega fv\,dx$ and the dual function

$$W^*(a) := \frac{|a|^q}{q} \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

The Euler-Lagrange equation of (2) consists in finding $u \in W_0^{1,p}(\Omega)$ with

$$\int_\Omega \sigma \cdot \nabla v dx - \widetilde{F}(v) = 0 \quad \text{for all } v \in W_0^{1,p}(\Omega). \tag{4}$$

The finite element analysis for (1) has been well done. We can find some previous work in sense of traditional $W^{1,p}(\Omega)$-norm in [12, 15, 23, 13]. Sharper error estimates were derived in [18, 14, 3] by developing the so called quasi-norm techniques, and these techniques were extended to establish improved a posteriori error estimators of residual type for the adaptive finite element methods [11, 19]. Liu [17, 16] generalized the quasi-norm techniques to a new measure framework for $2 \leq p < \infty$,

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and obtained the reliable error analysis for conforming FEM, nonconforming FEM, and dRT-MFEM. Nevertheless, the research for $1 < p < 2$, which including a singular operator, was not covered in the above references. In this paper, we mainly focus on the dRT-MFEM of p-Laplace equation for $1 < p < 2$.

Marini representation [2, 21] was proposed for the purpose of the cost-free approximation of Raviart-Thomas MFEM for linear problem. Arbogast [1] improved the method for general variable coefficients elliptic PDEs. A one-point quadrature rule in the dual Raviart-Thomas MFEM leads to the dRT-MFEM in [10], which developed the Marini representation for nonlinear optimal design problem, the first guaranteed energy bound and an optimal a posteriori error estimate were obtained. Liu [17] generalized the dRT-MFEM for p-Laplace equation ($2 \leq p < \infty$), and provided the reliable error analysis. This paper will study the dRT-MFEM of p-Laplace problem for $1 < p < 2$, and show the error estimators without a gap between the upper bound and the lower bound.

The remaining parts of this paper are organized as follows. Section 2 introduces the newly defined measure to quantify the quality of approximations, and proves the convex control of energy density function W. Section 3 states the dRT-MFEM for the p-Laplace problem. A priori and a posteriori error estimators based on the newly defined measure are presented in Section 4. Some numerical experiments conclude the paper in Section 5 with empirical evidence of the expected convergence.

Standard notation applies throughout this paper to Lebesgue and Sobolev spaces $L^q(\Omega)$, $H^s(\Omega)$, and $H(\mathrm{div}, \Omega)$, as well as to the associated norms $\|\cdot\|_{q,\Omega} := \|\cdot\|_{L^q(\Omega)}$, $|||\cdot|||_{q,\Omega} := \|\nabla \cdot\|_{L^q(\Omega)}$, and $|||\cdot|||_{NC,q,\Omega} := \|\nabla_{NC} \cdot\|_{L^q(\Omega)}$ with the piecewise gradient $\nabla_{NC} \cdot |_T := \nabla(\cdot|_T)$ for all T in a regular triangulation $\mathcal{T}$ of the polygonal domain Ω. Here and throughout,the expression "$\lesssim$" abbreviates an inequality up to some multiplicative generic constant, i.e., $A \lesssim B$ means $A \leq CB$ with some generic constant $0 \leq C < \infty$, which depends on the interior angles of the triangles but not their sizes.

2. The convexity control of W

We firstly recall the concept of distance. Define

$$F(a) := |a|^{p/2-1}a \quad \forall a \in L^2(\Omega; \mathbb{R}^2).$$

Let $\alpha := DW(a)$, $\beta := DW(b)$ for $a, b \in L^2(\Omega; \mathbb{R}^2)$, the distance of $F(a)$ and $F(b)$ can be defined as follows [16]

$$\|F(a) - F(b)\|^2_{2,q,\Omega} := \int_\Omega \frac{|\alpha - \beta|^2}{(|\alpha| + |\beta|)^{2-q}} dx \quad \forall a, b \in \mathbb{R}^2. \tag{5}$$

The remaining parts of this section are devoted to the convexity control of energy density function W, which is formulated in the following lemma 2.2.

Lemma 2.1. *Given $1 < p < 2$ and the conjugate q, there exist positive constants $s_1(p)$, $s_2(p)$, $m_1(p)$, $m_2(p)$, $l_1(p)$, $l_2(p)$ such that for any $a, b \in L^2(\Omega; \mathbb{R}^2)$, $\alpha := DW(a)$, $\beta := DW(b)$ satisfy*

$$\begin{aligned} s_1(p) \left(DW(b) - DW(a)\right) \cdot (b - a) &\leq |DW(b) - DW(a)|^2 \left(|\alpha| + |\beta|\right)^{q-2} \\ &\leq s_2(p) \left(DW(b) - DW(a)\right) \cdot (b - a) . \end{aligned} \tag{6}$$

$$\begin{aligned} m_1(p) \left(|b| + |a|\right)^{p-2} |b - a|^2 &\leq \left(DW(b) - DW(a)\right) \cdot (b - a) \\ &\leq m_2(p) \left(|b| + |a|\right)^{p-2} |b - a|^2. \end{aligned} \tag{7}$$

$$l_1(p)\,(W(b)-W(a)-\alpha\cdot(b-a)) \le (DW(b)-DW(a))\cdot(b-a) \le l_2(p)\,(W(b)-W(a)-\alpha\cdot(b-a))\,. \tag{8}$$

where $s_1(p)=\min\{(p-1)\,2^{q-2},1\}$, $s_2(p)=2^{q-2}$, $m_1(p)=(p-1)\,2^{2-p}$, $m_2(p)=2^{2-p}$, $l_1(p)=\frac{p+1}{2}$, *and* $l_2(p)=\frac{3p-p^2}{p-1}$.

Proof. We firstly consider the case when $|b|>|a|>0$. Set $t:=|b|/|a|(t>1)$, $r_1:=a/|a|$, $r_2:=b/|b|$, and $r=r_1\cdot r_2$ $(-1\le r\le 1)$. It is not difficult to deduce that

$$\frac{|DW(b)-DW(a)|^2\,(|\alpha|+|\beta|)^{q-2}}{(DW(b)-DW(a))\cdot(b-a)}=\frac{\left(t^{2p-2}-2rt^{p-1}+1\right)\left(1+t^{p-1}\right)^{q-2}}{t^p-t^{p-1}r-tr+1}:=g(t,r). \tag{9}$$

A direct calculation verifies that $\partial g/\partial r$ as a function of r has one sign (which depends on t and p), hence it is monotone increasing or decreasing for all $t>1$,

$$\min\{g(t,1),g(t,-1)\}\le g(t,r)\le\max\{g(t,1),g(t,-1)\}<\infty.$$

$$g(t,1)=\frac{\left(t^{p-1}-1\right)\left(1+t^{p-1}\right)^{q-2}}{t-1}.$$

$$g'(t,1)=\frac{\left(1+t^{p-1}\right)^{q-3}\left[(2p-3)\,t^{p-1}+(3-2p)\,t^{p-2}-t^{2p-3}+1\right]}{(t-1)^2}.$$

Let

$$Q:=(2p-3)\,t^{p-1}+(3-2p)\,t^{p-2}-t^{2p-3}+1.$$

If $p\in(1,\frac{3}{2})$, $dQ=(2p-3)\,t^{p-3}[(p-1)\,t-t^{p-1}-(p-2)]<0$ implies that $g'(t,1)<0$ and thus

$$g_{\max}(t,1)=\lim_{t\to1}g(t,1)=(p-1)\,2^{q-2},\quad g_{\min}(t,1)=\lim_{t\to+\infty}g(t,1)=1.$$

If $p\in(\frac{3}{2},2)$, $dQ>0$ implies that $g'(t,1)>0$ and

$$g_{\max}(t,1)=\lim_{t\to+\infty}g(t,1)=1,\quad g_{\min}(t,1)=\lim_{t\to1}g(t,1)=(p-1)\,2^{q-2}.$$

Due to the bound of $g(t,-1)$, we have

$$g(t,-1)=\frac{\left(1+t^{p-1}\right)^{q-1}}{(1+t)}.$$

$$g'(t,-1)=\frac{\left(1+t^{p-1}\right)^{q-2}\left(t^{p-2}-1\right)}{(1+t)^2}.$$

The fact $g'(t,-1)<0$ implies that $g(t,-1)$ is monotone decreasing, and

$$g_{\max}(t,-1)=\lim_{t\to1}g(t,-1)=2^{q-2},\quad g_{\min}(t,-1)=\lim_{t\to+\infty}g(t,-1)=1.$$

Denote

$$s_1(p):=\min\{g(t,1),g(t,-1)\}=\min\{(p-1)2^{q-2},1\},$$
$$s_2(p):=\max\{g(t,1),g(t,-1)\}=2^{q-2},$$

then (6) is obtained. In the case of $|a|>|b|>0$, the symmetry of a and b leads to the same conclusion, inequality (6) is proved.

We can obtain (7) with similar technique, we write it here for complement.

$$\frac{(DW(b)-DW(a))\cdot(b-a)}{(|b|+|a|)^{p-2}\,|b-a|^2}=\frac{t^p-t^{p-1}r-tr+1}{(1+t)^{p-2}\,(t^2-2rt+1)}:=h(t,r). \tag{10}$$

$$\min\{h(t,1),h(t,-1)\}\le h(t,r)\le\max\{h(t,1),h(t,-1)\}<\infty.$$

$$h(t,1)=\frac{t^{p-1}-1}{(1+t)^{p-2}(t-1)}.$$

$$h_{\max}(t,1)=\lim_{t\to+\infty}h(t,1)=1,\quad h_{\min}(t,1)=\lim_{t\to 1}h(t,1)=(p-1)\,2^{2-p}.$$

$$h(t,-1)=\left(t^{p-1}+1\right)(1+t)^{1-p}.$$

$$h_{\max}(t,-1)=\lim_{t\to 1}h(t,-1)=2^{2-p},\quad h_{\min}(t,-1)=\lim_{t\to+\infty}h(t,-1)=1.$$

Denote

$$m_1(p):=\min\{h(t,1),h(t,-1)\}=(p-1)2^{2-p},$$
$$m_2(p):=\max\{h(t,1),h(t,-1)\}=2^{2-p},$$

then (7) is proved.

Now it comes to the proof of (8). For any $t>0$,

$$\frac{(DW(b)-DW(a))\cdot(b-a)}{(W(b)-W(a)-DW(a)\cdot(b-a))}:=\frac{t^p-t^{p-1}r-tr+1}{t^p/p+1/q-tr}=f(t,r). \tag{11}$$

$$\min\{f(t,1),f(t,-1)\}\le f(t,r)\le\max\{f(t,1),f(t,-1)\}<\infty.$$

$$f(t,1)=\frac{t^p-t^{p-1}-t+1}{t^p/p+1/q-t}.$$

$$f'(t,1)=\frac{\left(pt^{p-1}-(p-1)\,t^{p-2}-1\right)\left(t^p/p+1/q-t\right)-\left(t^p-t^{p-1}-t+1\right)\left(t^{p-1}-1\right)}{\left(t^p/p+1/q-t\right)^2}.$$

The derivative of numerator of $f'(t,1)$ can be written as

$$(p-1)\,t^{p-3}\left[\frac{2}{p}t^p-(p-1)\,t^2+(2p-4)\,t-\frac{(p-1)\,(p-2)}{p}\right].$$

Let

$$L:=\frac{2}{p}t^p-(p-1)\,t^2+(2p-4)\,t-\frac{(p-1)\,(p-2)}{p},$$

$d^2L=2\,(p-1)\left(t^{p-2}-1\right)>0$ on $(0,1)$ implies that $f'(t,1)<0$, and

$$f_{\max}(t,1)=\lim_{t\to 0}f(t,1)=q,\quad f_{\min}(t,1)=\lim_{t\to 1}f(t,1)=2.$$

$d^2L=2\,(p-1)\left(t^{p-2}-1\right)<0$ on $(1,+\infty)$ implies that $f'(t,1)<0$, and

$$f_{\max}(t,1)=\lim_{t\to 1}f(t,1)=2,\quad f_{\min}(t,1)=\lim_{t\to+\infty}f(t,1)=p.$$

Hence, $f_{\max}(t,1)=q$, $f_{\min}(t,1)=p$.

$$f(t,-1)=\frac{t^p+t^{p-1}+t+1}{t^p/p+1/q+t}=p+p\frac{t^{p-1}+(1-p)t+2-p}{t^p+pt+p-1}.$$

If $t\in(0,1)$,

$$f(t,-1)<p+p\cdot\frac{4-2p}{t^p+pt+p-1}<p+p\frac{4-2p}{p-1}=\frac{3p-p^2}{p-1},$$

$$f(t,-1)>p+p\cdot\frac{2-p}{t^p+pt+p-1}>p+p\frac{2-p}{2p}=\frac{p+2}{2}.$$

If $t\in(1,+\infty)$,

$$f(t,-1)<p+\frac{p(4-2p)}{2p}=2,$$

$$f(t,-1)>p+\frac{p-p^2}{2p}=\frac{1+p}{2}.$$

That is, $\frac{p+1}{2} < f(t,-1) < \frac{3p-p^2}{p-1}$, and therefore, $\frac{p+1}{2} < f(t,r) < \frac{3p-p^2}{p-1}$. This leads to (8) with $l_1(p) = \frac{p+1}{2}$ and $l_2(p) = \frac{3p-p^2}{p-1}$, which concludes the proof. $\square$

Lemma 2.2. *Given $1 < p < 2$ and the conjugate q, there exist positive constants $c_1(p)$, $c_2(p)$, such that for any $a, b \in L^2(\Omega;\mathbb{R}^2)$, $\alpha := DW(a)$, $\beta := DW(b)$ satisfy*

$$\begin{aligned}(12)\quad & c_1(p)\int_\Omega (W(b) - W(a) - \alpha\cdot(b-a))\,dx \le \|F(a) - F(b)\|^2_{2,q,\Omega} \\ & \qquad \le c_2(p)\int_\Omega (W(b) - W(a) - \alpha\cdot(b-a))\,dx.\end{aligned}$$

Any $\alpha, \beta \in L^2(\Omega;\mathbb{R}^2)\setminus\{0\}$ and any $b \in \partial W^*(\beta)$ satisfy

$$\begin{aligned}(13)\quad & c_1(p)\int_\Omega (W^*(\alpha) - W^*(\beta) - b\cdot(\alpha-\beta))\,dx \\ \le & \|F(a) - F(b)\|^2_{2,q,\Omega} \\ \le & c_2(p)\int_\Omega (W^*(\alpha) - W^*(\beta) - b\cdot(\alpha-\beta))\,dx.\end{aligned}$$

Here, $c_1(p) = \min\{\frac{p^2-1}{2}\cdot 2^{q-2}, \frac{p+1}{2}\}$ and $c_2(p) = \frac{3p-p^2}{p-1}\cdot 2^{q-2}$.

Proof. Given $a, b \in \mathbb{R}^2\setminus\{0\}$ with $a \ne b$, set $t := |b|/|a|$ and $z := a\cdot b/(|a|\cdot|b|)$ $(-1 \le z \le 1)$, it is obvious that

$$\begin{aligned}&\frac{|\alpha-\beta|^2}{(|\alpha|+|\beta|)^{2-q}(W(b)-W(a)-\alpha\cdot(b-a))} \\ &= \frac{1+t^{2(p-1)}-2zt^{p-1}}{(1+t^{p-1})^{2-q}(t^p/p+1/q-zt)} := f_1(t,z).\end{aligned}$$

For all $0 < t < \infty$, there exists constants $c_1(p)$, $c_2(p)$ satisfy that

$$\min\{f_1(t,1), f_1(t,-1)\} := c_1(p) \le f_1(t,z) \le c_2(p) := \max\{f_1(t,1), f_1(t,-1)\} < \infty.$$

The combination of (6) and (8) in Lemma 2.1 leads to $c_1(p) = \min\{\frac{p^2-1}{2}\cdot 2^{q-2}, \frac{p+1}{2}\}$ and $c_2(p) = \frac{3p-p^2}{p-1}\cdot 2^{q-2}$.

The duality relationship

$$W^*(\alpha) + W(a) = a\cdot\alpha, W^*(\beta) + W(b) = b\cdot\beta$$

and (12) lead to (13), which proves the Lemma. $\square$

3. DRT MFEM for p-Laplace equation

3.1. Finite element discretization. Let $\mathcal{T}$ be a shape-regular triangulation of the simply-connected bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^2$ with polygonal boundary $\partial\Omega$ into closed triangles. Let $\mathcal{E}$ denote the set of all edges, $\mathcal{N}$ denote the set of vertices and $h_T := \mathrm{diam}(T)$ for $T \in \mathcal{T}$. Let

$$\mathcal{P}_k(\mathcal{T}) = \{v_k : \Omega \to \mathbb{R} |\ \forall T \in \mathcal{T},\ v_k|_T \text{ is a polynomial of total degree} \le k\}$$

denote the set of piecewise polynomials and $h_\mathcal{T}|_T = h_T$ for all $T \in \mathcal{T}$.

Let $\Pi_0 : L^q(\Omega) \to \mathcal{P}_0(\mathcal{T})$ denote the L^q projection onto $\mathcal{T}$ piecewise constant, i.e., $(\Pi_0 f)|_T = ⨍_T f dx$ for all $T \in \mathcal{T}$ and $\mathrm{osc}(f,\mathcal{T}) := \|h_\mathcal{T}(f - \Pi_0 f)\|_{q,\Omega}$.

3.2. Discrete Raviart-Thomas mixed FEM. The dual energy E^* is defined as

$$E^*(\tau) := -\int_\Omega W^*(\tau)dx \quad \text{for } \tau \in L^q(\Omega;\mathbb{R}^2).$$

with $W^*(A) := \sup_{B\in\mathbb{R}^2}(A\cdot B - W(B))$ [22]. The dual problem of (2) maximizes the energy E^* in $Q(f) := \{\tau \in H(\mathrm{div},\Omega) \mid f + \mathrm{div}(\tau) = 0 \text{ } a.e. \text{ in } \Omega\}$, written

$$\sigma = \arg\max E^*(Q(f)).$$

The maximizer $\sigma := DW(\nabla u)$ is unique [15] for minimizer u of E in $W_0^{1,p}(\Omega)$.

Define the Raviart-Thomas finite element space

$$RT_0(\mathcal{T}) := \big\{p \in H(\mathrm{div},\Omega) \mid \forall T \in \mathcal{T}, \exists a \in \mathbb{R}^2, b \in \mathbb{R}, \forall x \in T, p = a + bx\big\}$$

and $Q(f,\mathcal{T}) := \{\tau_{RT} \in RT_0(\mathcal{T}) \mid \Pi_0 f + \mathrm{div}(\tau_{RT}) = 0 \text{ } a.e. \text{ in } \Omega\}$.

The discrete Raviart-Thomas mixed finite element approximation σ_{dRT} to the dual solution σ maximizes the energy E_d^* in $Q(f,\mathcal{T})$, written

$$\sigma_{dRT} = \arg\max E_d^*(Q(f,\mathcal{T})). \tag{14}$$

Here,

$$E_d^*(\tau_{RT}) = -\int_\Omega W^*(\Pi_0\tau_{RT})dx \quad \text{for } \tau_{RT} \in Q(f,\mathcal{T}).$$

The strict convexity of W^* in lemma 2.2 shows that the maximizer σ_{dRT} is unique in $Q(f,\mathcal{T})$. An a priori and a posteriori error analysis follows in Section 4.

3.3. The equivalence of dRT-MFEM with CR-NCFEM. The Crouzeix-Raviart finite element space is defined as

$$CR_0^1(\mathcal{T}) := \{v_h \in \mathcal{P}_1(\mathcal{T}) \mid v_h \text{ is continuous at midpoints of interior edges and vanishes at midpoints of boundary edges}\}.$$

The Crouzeix-Raviart finite element approximation u_{CR} to (2) minimizes the energy E_{NC} in $CR_0^1(\mathcal{T})$, written

$$u_{CR} \in \arg\min E_{NC}(CR_0^1(\mathcal{T})). \tag{15}$$

Here,

$$E_{NC}(v_{CR}) := \int_\Omega W(\nabla_{NC} v_{CR})dx - \widetilde{F_h}(v_{CR}) \quad \text{for } v_{CR} \in CR_0^1(\mathcal{T}) \tag{16}$$

and $\widetilde{F_h}(\cdot) := \widetilde{F} \circ \Pi_0(\cdot) = \int_\Omega (\Pi_0 f)\cdot dx$. The discrete stress $\sigma_{CR} := DW(\nabla_{NC} u_{CR})$ is unique [17]. The Euler-Lagrange equations of (15) consists in finding $u_{CR} \in CR_0^1(\mathcal{T})$ with

$$\int_\Omega \sigma_{CR}\cdot\nabla_{NC} v_{CR}\, dx - \widetilde{F_h}(v_{CR}) = 0 \quad \text{for all } v_{CR} \in CR_0^1(\mathcal{T}). \tag{17}$$

Recall the Crouzeix-Raviart interpolation operator $I_{NC} : W_0^{1,p}(\Omega) \to CR_0^1(\mathcal{T})$ $(1 < p < 2)$,

$$(I_{NC} v)(\mathrm{mid}(E)) := \fint_E v ds \text{ for all } E \in \mathcal{E}.$$

Lemma 3.1. (Property of the Crouzeix-Raviart interpolant)[12, 8, 7] *Any $v \in W^{1,p}(\Omega)$ with its interpolation $I_{NC} v$ and the constant κ satisfy* $\Pi_0\nabla v = \nabla_{NC}(I_{NC} v)$ *and*

$$\|v - I_{NC} v\|_{p,\Omega} \le \kappa \|h_\mathcal{T}(I - \Pi_0)\nabla v\|_{p,\Omega} \le \kappa \|h_\mathcal{T}\nabla v\|_{p,\Omega}.$$

Recall the **Friedrichs** inequality

$$\|v\|_{p,\Omega} \le C_F |||v|||_{p,\Omega} \text{ for any } v \in W_0^{1,p}(\Omega)$$

with $C_F \le \text{width}(\Omega)/\pi$ and the discrete **Friedrichs** inequality (see [4]) with some constant $C_{dF} \approx 1$

$$\|v_{CR}\|_{p,\Omega} \le C_{dF} |||v_{CR}|||_{NC,p,\Omega} \text{ for any } v_{CR} \in CR_0^1(\mathcal{T}).$$

Lemma 3.2. (Conforming $\mathcal{P}_3$ companion)[10] *Given any $v_{CR} \in CR_0^1(\mathcal{T})$, there exists some $v_3 \in \mathcal{P}_3(\mathcal{T}) \cap W_0^{1,p}(\Omega)$ with $v_{CR} = I_{NC} v_3$, $\Pi_0 v_{CR} = \Pi_0 v_3$, and*

$$\|h_{\mathcal{T}}^{-1}(v_{CR} - v_3)\|_{p,\Omega} + |||v_{CR} - v_3|||_{NC,p,\Omega} \lesssim \min_{v \in W_0^{1,p}(\Omega)} |||v - v_{CR}|||_{NC,p,\Omega}.$$

Remark 3.1. *The construction process can be described as follows: Given $v_{CR} \in CR_0^1(\mathcal{T})$, define some conforming approximation $v_1 \in \mathcal{P}_1(\mathcal{T}) \cap C_0(\Omega)$ by the averaging of the v_{CR} at node $z \in \mathcal{N}$*

$$v_1(z) := \frac{\sum_{T \in \mathcal{T}(z)} v_{CR}(z)}{|\mathcal{T}(z)|}$$

Adding edge-bubble functions to v_1 defines $v_2 \in \mathcal{P}_2(\mathcal{T}) \cap C_0(\Omega)$ which equals v_1 at all nodes $\mathcal{N}$ and satisfies $\int_E v_{CR}\, ds = \int_E v_2\, ds$ for all $E \in \mathcal{E}$. The last step adds the cubic bubble functions to v_2 such that the resulting function $v_3 \in \mathcal{P}_3(\mathcal{T}) \cap C_0(\Omega)$ equals v_2 along the edges and satisfies

$$\int_T v_{CR}\, dx = \int_T v_3\, dx \quad \text{for all } T \in \mathcal{T}.$$

integration by parts shows

$$\int_T \nabla v_{CR}\, dx = \int_T \nabla v_3\, dx \quad \text{for all } T \in \mathcal{T}.$$

Denote the postprocessing of σ_{CR} by σ_{CR}^*

$$\sigma_{CR}^* := \sigma_{CR} - \frac{\Pi_0 f}{2}(\cdot - \text{mid}(\mathcal{T})) \in \mathcal{P}_1(\mathcal{T}; \mathbb{R}^2).$$

Here, the piecewise affine function $\cdot - \text{mid}(\mathcal{T}) \in \mathcal{P}_1(\mathcal{T})$ equals $x - \text{mid}(T)$ at $x \in T \in \mathcal{T}$ with barycenter $\text{mid}(T)$. It is proved in [10] that $\sigma_{CR}^* \in Q(f, \mathcal{T}) \subseteq H(\text{div}, \Omega)$.

The following conclusion was obtained in [17] for $2 \le p < \infty$, here we focus on the case when $1 < p < 2$. we need to emphasize that the measurement (5) is different with that of [17].

Lemma 3.3. *It holds $\sigma_{CR}^* = \sigma_{dRT}$ and $\max E_d^*(Q(f, \mathcal{T})) = \min E_{NC}(CR_0^1(\mathcal{T}))$.*

Proof. The choice of $\alpha := \Pi_0 \sigma_{dRT}|_T = \sigma_{dRT}(\text{mid}(T))$, $\beta := \Pi_0 \sigma_{CR}^* = \sigma_{CR}$, and $b := \nabla_{NC} u_{CR}$ in Lemma 2.2 leads to

$$\begin{aligned}
&\|F(\partial W^*(\Pi_0 \sigma_{dRT})) - F(\partial W^*(\Pi_0 \sigma_{CR}^*))\|_{2,q,\Omega}^2 \\
&\le c_2(p)\left(E^*(\Pi_0 \sigma_{CR}^*) - E^*(\Pi_0 \sigma_{dRT}) - \int_\Omega \nabla_{NC} u_{CR} \cdot (\sigma_{dRT} - \sigma_{CR}) dx\right).
\end{aligned}$$

$\sigma_{dRT} \in Q(f, \mathcal{T})$ show that the last term vanishes, that is,

$$\|F(\partial W^*(\Pi_0 \sigma_{dRT})) - F(\partial W^*(\Pi_0 \sigma_{CR}^*))\|_{2,q,\Omega}^2 \le c_2(p)(E_d^*(\sigma_{CR}^*) - E_d^*(\sigma_{dRT})).$$

Hence, $\sigma_{CR}^* = \sigma_{dRT}$.

The relation $W^*(\sigma_{CR}) + W(\nabla_{NC} u_{CR}) = \sigma_{CR} \cdot \nabla_{NC} u_{CR}$ implies that

$$\int_\Omega W(\nabla_{NC} u_{CR}) dx - \int_\Omega \sigma_{CR} \cdot \nabla_{NC} u_{CR} dx = -\int_\Omega W^*(\sigma_{CR}) dx = E_d^*(\sigma_{dRT}).$$

We have $E_d^*(\sigma_{dRT}) = E_{NC}(u_{CR})$. This concludes the proof. $\square$

4. Main results

4.1. Upper Bound.

Theorem 4.1. (A priori error estimate). *The discrete stress σ_{dRT} satisfies*

$$
\begin{aligned}
&\|F(\nabla u) - F(\partial W^*(\sigma_{dRT}))\|_{2,q,\Omega}^2 + \|F(\nabla u) - F(\nabla_{NC} u_{CR})\|_{2,q,\Omega}^2 \\
&\le c_2(p)|E_d^*(\sigma_{dRT}) - E^*(\sigma_{dRT})| \\
&\quad + c_2(p)\left(|\int_\Omega (\sigma - \sigma_{dRT}) \cdot (I - \Pi_0)\nabla u_3 dx + \int_\Omega (u - u_3)(I - \Pi_0) f dx|\right).
\end{aligned} \tag{18}
$$

Proof. The choice $\alpha := \sigma_{dRT}$, $\beta := \sigma$, and $b := \nabla u$ in Lemma 2.2 leads to

$$
\begin{aligned}
&\|F(\nabla u) - F(\partial W^*(\sigma_{dRT}))\|_{2,q,\Omega}^2 \\
&\le c_2(p)\left(E^*(\sigma) - E^*(\sigma_{dRT})\right) - c_2(p)\int_\Omega \nabla u \cdot (\sigma_{dRT} - \sigma)\, dx \\
&= c_2(p)\left(E^*(\sigma) - E^*(\sigma_{dRT})\right) + c_2(p)\int_\Omega u(I - \Pi_0) f dx.
\end{aligned} \tag{19}
$$

The choice $\alpha := \sigma$, $\beta := \Pi_0 \sigma_{dRT} = \sigma_{CR}$, and $b := \nabla_{NC} u_{CR}$ in Lemma 2.2 leads to

$$
\begin{aligned}
&\|F(\nabla u) - F(\nabla_{NC} u_{CR})\|_{2,q,\Omega}^2 + c_2(p)\left(E^*(\sigma) - E_d^*(\sigma_{dRT})\right) \\
&\le -c_2(p)\int_\Omega \nabla_{NC} u_{CR} \cdot (\sigma - \Pi_0 \sigma_{dRT})\, dx.
\end{aligned}
$$

The conforming $\mathcal{P}_3$ companion $u_3 \in \mathcal{P}_3(\mathcal{T}) \cap V$ with $u_{CR} = I_{NC} u_3$ from Lemma 3.2 shows

$$
\begin{aligned}
&-\int_\Omega \nabla_{NC} u_{CR} \cdot (\sigma - \Pi_0 \sigma_{dRT})\, dx \\
&= \int_\Omega (\sigma - \sigma_{dRT}) \cdot (I - \Pi_0)\nabla u_3 dx + \int_\Omega u_3 \operatorname{div}(\sigma - \sigma_{dRT})\, dx.
\end{aligned}
$$

We can obtain

$$
\begin{aligned}
&\|F(\nabla u) - F(\nabla_{NC} u_{CR})\|_{2,q,\Omega}^2 + c_2(p)(E^*(\sigma) - E_d^*(\sigma_{dRT})) \\
&\le c_2(p)\left(\int_\Omega (\sigma - \sigma_{dRT}) \cdot (I - \Pi_0)\nabla u_3 dx - \int_\Omega (u_3 - \Pi_0 u_3)(I - \Pi_0) f dx\right).
\end{aligned} \tag{20}
$$

The sum of (19) and (20) implies

$$
\begin{aligned}
&\|F(\nabla u) - F(\partial W^*(\sigma_{dRT}))\|_{2,q,\Omega}^2 + \|F(\nabla u) - F(\nabla_{NC} u_{CR})\|_{2,q,\Omega}^2 \\
&\le c_2(p)|E_d^*(\sigma_{dRT}) - E^*(\sigma_{dRT})| \\
&\quad + c_2(p)\left(|\int_\Omega (\sigma - \sigma_{dRT}) \cdot (I - \Pi_0)\nabla u_3 dx - \int_\Omega (u - u_3)(I - \Pi_0) f dx|\right).
\end{aligned} \tag{21}
$$

Therefore, we obtain the proof of the theorem. $\square$

Theorem 4.2. (A posteriori error estimate).*The discrete stress σ_{dRT} satisfies*

$$
\begin{aligned}
&\frac{1}{p}\|F(\nabla u) - F(\partial W^*(\sigma_{dRT}))\|_{2,q,\Omega}^2 + \|F(\nabla u) - F(\nabla_{NC} u_{CR})\|_{2,q,\Omega}^2 \\
&\le c_2(p)|E_d^*(\sigma_{dRT}) - E^*(\sigma_{dRT})| + c_2(p)\frac{(pC_F\|f\|_{q,\Omega})^{\frac{1}{p-1}} + \|\!|u_3|\!\|_{p,\Omega}}{j_{1,1}}\operatorname{osc}(f, \mathcal{T}) \\
&\quad + \frac{c_2^p(p)}{p}\|\!|I_{NC} u_3 - u_3|\!\|_{NC,p,\Omega}^p,
\end{aligned} \tag{22}
$$

where $C_F \leq \mathrm{width}(\Omega)/\pi$.

Proof. Using the Friedrichs inequality in (3) and the fact $E(u) \leq E(0) = 0$ imply that

$$\left\lVert\!\left\lvert u \right\rvert\!\right\rVert_{p,\Omega} \leq \left(pC_F \|f\|_{q,\Omega}\right)^{\frac{1}{p-1}}. \tag{23}$$

The piecewise Poincaré inequality applies in the following term with the constant $h_T/j_{1,1}$ from [20] shows,

$$\begin{aligned}\int_\Omega (u-u_3)\,(f-\Pi_0 f)\,dx \leq & \frac{\left\lVert\!\left\lvert u-u_3 \right\rvert\!\right\rVert_{p,\Omega}}{j_{1,1}} \mathrm{osc}(f,\mathcal{T}) \\ \leq & \frac{\left(pC_F\|f\|_{q,\Omega}\right)^{\frac{1}{p-1}} + \left\lVert\!\left\lvert u_3 \right\rvert\!\right\rVert_{p,\Omega}}{j_{1,1}} \mathrm{osc}(f,\mathcal{T}).\end{aligned}$$

The Young inequality, $\Pi_0 \nabla u_3 = \nabla_{NC} I_{NC} u_3$ and $|\sigma - \sigma_{dRT}|^{q-2} \leq (|\sigma| + |\sigma_{dRT}|)^{q-2}$ for $q > 2$ show that

$$\begin{aligned}& c_2(p)\int_\Omega (\sigma-\sigma_{dRT})\cdot (I-\Pi_0)\,\nabla u_3 dx \\ & \leq \frac{1}{q}\int_\Omega |\sigma-\sigma_{dRT}|^q dx + \frac{c_2^p(p)}{p}\int_\Omega |(I-\Pi_0)\nabla u_3|^p dx \\ & \leq \frac{1}{q}\int_\Omega |\sigma-\sigma_{dRT}|^2 (|\sigma|+|\sigma_{dRT}|)^{q-2} dx + \frac{c_2^p(p)}{p}\left\lVert\!\left\lvert I_{NC}u_3 - u_3 \right\rvert\!\right\rVert^p_{NC,p,\Omega} \\ & = \frac{1}{q}\|F(\nabla u) - F(\partial W^*(\sigma_{dRT}))\|^2_{2,q,\Omega} + \frac{c_2^p(p)}{p}\left\lVert\!\left\lvert I_{NC}u_3 - u_3 \right\rvert\!\right\rVert^p_{NC,p,\Omega}.\end{aligned}$$

The combination of the preceding displayed inequalities concludes the proof. □

4.2. Lower Bound.

Theorem 4.3. *(*A priori error estimate*). The discrete stress* σ_{dRT} *satisfies*

$$\begin{aligned}& \|F(\nabla u) - F(\partial W^*(\sigma_{dRT}))\|^2_{2,q,\Omega} + \|F(\nabla u) - F(\nabla_{NC} u_{CR})\|^2_{2,q,\Omega} \\ & \geq c_1(p)(E_d^*(\sigma_{dRT}) - E^*(\sigma_{dRT})) + c_1(p)\left(\tilde{F}_h(I_{NC}u) - \tilde{F}(u)\right) \\ & \quad + c_1(p)\int_\Omega u(I-\Pi_0) f dx.\end{aligned}$$

Proof. The choice $\alpha := \sigma_{dRT}$, $\beta := \sigma$, and $b := \nabla u$ in Lemma 2.2 leads to

$$\begin{aligned}& \|F(\nabla u) - F(\partial W^*(\sigma_{dRT}))\|^2_{2,q,\Omega} \\ & \geq c_1(p)\,(E^*(\sigma) - E^*(\sigma_{dRT})) - c_1(p)\int_\Omega \nabla u\cdot (\sigma_{dRT} - \sigma)\,dx \\ & = c_1(p)\,(E^*(\sigma) - E^*(\sigma_{dRT})) + c_1(p)\int_\Omega u(I-\Pi_0) f dx.\end{aligned} \tag{24}$$

The choice $\alpha := \sigma$, $\beta := \sigma_{CR}$, and $b := \nabla_{NC} u_{CR}$ in Lemma 2.2 leads to

$$\begin{aligned}& \|F(\nabla u) - F(\nabla_{NC} u_{CR})\|^2_{2,q,\Omega} + c_1(p)\,(E^*(\sigma) - E^*(\sigma_{CR})) \\ & \geq -c_1(p)\int_\Omega (\sigma - \sigma_{CR})\cdot \nabla_{NC} u_{CR} dx.\end{aligned} \tag{25}$$

Where,

$$
\begin{aligned}
&- c_1(p) \int_\Omega (\sigma - \sigma_{CR}) \cdot \nabla_{NC} u_{CR} dx \\
&= c_1(p) \int_\Omega (\sigma_{CR} - \sigma) (\nabla_{NC} u_{CR} - \nabla u) \, dx + c_1(p) \int_\Omega (\sigma_{CR} - \sigma) \cdot \nabla u dx \\
&= \mathrm{Lower_A} + \mathrm{Lower_B}.
\end{aligned}
$$

By (7) of Lemma 2.1, we have

$$
\begin{aligned}
\mathrm{Lower_A} &\geq (p-1) 2^{2-p} c_1(p) \int_\Omega (|\nabla u| + |\nabla_{NC} u_{CR}|)^{p-2} |\nabla u - \nabla_{NC} u_{CR}|^2 dx \\
&\geq 0.
\end{aligned} \tag{26}
$$

$$
\mathrm{Lower}_B = c_1(p) \int_\Omega (\sigma_{CR} - \sigma) \cdot \nabla u dx = c_1(p) \left(\tilde{F}_h(I_{NC} u) - \tilde{F}(u) \right). \tag{27}
$$

The combination of (26) and (27) implies that

(28)

$$
\|F(\nabla u) - F(\nabla_{NC} u_{CR})\|^2_{2,q,\Omega} + c_1(p)(E^*(\sigma_{CR}) - E^*(\sigma)) \geq c_1(p) \left(\tilde{F}_h(I_{NC} u) - \tilde{F}(u) \right).
$$

Combining the previous results leads to

$$
\begin{aligned}
&\|F(\nabla u) - F(\partial W^*(\sigma_{dRT}))\|^2_{2,q,\Omega} + \|F(\nabla u) - F(\nabla_{NC} u_{CR})\|^2_{2,q,\Omega} \\
&\geq c_1(p)(E^*_d(\sigma_{dRT}) - E^*(\sigma_{dRT})) + c_1(p) \left(\tilde{F}_h(I_{NC} u) - \tilde{F}(u) \right) \\
&\quad + c_1(p) \int_\Omega u(I - \Pi_0) f dx.
\end{aligned} \tag{29}
$$

which concludes the proof. □

Theorem 4.4. *(*A posteriori error estimate*). The discrete stress σ_{dRT} satisfies*

$$
\begin{aligned}
&\frac{1}{C} \sum_{T \in \mathcal{T}} \|h_T \Pi_0 f\|^q_{q,T} \\
&\leq \|F(\nabla u) - F(\partial W^*(\sigma_{dRT}))\|^2_{2,q,\Omega} + \|F(\nabla u) - F(\nabla_{NC} u_{CR})\|^2_{2,q,\Omega} + \varepsilon_1,
\end{aligned} \tag{30}
$$

Where,

$$
\begin{aligned}
\varepsilon_1 := &\sum_{T \in \mathcal{T}} \|h_T (\Pi_0 f - f)\|^q_{q,T} + c_1(p) (p C_F \|f\|_{q,\Omega})^{\frac{1}{p-1}} \operatorname{osc}(f, \mathcal{T}) \\
&+ c_1(p) |E^*(\sigma) - E^*(\sigma_{dRT})|.
\end{aligned}
$$

Proof. We have known that for each $T \in \mathcal{T}$ there exists a bubble function $w_T \in W^{1,p}_0(T)$ with $w_T \geq 0$ and

$$
\int_T w_T dx = |T|, \quad \|w_T\|_\infty \leq C, \quad \|\nabla w_T\|_\infty \leq \frac{C}{h_T}, \tag{31}
$$

where $C > 0$ depends only on the shape regularity of $\mathcal{T}$. Then for $s \in \mathbb{R}$

$$
\int_T (\sigma - \sigma_{CR}) \cdot \nabla(s w_T) dx = \int_T f s w_T dx. \tag{32}
$$

For $\Pi_0 f \in \mathbb{R}$ there exists $s_T \in \mathbb{R}$ such that

$$
s_T (h_T \Pi_0 f) = \frac{|h_T (\Pi_0 f)|^q}{q} + \frac{|s_T|^p}{p}. \tag{33}
$$

Taking $s = h_T s_T$ in (32) leads to

$$
\begin{aligned}
&|T|\left(\frac{|h_T(\Pi_0 f)|^q}{q} + \frac{|s_T|^p}{p}\right) = |T|\Pi_0 f h_T s_T \\
(34)\qquad &= \int_T (\sigma - \sigma_{CR}) \cdot \nabla(h_T s_T w_T)dx + \int_T (\Pi_0 f - f) \cdot h_T s_T w_T dx.
\end{aligned}
$$

The Young's inequality, (31) and $|\sigma - \sigma_{CR}|^{q-2} \le (|\sigma| + |\sigma_{CR}|)^{q-2}$ for $q > 2$ imply

$$
\begin{aligned}
&\int_T (\sigma - \sigma_{CR}) \cdot \nabla(h_T s_T w_T)dx \le C \int_T |\sigma - \sigma_{CR}||s_T|dx \\
&\le \frac{C_\delta}{q}\int_T |\sigma - \sigma_{CR}|^q dx + \frac{\delta C}{p}|T||s_T|^p \\
(35)\qquad &\le \frac{C_\delta}{q}\int_T |\sigma - \sigma_{CR}|^2(|\sigma| + |\sigma_{CR}|)^{q-2}dx + \frac{\delta C}{p}|T||s_T|^p \\
&= \frac{C_\delta}{q}\|F(\nabla u) - F(\nabla_{NC} u_{CR})\|^2_{2,q,T} + \frac{\delta C}{p}|T||s_T|^p.
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_T (\Pi_0 f - f) \cdot h_T s_T w_T dx \le C \int_T (\Pi_0 f - f) \cdot h_T s_T dx \\
(36)\qquad &\qquad \le C_\delta \int_T \frac{|h_T(\Pi_0 f - f)|^q}{q}dx + \delta|T|\frac{|s_T|^p}{p}.
\end{aligned}
$$

Now, taking $\delta > 0$ small enough, (34)-(36) imply that

(37)
$$
\int_T \frac{|h_T(\Pi_0 f)|^q}{q}dx \le \frac{C_\delta}{q}\|F(\nabla u) - F(\nabla_{NC} u_{CR})\|^2_{2,q,T} + C_\delta \int_T \frac{|h_T(\Pi_0 f - f)|^q}{q}dx.
$$

Hence,

$$
(38)\quad \|F(\nabla u) - F(\nabla_{NC} u_{CR})\|^2_{2,q,\Omega} \ge \frac{1}{C_\delta}\sum_{T\in\mathcal{T}} \|h_T \Pi_0 f\|^q_{q,T} - \sum_{T\in\mathcal{T}} \|h_T\left(\Pi_0 f - f\right)\|^q_{q,T}.
$$

The combination of (38) and (24) gives

(39)
$$
\begin{aligned}
&\|F(\nabla u) - F(\partial W^*(\sigma_{dRT}))\|^2_{2,q,\Omega} + \|F(\nabla u) - F(\nabla_{NC} u_{CR})\|^2_{2,q,\Omega} \\
&\ge c_1(p)(E^*(\sigma) - E^*(\sigma_{dRT})) + \frac{1}{C_\delta}\sum_{T\in\mathcal{T}} \|h_T \Pi_0 f\|^q_{q,T} - \sum_{T\in\mathcal{T}} \|h_T\left(\Pi_0 f - f\right)\|^q_{q,T} \\
&\quad + c_1(p)\int_\Omega u(I - \Pi_0) f dx \\
&\ge c_1(p)(E^*(\sigma) - E^*(\sigma_{dRT})) + \frac{1}{C_\delta}\sum_{T\in\mathcal{T}} \|h_T \Pi_0 f\|^q_{q,T} - \sum_{T\in\mathcal{T}} \|h_T\left(\Pi_0 f - f\right)\|^q_{q,T} \\
&\quad - c_1(p)\left(pC_F\|f\|_{q,\Omega}\right)^{\frac{1}{p-1}} \operatorname{osc}(f, \mathcal{T}).
\end{aligned}
$$

The assertion is proved. $\square$

5. Numerical experiments

5.1. Regularity. The data structure and the discrete Euler-Lagrange equation are realized as in [5] and then minimized with the Matlab standard function fminunc and the input of W, DW, and D^2W at x. When $1 < p < 2$, the Euler-Lagrange equation gives rise to a singular differential operator which requires a careful numerical treatment. Hence, we perturb the p-Laplace by using $\rho(t) = (\mu + t)^{p-2}$

with $\mu = 10^{-6}$ replacing $\rho(t) = t^{p-2}$ to avoid singular second derivatives D^2W. The first and second derivatives can be written in the following form,

$$DW(\nabla u) = |\mu + \nabla u|^{p-2}\nabla u$$
$$D^2W(\nabla u) = |\mu + \nabla u|^{p-2}I_2 + (p-2)\,|\mu + \nabla u|^{p-4}\nabla u^{\mathrm{T}}\nabla u$$

with I_2 is identity matrix of order two.

5.2. A posteriori error control. The numerical experiments concern the practical application of the a posteriori error estimates(22), (30) and their efficiency. Denote the left-hand side (LHS) of the estimate by LHS(22). The global upper bounds (GUB) read

$$\begin{aligned}\mathrm{GUB}(22) =& c_2(p)|E_d^*(\sigma_{dRT}) - E^*(\sigma_{dRT})| + c_2(p)\frac{(pC_F\|f\|_{q,\Omega})^{\frac{1}{p-1}} + |||u_3|||_{p,\Omega}}{j_{1,1}}\mathrm{osc}(f,\mathcal{T})\\ &+ \frac{c_2^p(p)}{p}|||I_{NC}u_3 - u_3|||^p_{NC,p,\Omega}.\end{aligned}$$

Denote the right-hand side (RHS) of the estimate by RHS(30). The global lower bounds (GLB) read

$$\mathrm{GLB}(30) = \frac{1}{C}\sum_{T\in\mathcal{T}}\|h_T\Pi_0 f\|^q_{q,T}.$$

The triangulations are either uniform with successive red-refinement or with an adaptive mesh-refinement algorithm with initial mesh $\mathcal{T}_0$ and then, for any triangle T of a triangulation $\mathcal{T}_\ell$ at level $\ell = 0,1,2,3,\cdots$, set

$$\eta^2(T) = |||I_{NC}u_3 - u_3|||^2_{NC,2,T} + \mathrm{osc}^q(f,T).$$

Given all those contributions, mark some set $\mathcal{M}_\ell$ of triangles in $\mathcal{T}_\ell$ of minimal cardinality with the bulk criterion

$$1/2\sum_{T\in\mathcal{T}_\ell}\eta_\ell^2(T) \le \sum_{T\in\mathcal{M}_\ell}\eta_\ell^2(T).$$

The refinement of all triangles in $\mathcal{M}_\ell$ plus minimal further refinements to avoid hanging nodes lead to the triangulation $\mathcal{T}_{\ell+1}$ within the newest-vertex bisection [6, 9]. The convergence history plots display the left-hand side LHS(22), the upper bound GUB(22), the right-hand side RHS(30), and the lower bound GLB(30) as function of the number of degrees of freedom (ndof) in a log-log scale.

5.3. Example 1. Consider the p-Laplace equation on the square domain $\Omega := [-1,1]^2$ with the exact solution

$$u(r) = \begin{cases} \left(\frac{1}{4} - r^2\right)^2 e^{-\frac{r^2}{s}} & \text{for } r \le \frac{1}{2}\\ 0 & \text{for } r > \frac{1}{2}\end{cases}$$

and right-hand side

$$\begin{aligned}f(r) =& -2^{p-1}d^{p-2}\left(2 + \frac{1}{s}d\right)^{p-2} e^{-(p-1)\frac{r^2}{s}}r^{p-2} \times [2\,(p-1)\left(2 + \frac{1}{s}d\right)r^2\\ &+ 2\,(p-1)\,d\left(3 + \frac{1}{s}d\right)\frac{r^2}{s} - pd\left(2 + \frac{1}{s}d\right)]\end{aligned}$$

where, $d = \frac{1}{4} - r^2$, we test this example for $p = \frac{3}{2}$, $s = 0.02$. The reference value for the minimal energy $E = -0.00759423$ was computed by Aitken extrapolation. FIGURE 1. displays the global upper bounds(GUB) and the corresponding error terms(LHS) of the estimate from (22) for uniform and adaptive mesh-refinement.

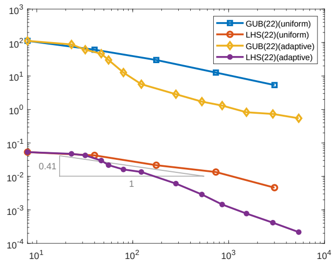

FIGURE 1. Convergence history of upper bound for dRT method on square domain.

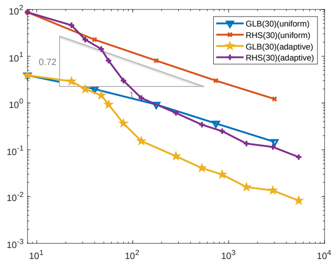

FIGURE 2. Convergence history of lower bound for dRT method on square domain.

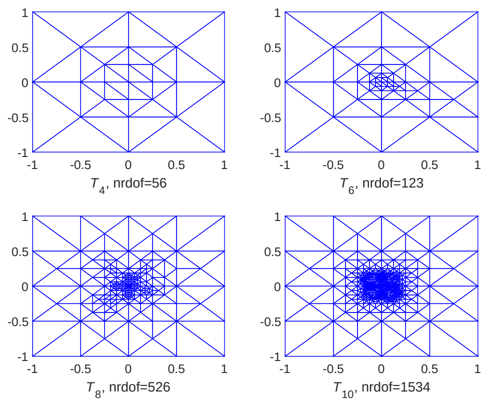

FIGURE 3. Adaptively generated triangulation $\mathcal{T}_l$ for $l = 4, 6, 8, 10$ on square domain.

FIGURE 2. displays the global lower bound(GLB) and the corresponding error terms(RHS) of the estimates from (30) for uniform and adaptive mesh-refinement. FIGURE 3. shows the corresponding sequences of triangulations generated by adaptive FEM for (22).

5.4. Example 2. Consider the p-Laplace equation on the square domain $\Omega := (0,1)^2$ with the exact solution

$$u(r) = \begin{cases} 0 & \text{for } r \leq b \\ (r-b)^4 & \text{for } r > b \end{cases}$$

and right-hand side

$$f(r) = \begin{cases} 0 & \text{for } r \leq b \\ 4^{p-1}(r-b)^{3p-4}\left(2 + \frac{b}{r} - 3p\right) & \text{for } r > b \end{cases}$$

for $p = \frac{4}{3}$, $b = 1.3$. The extrapolated energy reads $E = 0.00000097$. FIGURE 4. displays GUB and LHS of the estimate from (22) for uniform and adaptive mesh-refinement. FIGURE 5. displays GLB and RHS of the estimates from (30) for uniform and adaptive mesh-refinement. FIGURE 6. presents the corresponding sequences of triangulations generated by adaptive FEM for (22).

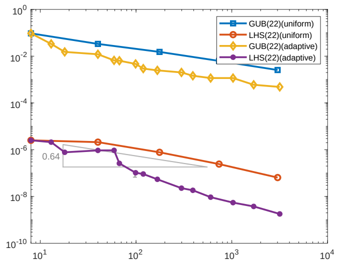

FIGURE 4. Convergence history of upper bounds for dRT method on square domain.

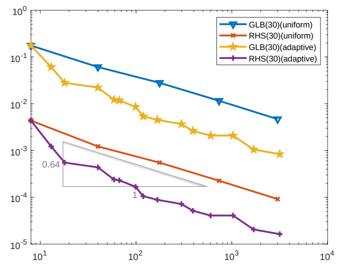

FIGURE 5. Convergence history of lower bound for dRT method on square domain.

5.5. Conclusions. The proposed the dRT-MFEM for p-Laplace equation $(1 < p < 2)$ under the new measure is equivalent to CR-NCFEM, the a posteriori error estimates provide reliable upper bound and efficient lower bound, and the error of the energy can be presented at the same time. The numerical examples show that the convergence results are consistent with the theoretical analysis. However, our case has good smoothness and the selected rectangular domain has no singularity, error estimate is not so sharp. In the follow-up study, we will make up for this deficiency.

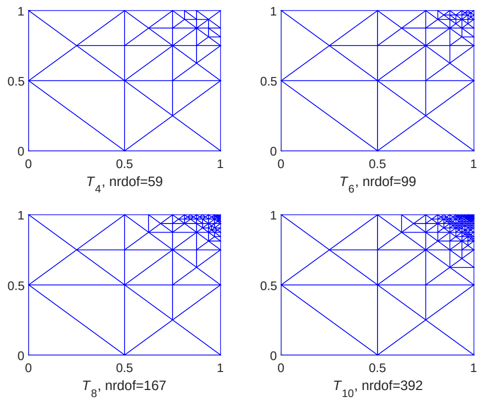

FIGURE 6. Adaptively generated triangulation $\mathcal{T}_l$ for $l = 4, 6, 8, 10$ on square domain.

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